Minimum-Time Paths for a Small Aircraft in the Presence of Regionally-Varying Strong Winds

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We consider the minimum-time path-planning problem for a small aircraft flying horizontally in the presence of obstacles and regionally-varying strong winds. The aircraft speed is not necessarily larger than the wind speed, a fact that has major implications in terms of the existence of feasible paths. First, it is possible that there exist configurations in close proximity to an obstacle from which a collision may be inevitable. Second, it is likely that points inside the obstacle-free space may not be connectable by means of an admissible bidirectional path. The assumption of a regionally-varying wind field has also implications on the optimality properties of the minimum-time paths between reachable configurations. In particular, the minimum-time-to-go and minimum-time-to-come between two points are not necessarily equal. To solve this problem, we consider a convex subdivision of the plane into polygonal regions that are either free of obstacles or they are occupied with obstacles, and such that the vehicle motion within each obstacle-free region is governed by a separate set of equations. The equations of motion inside each obstacle-free region are significantly simpler when compared with the original system dynamics. This approximation simplifies both the reachability/accesibility analysis, as well as the characterization of the locally minimum-time paths. Furthermore, it is shown that the minimum-time paths consist of concatenations of locally optimal paths with the concatenations occurring along the common boundary of neighboring regions, similarly to Snell's law of refraction in optics. Armed with this representation, the problem is subsequently reduced to a directed graph search problem, which can be solved by employing standard algorithms.

I.   Introduction

We consider the problem of steering a partially controllable lightweight aircraft traveling horizontally in the presence of obstacles and spatially varying (strong) winds. The aircraft's mission is to reach the terminal position among a finite set of goal destinations that is the "closest," in terms of travel time, to the aircraft. Our problem is essentially a variation of the Zermelo's navigation problem, named after Zermelo who posed the problem for the first time in 1931 [1]. A typical application of this problem is the minimum-time landing of a small aircraft in an emergency situation. In such a situation, the time for the vehicle to initiate and perform successfully a safe landing is limited. Knowledge of the closest landing site among a number of landing sites/airports in the vicinity of the vehicle, as well as the route to the closest landing site are of paramount importance. The problem is complicated by that fact that the vehicle is partially controllable because the vehicle's forward speed is not necessarily larger than the wind speed. Such a situation typically arises when, for example, a small, lightweight aircraft operates within an area with locally strong winds and/or when the aircraft's ability to steer towards the desired route is diminished owing to, say, a mechanical failure (e.g., loss of thrust).

Path-planning problems with spatially-varying parameters are known in the literature as *weighted region problems*.[2–5] The weighted region problem, first introduced by Mitchell in Ref. [2], and Mitchell and Papadimitriou in Ref. [3], deals with the characterization of the shortest paths from an initial point to a terminal point (or set of terminal points) within a polygonal subdivision of the configuration space with respect to a weighted Euclidean distance function. The authors of Refs. [2, 3] assume that the distance

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weight (cost per unit distance) is homogeneous within each cell of the subdivision. Thus the solution paths of the problems posed in Refs. [2, 3] can be interpreted as the minimum-time paths of a light beam that travels through different isotropic media (where the media are the polygonal cells of the subdivision). The optimal paths of the weighted region problem are concatenations of regionally optimal paths, where the concatenations occur along the boundaries of neighboring regions according to Snell's law of refraction.[3, 6] The paper by Rowe and Alexander[4] presents a generalization of the results presented in Ref. [3] regarding the characterization of the synthesis of optimal paths to a specific goal destination for the weighted region problem, therein stated as a *map of optimal paths*.

The characterization of minimum-time paths for planning problems where the travel speed depends on the direction of motion (anisotropic problems) are, in general, more challenging compared to isotropic problems, mainly due to the fact that the time of travel in anisotropic problem does not qualify as a distance function (it does not enjoy the symmetry property and/or it does not satisfy the triangle inequality). Consequently, standard techniques from the well-studied isotropic problems may not apply to the anisotropic problems. An example of a path planning problem with respect to a distance function that is both regionally and directionally weighted is presented in Ref. [7]. The approach of Ref. [7] is limited in scope to the specific problem treated therein, and thus it does not propose a framework to address more general problems. An analytic solution to a more general class of anisotropic path planning problems than the one treated in Ref. [7] (albeit non-regionally weighted) was first presented in Ref. [8]. In particular, in Ref. [8] it is demonstrated that the solution to the anisotropic path planning problem in the absence of obstacles is composed of a concatenation of at most two line segments. All the results in Ref. [8] however, are limited to the distance weight being a piece-wise linear function of the direction of motion while the solution for general weight functions was only conjectured.

The main limitation of all the aforementioned methods is that they do not provide a sufficiently powerful methodology for addressing more general path-planning problems for vehicles with complex dynamics. By contrast, the theory of optimal control offers a powerful theoretical framework and a plethora of numerical techniques to deal with any path-planning problem, at least in principle. It turns out, however, that in many cases the solution of the path-planning problem formulated as an optimal control problem may be computationally expensive. One technique to relax the optimal control problem is to assume that the vehicle obeys multi-regional dynamics, that is, the vehicle's motion is governed by different and simpler equations of motion within each region of a given partition of its state space. Since the dynamics of the vehicle within each region are simpler, it is likely that the path-planning problem in this multi-regional formulation admits either an analytic solution, or the construction of the solution by means of numerical schemes is a more tractable problem compared to the original problem's formulation. In particular, the optimal solution depends on n parameters, and it is a concatenation of $n + 1$ locally optimal paths, where n is the number of times the vehicle crosses the common boundary of any two neighboring regions during its progression to the goal destination. These parameters determine essentially the behavior of the optimal path on the boundary of neighboring regions, where the dynamics of the system switch during the progression of the vehicle towards the goal destination.[9] It turns out that standard techniques from the calculus of variations[10] furnish conditions that suffice for the characterization of optimal paths that traverse different regions. The theory of optimal hybrid systems[9, 11] provides an alternative solution to this problem. Examples where the approach of hybrid optimal control for systems with multi-regional dynamics is employed can be found in Refs. [12, 13].

In this work we follow the multi-regional optimal control approach while some ideas and techniques from the solution of weighted region problem are also employed in order to address the path planning problem in the presence of strong winds. In particular, given a convex, polygonal subdivision of the state space into a number of regions (cells), we assume that the wind field inside each cell is constant and is equal to the average wind velocity in that region. Consequently, the vehicle dynamics, which depend explicitly on the wind flow in the vicinity of the vehicle, undergo discontinuous jumps as the vehicle travels though different regions. Next, we associate this polygonal subdivision with a connectivity graph $\mathcal{G}$. The connectivity graph turns out to be insufficient to describe accurately the reachability/accessibility properties of the vehicle. Specifically, winds that are locally stronger than the vehicle's forward speed restrict the directions of motion to a pencil of directions. Consequently, adjacent nodes of $\mathcal{G}$ may not correspond to system configurations that are necessarily connected by means of an admissible bidirectional path. Furthermore, it is possible that in case the aircraft flies close to a physical obstacle (for example, a hill), locally strong winds may force the vehicle to crush. Thus, the obstacle space needs to be expanded appropriately so that it contains not only

the positions occupied by physical obstacles but also the locations from which a collision of the vehicle may be inevitable. After a number of node and edge deletions the graph $\mathcal{G}$ is transformed to a directed graph $\vec{\mathcal{G}}$, which describes more accurately the system topological properties compared to $\mathcal{G}$. Finally, we assign a cost to each arc of this directed graph, which is the minimum travel time required for steering the aircraft from some position inside the cell that corresponds to the first node of the arc to some position inside the cell that corresponds to the last node of the arc (this position is, for example, the centroid of each cell). We show that the minimum-time path between these two points is a concatenation of at most two line segments with the path concatenation taking place along the common boundary of the two neighboring regions. The previous problem formulation reduces the original steering problem to the shortest path problem over a weighted graph, the solution of which can be obtained by using any of the standard graph search algorithms.[14]

The rest of this paper is organized as follows. In Section II, we formulate the path planning problem as a multi-regional optimal control problem. In Section III we solve a special case of the path-planning problem, which we use as an archetype for reducing the original problem to a directed network minimization problem in Section IV. Section V presents simulation results, and finally Section VI concludes the paper with a summary of remarks.

II.   The Zermelo's Navigation Problem with Obstacles and its Formulation as a Multi-Regional Optimal Control Problem

In this section we formulate the path planning problem as an optimal control problem. In particular, we consider a lightweight aircraft whose motion is described by the following equation

$$\dot{\mathsf{x}} = u + w(\mathsf{x}), \tag{1}$$

where $\mathsf{x} \triangleq (x, y)$ denotes the cartesian coordinates of a reference point of the vehicle, $u \triangleq (u_1, u_2)$ is the control input and $w \triangleq (\mu, \nu)$ is the velocity field induced by the winds. We assume that the state space of the system, denoted as $\mathcal{M}$, is a subset of $\mathbb{R}^2$ while the admissible input set, denoted as $\mathcal{U}$, is comprised of all piecewise continuous functions that take values over the input value set $U = \{(u_1, u_2) : u_1^2 + u_2^2 \leq 1\}$. Furthermore, we assume that the state space can be partitioned into two disjoint sets, namely the free space $\mathcal{F}$ and the obstacle space $\mathcal{O}$. Additionally, the space $\mathcal{F}$ is assumed to be open and connected, which implies that there always exists a path between two arbitrary points in $\mathcal{F}$ (though such a path may not always be admissible for the system (1)).

Next, we formulate the problem of steering the system (1) from a given initial position x_A to a terminal point taken from a set of goal destinations $\{\mathsf{x}_\mathsf{f}^j, j \in \mathcal{J}\}$, where $\mathcal{J}$ is a finite index set, in minimum time, and such that the ensuing path lies inside $\mathcal{F}$ at all times. We refer to this problem as the *Zermelo's Navigation Problem with Obstacles and Multiple Targets*.

Problem 1 *Given the system described by equation* (1) *determine the control* $u^* \in \mathcal{U}$ *such that*

1. *The control* u^* *minimizes the cost functional* $J(u) \triangleq T_\mathsf{f}$ *where* T_f *is the free final time.*

2. *The trajectory* $\mathsf{x}^* : [0, T_\mathsf{f}] \mapsto \mathcal{M}$ *generated by the control* u^* *satisfies*

   (a) *the collision free condition:*
   $$\mathsf{x}^*(t) \in \mathcal{F}, \quad \text{for all} \ \ t \geq 0, \tag{2}$$

   (b) *the boundary conditions:*
   $$\mathsf{x}^*(0) = \mathsf{x}_\mathsf{A}, \quad \mathsf{x}^*(T_\mathsf{f}) \in \{\mathsf{x}_\mathsf{f}^j, j \in \mathcal{J}\}. \tag{3}$$

Figure 1 illustrates Problem 1. In particular, the vehicle starting from point A has to reach the "closest," in terms of travel time, terminal point from the point-set $\{\mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}\}$. The black arrows denote the wind velocity field, whereas the plane is colored with respect to the elevation map of the terrain. Bright and dark colors correspond to low and high elevation regions, respectively. Since we assume that the vehicle flies at a constant altitude within the altitude zone that corresponds to bright green, for example, then each region colored with the same or a brighter color corresponds to the free space $\mathcal{F}$, whereas the dark green regions

correspond to the obstacle space $\mathcal{O}$. We observe in Fig. 1 that the terminal sites B and C, although they are closer to the vehicle's initial position, they may not be accessible from A due to locally strong winds.

We assume that the state space is decomposed into a given collection of non-overlapping convex, polygonal, closed cells, denoted by $\mathcal{C}(\mathcal{M}) = \{c_i, i \in \mathcal{I}\}$, where $\mathcal{I}$ is a finite index set, such that the wind velocity in each cell c_i of this subdivision can be approximated by a constant vector, say $w = w_i$, where $w_i \triangleq (\mu_i, \nu_i)$ and

$$w_i \triangleq \frac{\iint_{c_i} \tilde{w}_i(\mathsf{x}) d\mathsf{x}}{\iint_{c_i} d\mathsf{x}}, \quad i \in \mathcal{I}, \tag{4}$$

where $\tilde{w}_i$ denotes the measured wind velocity at position $\mathsf{x} \in c_i$. Subsequently, the given polygonal subdivision of $\mathcal{M}$ induces a uniform (discontinuous) wind velocity distribution. Within each cell c_i in the subdivision $\mathcal{C}(\mathcal{M})$ we have the regional dynamics

$$\dot{\mathsf{x}} = u + w_i, \qquad \mathsf{x} \in c_i. \tag{5}$$

The dynamics change each time the vehicle crosses the face f_{ij} of two neighboring cells c_i and c_j. The system dynamics jump only along f_{ij}, and when furthermore, the vehicle exits from cell c_i and enters cell c_j, or vice versa, that is, the vehicle does not travel along f_{ij}. Note that the system can equivalently be described using a hybrid model approach, similarly to the one adopted by Sanfelice and Frazzoli in Ref. [13]. To this end, we formulate the *Multi-Regional Zermelo's Navigation Problem with Obstacles and Multiple Targets* as follows.

Problem 2 *Given a polygonal subdivision of $\mathcal{C}(\mathcal{M}) = \{c_i, i \in \mathcal{I}\}$ of $\mathcal{M}$, solve Problem 1 for the system with regional dynamics described by* (5).
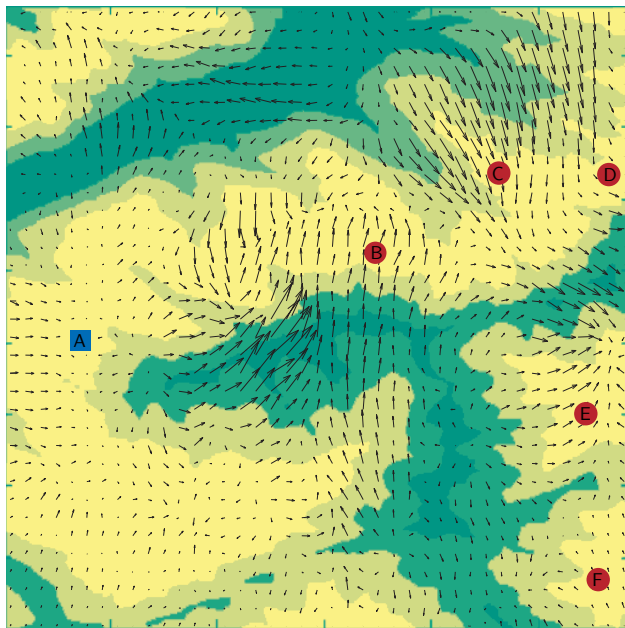

Figure 1. The Zermelo Navigation Problem with Obstacles and Multiple Targets. The obstacle space corresponds to dark green regions. The black arrows denote the wind velocity field.

III. Zermelo's Navigation Problem Between Two Patches

Before we deal with Problem 2, we first address a simpler problem, the solution of which will provide us with the archetypes for generating an approximation of the solution of Problem 2. In particular, let $\mathcal{O} = \varnothing$ and $\mathcal{M} = \mathcal{F} = \mathbb{R}^2$, and consider two distinct points x_A and x_B. Let us consider the bisector of x_A and x_B, that is, the line which is equidistant from x_A and x_B and divides the Euclidean plane into two closed half-planes, $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Henceforth, we denote the bisector of A and B by $\partial \mathcal{P}_{12}$. We assume that both

$\mathcal{P}_1$ and $\mathcal{P}_2$ are equipped with a constant wind velocity field $w_1 = (\mu_1, \nu_1)$ and $w_2 = (\mu_2, \nu_2)$ respectively. Next, we consider the minimum-time problem from x_A to x_B. We call this problem the *Zermelo's Navigation Problem between Two Patches* whose formulation is as follows:

Problem 3 *Find the control u^* that solves Problem 2 in case $\mathcal{M} = \mathbb{R}^2$, $\mathcal{C} = \{\mathcal{P}_1, \mathcal{P}_2\}$ and the boundary conditions $x(0) = x_A$ and $x(T_f) = x_B$.*

Figure 2 illustrates Problem 3. To simplify the presentation we consider a reference frame such that the bisector $\partial \mathcal{P}_{12}$ coincides with the y-axis, and the points A and B have coordinates $(-\chi, 0)$ and $(\chi, 0)$, where χ is positive constant, respectively. Before we proceed to the solution of Problem 3, we first examine the existence of feasible solutions to the problem.
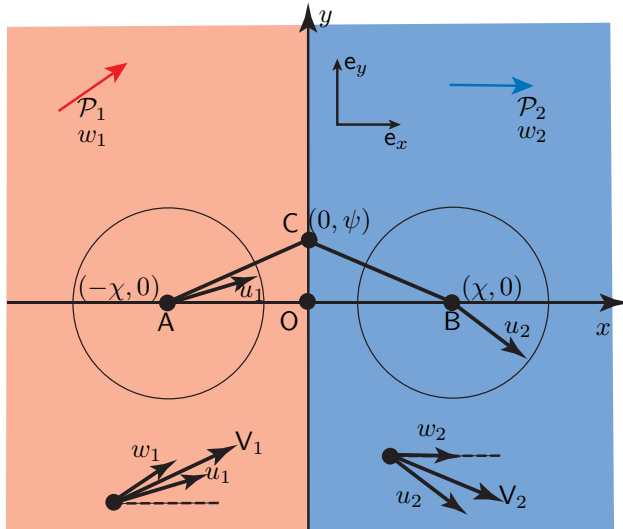

Figure 2. The Zermelo navigation problem between two patches.

III.A. Reachability Analysis for the Zermelo Navigation Problem between Two Patches

One of the distinctive characteristics of Zermelo's navigation problem is the dependence of the controllability/reachability properties of the original system (1) on the strength of the local flow induced by the winds in the vicinity of the vehicle (for more details see [15]). In particular, as we demonstrate shortly afterwards, the system is completely controllable, that is, the vehicle can be steered everywhere from an arbitrary initial position, if and only if the local wind speed is less than the vehicle's forward speed (normalized to unit). The situation is illustrated in Fig. 3. In particular, Fig. 3(a) illustrates that if the wind speed is less than or equal to one, then the vehicle can move to every direction except from $-w$ in the case $\|w\| = 1$. On the contrary, as it is illustrated in Fig. 3(b), if the wind speed is greater than the forward speed of the vehicle, then the vehicle's inertial velocity $V = w + u$, is constrained to lie for all $u \in \mathcal{U}$ and for all $t \geq 0$ within a pencil of directions, to which we shall henceforth refer to as the *cone of admissible directions of motion*, denoted as $\mathcal{K}$. It follows readily from Fig. 3(b) that $\mathcal{K}$ is defined as

$$\mathcal{K}(w) \triangleq \{v \in \mathbb{R}^2 : |\angle(w, v)| \leq \arcsin(1/\|w\|)\}, \tag{6}$$

where $\angle(a, b) : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto [-\pi, \pi]$ denotes the angle from a to b and $\mathbb{R}^2$ is the set of two-dimensional vectors. Note that $\mathcal{K}(w)$ is a right (non-oblique) cone of angle $\vartheta \triangleq 2 \arcsin(1/\|w\|)$, whose apex is x (current position of the vehicle) and its axis is parallel to w.

To this end, let us consider the restriction on $\mathcal{P}_1$ of the reachable set from point $A \in \mathcal{P}_1$, which is given by

$$\mathcal{R}_{\mathcal{P}_1}(x_A) \triangleq \{x \in \mathcal{P}_1 : x = x_A + sv, \quad \text{for all} \quad v \in \mathcal{K}(w_1) \text{ and } s \geq 0\}. \tag{7}$$

Let $\ell_1 \triangleq \mathcal{R}_{\mathcal{P}_1}(x_A) \cap \partial \mathcal{P}_{12}$, then the point B is reachable from A if and only if

$$x_B \in \bigcup_{x_C \in \ell_1} \mathcal{R}_{\mathcal{P}_2}(x_C), \tag{8}$$

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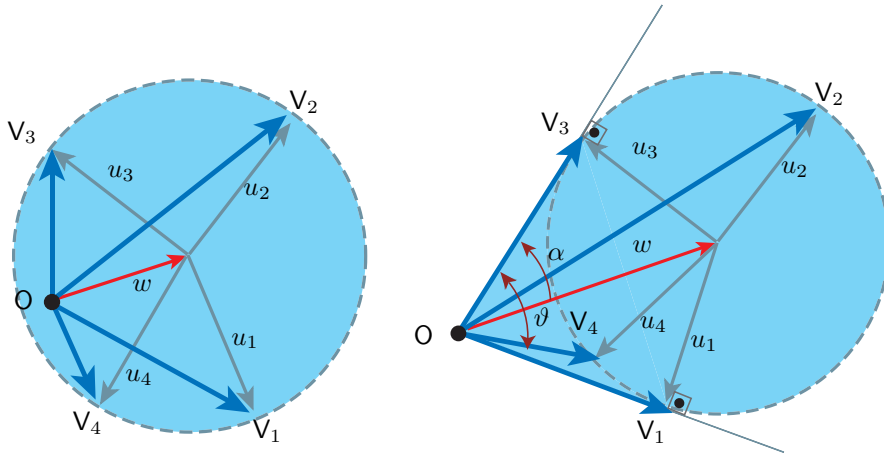
(a) Local controllability analysis for the Zermelo's navigation problem when $\|w\| \leq 1$. The vehicle can move to every direction except from $-w$ when $\|w\| = 1$.

(b) Local controllability analysis for the Zermelo's navigation problem when $\|w\| > 1$. The vehicle is constrained to move within a cone whose apex is O, its axis is parallel to w and its angle ϑ equals $2\arcsin 1/\|w\|$.

Figure 3. Local controllability analysis for the Zermelo's navigation problem.

where $\mathcal{R}_{\mathcal{P}_2}(\mathsf{x}_\mathsf{C})$ denotes the restriction on $\mathcal{P}_2$ of the reachable set from point $\mathsf{C} \in \partial\mathcal{P}_{12}$. The situation is illustrated in Fig. 4.

A more efficient way to verify the reachability of B from A, and thus the feasibility of Problem 3 is to characterize the intersection of the restriction of the accessibility set of B on $\mathcal{P}_2$, that is, the set of points in $\mathcal{P}2$ from which B can be reached, with ℓ_1. More precisely the restriction of the accessibility set of B on $\mathcal{P}_2$ is given by

$$\mathcal{A}_{\mathcal{P}_2}(\mathsf{x}_\mathsf{B}) \overset{\triangle}{=} \{\mathsf{x} \in \mathcal{P}_2 : \mathsf{x} = \mathsf{x}_\mathsf{B} - sv, \quad \text{for all } v \in \mathcal{K}(w_2) \text{ and } s \geq 0\}. \tag{9}$$

Let $\ell_2 = \partial\mathcal{P}_{12} \cap \mathcal{A}_{\mathcal{P}_2}(\mathsf{x}_\mathsf{B})$, then the point B is reachable from A if and only if the following condition holds

$$\ell_{12} \overset{\triangle}{=} \ell_1 \cap \ell_2 \neq \varnothing. \tag{10}$$

The situation is illustrated in Fig. 5.

Condition (10) can furnish an explicit characterization of the feasibility of Problem 3. In particular, let $\ell_1 = \overline{\Gamma\Delta}$ and $\ell_2 = \overline{\Gamma'\Delta'}$ as it is illustrated in Fig. 5. It follows readily that the coordinates of the points Γ, Δ, Γ', and Δ' are given by

$$\mathsf{x}_\Gamma = (0, \chi\tan(\angle(w_1, \mathsf{e}_x) - \arcsin(1/\|w_1\|))), \quad \mathsf{x}_\Delta = (0, \chi\tan(\angle(w_1, \mathsf{e}_x) + \arcsin(1/\|w_1\|))), \tag{11}$$
$$\mathsf{x}_{\Gamma'} = (0, \chi\tan(\angle(w_2, \mathsf{e}_x) - \arcsin(1/\|w_2\|))), \quad \mathsf{x}_{\Delta'} = (0, \chi\tan(\angle(w_2, \mathsf{e}_x) + \arcsin(1/\|w_2\|))). \tag{12}$$

Therefore, condition (10) implies that B is reachable from A, or, equivalently, the Problem 3 is feasible, if and only if

$$\Gamma' \in \Gamma\Delta, \quad \text{and/or } \Delta' \in \Gamma\Delta. \tag{13}$$

In light of (11)-(12) and after some algebraic manipulation, it follows that condition (13) is satisfied if and only if at least one of the following inequalities hold

$$\left|\angle(w_2, \mathsf{e}_x) - \angle(w_1, \mathsf{e}_x) - \arcsin(1/\|w_2\|)\right| \leq \arcsin(1/\|w_1\|), \tag{14}$$

$$\left|\angle(w_2, \mathsf{e}_x) - \angle(w_1, \mathsf{e}_x) + \arcsin(1/\|w_2\|)\right| \leq \arcsin(1/\|w_1\|). \tag{15}$$
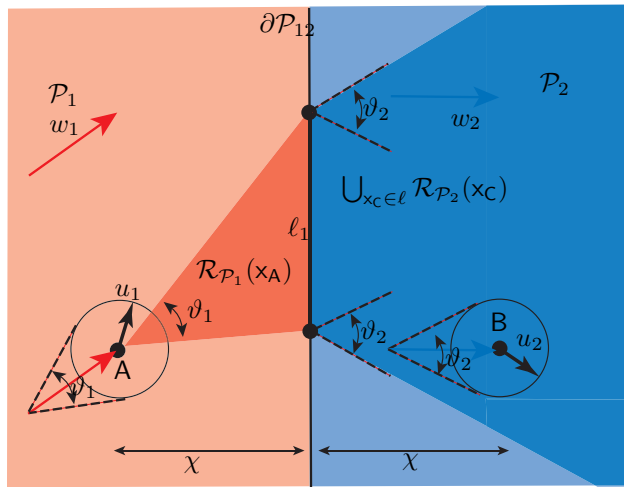
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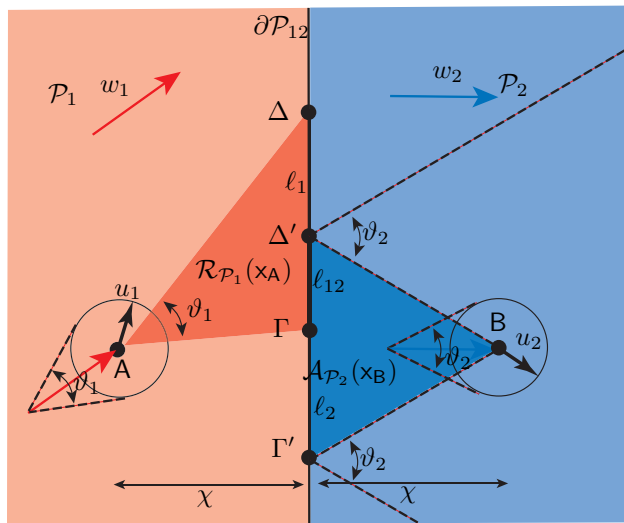**Figure 4.** Reachability analysis of Problem 3.

Figure 5. Feasibility of Problem 3.

III.B. Characterization of the Solution to the Zermelo's Navigation Problem between Two Patches

The solution of Problem 3 has a simple structure, which is illustrated in Fig. 2. In particular, the optimal path within $\mathcal{P}_1$ is the minimum-time path $\mathsf{x}_1^* : [0, T_{\mathsf{f},1}] \mapsto \mathcal{P}_1$ from A to some point C along the line segment ℓ_{12}, provided that $\ell_{12} \neq \varnothing$ (that is, Problem 3 is feasible), with coordinates $(0, \psi)$. Furthermore, the path that lies in $\mathcal{P}_2$ is necessarily the minimum-time path from C to the point B. Thus, the minimum-time path from A to B is a continuous path, which is the concatenation of two locally optimal paths, namely x_1^* and x_2^*, that is

$$\mathsf{x}^*(t; \psi) = \begin{cases} \mathsf{x}_1^*(t; \psi), & \text{for } t \in [0, T_{\mathsf{f},1}], \\ \mathsf{x}_2^*(t; \psi), & \text{for } t \in [T_{\mathsf{f},1}, T_{\mathsf{f}}]. \end{cases} \tag{16}$$

Therefore the solution of Problem 3 belongs necessarily to a family of paths that depend on one parameter, namely ψ.

There are two tools at our disposal to characterize ψ^* such that time of travel along the composite path from A to C and subsequently to B is minimized when $\psi = \psi^*$. The first approach is to apply the Maximum Principle for hybrid optimal control problems,[16] which will provide us the additional equation
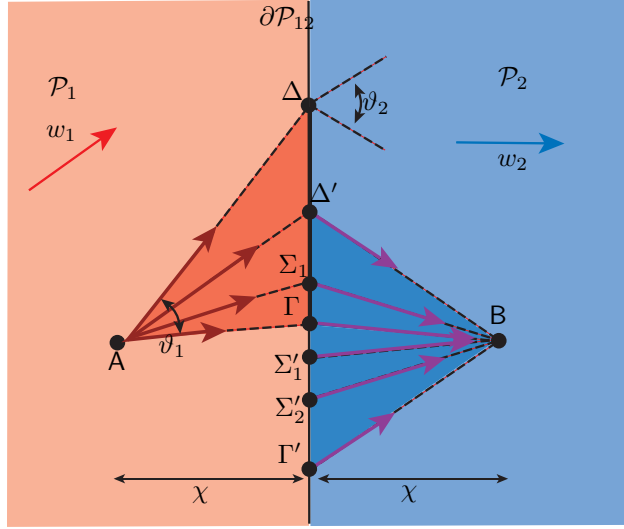
Figure 6. Constructing the composite optimal path of Problem 3.

(transversality condition) required for the characterization of ψ^*. The second technique is to mimic the solution of the minimum-time path planning problem through two different isotropic media (a problem whose solution is known to obey the *Snell's law of refraction*). In the latter approach it suffices to express the sum of the time of motion along the locally optimal paths from A to C and the path from C to B as functions of ψ, and subsequently determine the value of ψ at which the total travel time assumes its minimum value (unconstrained optimization problem). Although the first approach provides a more general framework to address similar problems, it turns out, however, that the second technique admits a much simpler answer to Problem 3, mainly due to the fact that the time of travel of Problem 3 enjoys the convexity property, which allow us, in turn, to apply standard optimization tools.

First, we characterize the minimum-time path from $A \in \mathcal{P}_1$ to a point $C \in \ell_{12}$. Because the Zermelo's navigation problem is a well studied problem in the literature (see for example Refs. [15, 17]), we shall give only some of the results from the solution, which are necessary for our analysis without providing all the details. In particular, the optimal path $x_1^*(t)$ from A to C is given by

$$x_1^*(t) = x_A + tv_1, \quad t \in [0, T_{f,1}], \tag{17}$$

where $T_{f,1}$ is the minimum time from A to C, and $v_1 \triangleq (x_C - x_A)/T_{f,1}$ is the velocity vector (constant) with which the vehicle traverses the minimum-time path. Furthermore, the minimum time $T_{f,1}$ from A to C is given by

$$T_{f,1}(\psi) = \begin{cases} T_1^+(\psi; \chi), & \text{if } \psi \in \mathcal{D}_{C,1}^+ \\ T_1^-(\psi), & \text{if } \psi \in \mathcal{D}_{C,1}^-, \end{cases} \tag{18}$$

where

$$T_1^\pm(\psi) = \frac{\nu_1 \psi + \mu_1 \chi \pm \sqrt{\Delta_1(\psi)}}{1 - \|w_1\|^2}, \quad \Delta_1(\psi) = (1 - \nu_1^2)\chi^2 + (1 - \mu_1^2)\psi^2 + 2\mu_1\nu_1\chi\psi, \tag{19}$$

and where

$$\mathcal{D}_{C,1}^\pm \triangleq \left\{ \psi : \left(\Delta_1(\psi) \geq 0 \right) \wedge \left(0 \leq T_1^\pm(\psi) \leq T_1^\mp(\psi) \text{ if } T_1^\mp(\psi) \geq 0 \right) \text{ or } 0 \leq T_1^\pm(\psi) \text{ otherwise} \right\}.$$

We immediately observe that $T_1^\pm$ can be written as the sum of convex functions of ψ with functions, which are, in turn, compositions of convex functions with affine functions of ψ (all the previous operations preserve convexity). Therefore $T_1^\pm$ is a convex function of ψ.

Furthermore, the optimal control input $u_1^* = (\cos\theta_1^*, \sin\theta_1^*)$, where $\theta_1^* \in [-\pi, \pi]$ is determined unambiguously from the following equations

$$\cos\theta_1^* = -\mu_1 + \chi/T_{f,1}, \quad \sin\theta_1^* = -\nu_1 + \chi/T_{f,1}. \tag{20}$$

Similarly, the expressions for the minimum time and the optimal control for the minimum-time problem from C to B are given by (17)-(20) after replacing ψ and the subscripts A, C and 1 by $-\psi$ and C, B and 2 respectively. It follows that the total time from A to B is given by

$$T_f(\psi) = \begin{cases} T_1^+ + T_2^+, & \text{if } \psi \in \mathcal{D}_{C,1}^+ \cap \mathcal{D}_{C,2}^+ \\ T_1^- + T_2^+, & \text{if } \psi \in \mathcal{D}_{C,1}^- \cap \mathcal{D}_{C,2}^+ \\ T_1^+ + T_2^-, & \text{if } \psi \in \mathcal{D}_{C,1}^+ \cap \mathcal{D}_{C,2}^- \\ T_1^- + T_2^-, & \text{if } \psi \in \mathcal{D}_{C,1}^- \cap \mathcal{D}_{C,2}^- \end{cases} \tag{21}$$

Note that both $T_1^\pm$ and $T_2^\pm$ are convex functions of ψ, and therefore each sum of the form $T_1^\pm + T_2^\pm$ is also a convex function of ψ (the sum of two convex functions is also convex). Thus the total time of travel from A to B, which is given by (21), is a piecewise convex function of ψ, and therefore the characterization of the global minimum within each of the four domains $\mathcal{D}_{C,1}^\pm \cap \mathcal{D}_{C,2}^\pm$ can be achieved by standard tools from convex optimization (e.g., gradient descent methods). Alternatively, and because the problem is one dimensional, it suffices to compute the unique root of the equation $\partial T_f/\partial\psi = 0$ within each of the four domains $\mathcal{D}_{C,1}^\pm \cap \mathcal{D}_{C,2}^\pm$. The analytic expression of the equation $\partial T_f/\partial\psi = 0$, when, say $x_C \in \mathcal{D}_{C,1}^+ \cap \mathcal{D}_{C,2}^+$ is given by

$$\frac{(1 - \|w_2\|^2)\nu_1 - (1 - \|w_1\|^2)\nu_2}{(1 - \|w_1\|^2)(1 - \|w_2\|^2)} + \sum_{i=1}^2 \frac{(-1)^{i+1}\mu_i\nu_i\chi + (1 - \mu_i^2)\psi}{(1 - \|w_i\|^2)\sqrt{\Delta_i(\psi)}} = 0, \tag{22}$$

where the equations for the three other domains can be derived similarly.

Fig. 6 illustrates the scheme for the generation of the minimum-time composite path from A to B. In particular, for each point $\Sigma \in \ell_{12} = \overline{\Gamma\Delta'}$, there exists a velocity $V_1 \in \mathcal{K}(w_1)$ that drives the vehicle from A to Σ in minimum time. In Fig. 6 we observe that the route from A to Δ' corresponds to the minimum-time path from A to the segment ℓ_{12}, whereas the minimum-time path from ℓ_{12} to B is the path from Γ to B, since along both of these two paths the vehicle travels with the maximum speed in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. It turns out, however, that the composite minimum-time path is the concatenation of the segments $\overline{A\Sigma_1}$ and $\overline{\Sigma_1 B}$, and therefore, the composite minimum-time path does not contain neither the segment $\overline{A\Delta'}$ nor $\overline{\Gamma B}$.

IV. Reducing the Multi-Regional Zermelo Problem to a Weighted Directed Graph Search Problem

In this Section we demonstrate how Problem 2 can be reduced to a shortest path problem over a weighted directed graph. The first step of this reduction process, is to transcribe a simple graph $\mathcal{G}$ to $\mathcal{C}(\mathcal{M})$. This requires, in turn, a rule to assign each cell of $\mathcal{C}(\mathcal{M})$ to a node of $\mathcal{G}$, and subsequently determine the adjacency relations between the nodes of $\mathcal{G}$ based on the proximity relations of their corresponding cells. In particular, a cell $c_i \in \mathcal{C}(\mathcal{M})$ is characterized as *free* if $c_i \cap \mathcal{O} = \varnothing$ and as *full* if $c_i \cap \mathcal{O} \neq \varnothing$. We denote as $\mathcal{C}_e(\mathcal{M}) \triangleq \{c_i \in \mathcal{M}, i \in \mathcal{I}_e\}$ and $\mathcal{C}_f(\mathcal{M}) \triangleq \{c_i \in \mathcal{M}, i \in \mathcal{I}_f\}$ the collections of free and full cells respectively, where $\mathcal{I} = \mathcal{I}_e \cup \mathcal{I}_f$ and $\mathcal{I}_e \cap \mathcal{I}_f = \varnothing$.

Furthermore, two cells c_i, c_j in $\mathcal{C}(\mathcal{M})$, with $i \neq j$, are characterized as neighboring if and only if they share a common face, which is denoted as f_{ij}. We furthermore denote as $\mathcal{N}_e(i)$ and $\mathcal{N}_f(i)$ the index sets of the respectively free and full cells that are neighboring to the free cell c_i. We subsequently associate $\mathcal{C}_e(\mathcal{M})$ with a topological (simple) graph $\mathcal{G}$, known in the literature[18] as the *connectivity graph*, as follows:

(i) A free cell $c_i \in \mathcal{C}_e(\mathcal{M})$ is associated uniquely with a node v_i; we write $v_i \sim c_i$.

(ii) Two nodes v_i and v_j are adjacent if and only their corresponding free cells c_i and c_j share a common face f_{ij}.

The vertex set of $\mathcal{G}$, denoted as $V = V(\mathcal{G})$, is the set $\{\mathsf{v}_i : \mathsf{v}_i \sim c_i, i \in \mathcal{I}_e\}$ and the edge set of $\mathcal{G}$, denoted as $E(\mathcal{G})$, is comprised of all the (unordered) pairs $(\mathsf{v}_i, \mathsf{v}_j)$, where v_i and v_j are adjacent. Note that $E(\mathcal{G})$ corresponds to the adjacency relationships of $V(\mathcal{G})$, as usual. Furthermore, we may associate each node $\mathsf{v}_i \in V(\mathcal{G})$ with any point $\mathsf{x}_i \in c_i$, $i \in \mathcal{I}_e$. We write $\mathsf{x}_i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i)$. Finally, if $\mathsf{x}_i \in c_i$ we write $\mathsf{v}_i = \mathsf{node}_\mathcal{G}(\mathsf{x}_i)$, where $\mathsf{v}_i \sim c_i$.

The next step is to update the adjacency relations of the graph $\mathcal{G}$ so that they reflect the local accessibility/reachability properties of the system (1). In particular, given two adjacent nodes v_i and v_j, we have to investigate whether it is possible to steer the system (1) from $\mathsf{x}_i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i)$ to $\mathsf{x}_j = \mathsf{cell}_\mathcal{G}(\mathsf{v}_j)$, and vice versa. In particular, given a node $\mathsf{v}_i \in V(\mathcal{G})$, then $\mathsf{v}_j \in V(\mathcal{G})$, with $j \in \mathcal{N}_e(i)$, is said to be connected from v_i if and only if

$$\ell_{ij} \triangleq \ell_i \cap \ell_j \neq \varnothing, \quad \ell_i \triangleq \mathcal{R}_{c_i}(\mathsf{x}_i) \cap f_{ij}, \quad \ell_j \triangleq \mathcal{A}_{c_j}(\mathsf{x}_j) \cap f_{ij}, \tag{23}$$

where $\mathcal{R}_{c_i}(\mathsf{x}_i)$, and $\mathcal{A}_{c_j}(\mathsf{x}_j)$ denote the restrictions of the reachable set from x_i to c_i and the accessible set of x_j to c_j, respectively. In light of (23), it follows that if $(\mathsf{v}_i, \mathsf{v}_j) \in E(\mathcal{G})$, then a path from $\mathsf{x}_i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i)$ to $\mathsf{x}_j = \mathsf{cell}_\mathcal{G}(\mathsf{v}_j)$, and/or vice versa, may not exist in general. Figure 7 illustrates how the wind field can affect the accessibility/reachability properties of the system (1) in contrast to the connectivity of the topological graph $\mathcal{G}$. In particular, Fig. 7(a) illustrates a case where the wind conditions do not alter the adjacency relations between two adjacent nodes v_i and v_j in $V(\mathcal{G})$, whereas Fig. 7(b) illustrates the case when v_i is connected to v_j but it is not connected from v_j. Finally, Fig. 7(c) illustrates the case when v_i is neither connected to nor connected from v_j. Therefore the topological graph $\mathcal{G}$ is insufficient to reflect whether points that lie within neighboring cells are connectable by bidirectional, admissible paths of the system (1) in the presence of locally strong winds. Therefore, we need to replace graph $\mathcal{G}$ with a directed graph (digraph) $\vec{\mathcal{G}}$. We initially take the vertex set of $\vec{\mathcal{G}}$, denoted by $V(\vec{\mathcal{G}})$, to be equal to $V(\mathcal{G})$, and we subsequently examine whether every pair of adjacent nodes of $\mathcal{G}$ corresponds to points of the system (1) that are reachable from and/or to each other. In particular, given a node v_i, we examine whether for each $j \in N_e(i)$ the point $\mathsf{x}_j = \mathsf{cell}_\mathcal{G}(\mathsf{v}_j)$ satisfies the condition (23). In the affirmative, then the arc $\overrightarrow{\mathsf{v}_i \mathsf{v}_j}$ belongs to $E(\vec{\mathcal{G}})$, where $E(\vec{\mathcal{G}})$ denotes the edge set of the directed graph $\vec{\mathcal{G}}$. By repeating this process for every node $\mathsf{v}_i \in V(\mathcal{G})$, we characterize completely the edge set of the directed graph $\vec{\mathcal{G}}$. Henceforth, given a node $\mathsf{v}_i \in V(\mathcal{G})$ with $\mathsf{v}_i \sim c_i$, where c_i is a free cell, we denote as $\mathcal{N}_e^+(i)$ and $\mathcal{N}_e^-(i)$ the index set of the adjacent to v_i nodes that are connectable from and to v_i, respectively. If for some node $\mathsf{v}_i \in V(\mathcal{G})$, both $\mathcal{N}_e^+(i)$ and $\mathcal{N}_e^-(i)$ are free, then we remove this node from $V(\vec{\mathcal{G}})$.

As we have already mentioned, it is likely that strong winds in the vicinity of the vehicle can force the latter to a collision with an obstacle. In particular, if the vehicle reaches a point $\mathsf{x}^i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i) \in c_i$, then a collision of the vehicle may be inevitable if and only if the following condition holds:

$$\left(\mathcal{N}_e^-(i) = \varnothing\right) \wedge \left(\mathcal{N}_f^-(i) \neq \varnothing\right) \wedge \left(\mathcal{R}_{c_i}(\mathsf{x}_i) \cap c_j \neq \varnothing, \ \ j \in \mathcal{N}_f^-(i)\right). \tag{24}$$

Condition (24) implies that a collision may be unavoidable if and only if the vehicle reaches a (free) cell c_i from which no other free cell is reachable, in the sense of condition (23), but only full cells, and furthermore the vehicle is constrained to move along directions that will lead eventually to a collision due to locally strong winds. Therefore the obstacle space needs to be expanded in accordance to (24) so that it includes not only the locations occupied by physical obstacles but also configurations that may lead to a collision. We characterize a cells c_i for which $\mathsf{x}_i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i)$ satisfies (24) as an *artificially full cell*. All nodes v_i that correspond to artificially full cells have to be deleted from $V(\vec{\mathcal{G}})$. The process of "expanding" the obstacle space $\mathcal{O}$ to contain artificially full cells as well is illustrated in Fig. 8.

The final step for reducing Problem 2 to a digraph search problem is to assign to each arc in $E(\vec{\mathcal{G}})$ an appropriate transition cost, and thus render $\vec{\mathcal{G}}$ a weighted directed graph. In particular, given an arc $\overrightarrow{\mathsf{v}_i \mathsf{v}_j} \in E(\vec{\mathcal{G}})$ we attach to it the cost $J(\overrightarrow{\mathsf{v}_i \mathsf{v}_j}) = T_\mathsf{f}(\psi)$, where T_f is the solution of a variation Problem 3. In particular, T_f is the minimum time required to steer the system (1) from $\mathsf{x}_i = \mathsf{cell}_\mathcal{G}(\mathsf{v}_i)$ to $\mathsf{x}_j = \mathsf{cell}_\mathcal{G}(\mathsf{v}_j)$ such that the ensuing path remains inside the two cells during the whole progression of the vehicle towards x_j. The latter problem can be associated with Problem 3 by replacing w_1 and w_2 with w_i and w_j, the points x_A and x_B with x_1 and x_2, respectively, and the bisector $\partial\mathcal{P}_{12}$ with the common face f_{ij} of the two cells c_i and c_j. Note that the solution ψ^* of this variation of Problem 3 equals either the solution of Problem 3, when the minimum-time path of the unconstrained Problem 3 intersects the line segment f_{ij}, or it equals one of the values of ψ that correspond to the end points of the line segment f_{ij} otherwise. After a transition
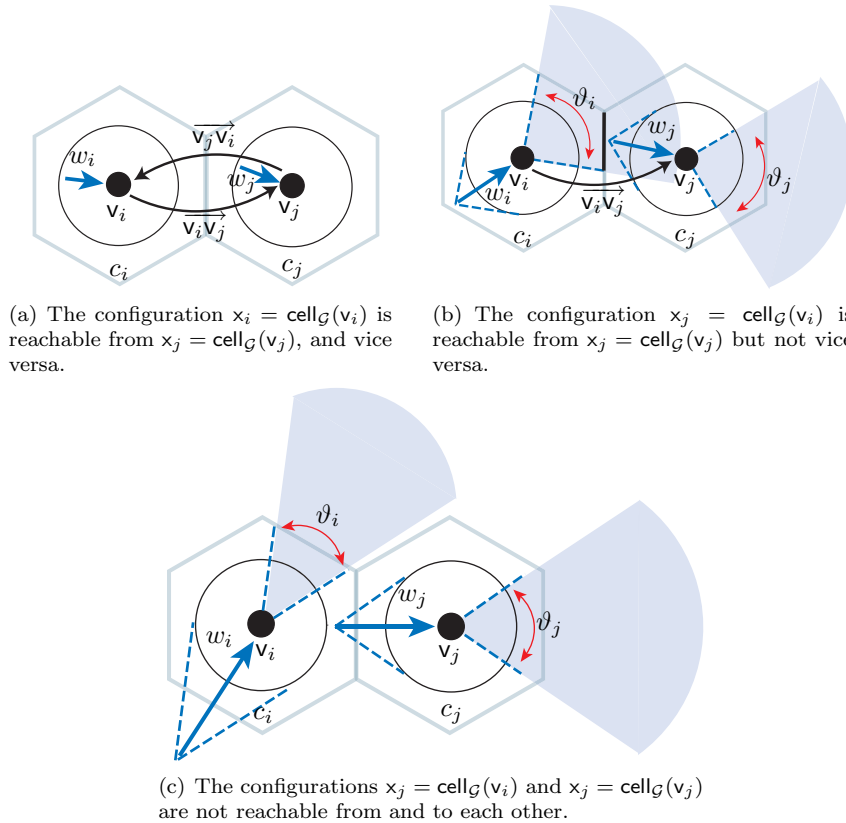
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(a) The configuration $x_i = \mathsf{cell}_{\mathcal{G}}(v_i)$ is reachable from $x_j = \mathsf{cell}_{\mathcal{G}}(v_j)$, and vice versa.

(b) The configuration $x_j = \mathsf{cell}_{\mathcal{G}}(v_i)$ is reachable from $x_j = \mathsf{cell}_{\mathcal{G}}(v_j)$ but not vice versa.

(c) The configurations $x_j = \mathsf{cell}_{\mathcal{G}}(v_i)$ and $x_j = \mathsf{cell}_{\mathcal{G}}(v_j)$ are not reachable from and to each other.

Figure 7. Graph connectivity in the presence of winds. The cone of admissible directions $\mathcal{K}(w_k)$, $k \in \{i, j\}$ is denoted by (light) blue color. Additionally, the tip of the vehicle's forward velocity tracks a black circle.

cost is assigned to every arc of $\vec{\mathcal{G}}$, Problem 2 is reduced to a shortest path problem over a weighted directed graph, which can be solved by employing any of the standard shortest path algorithms over graphs that are available in the literature (e.g., Dijkstra's algorithm). In particular, given the initial node $v_0 = \mathsf{node}_{\mathcal{G}}(x_0)$ and a set of goal nodes $V_g = \{v_f^j = \mathsf{node}_{\mathcal{G}}(x_f^j), j \in \mathcal{J}\}$ we can characterize a shortest path $\mathcal{P}$, if such a path exists, where

$$\mathcal{P} = (v_0, v_1, \cdots, v_f),$$

and where $v_f \in V_g$ is the node that renders the total transition cost from v_0 to V_g minimum. The path $\mathcal{P}$ corresponds to a sequence of points, namely $(x_0, x_1, \ldots, x_n, x_f)$, where $x_k = \mathsf{cell}_{\mathcal{G}}(v_k)$, $k \in \{1, \ldots, n\}$, are the way-points to be visited by the vehicle during its progression towards the goal destination $x_f \in \{x_f^j, j \in \mathcal{J}\}$.

V.   Simulations

In this section we present simulation results to illustrate the previous developments. In particular, we consider the problem of steering a small airplane to its closest final destination among a set of three airports/landing sites. We assume that we are given a polygonal decomposition of the environment which is induced, in turn, by the Voronoi diagram of a finite set of Voronoi generators. Each cell of this Voronoi diagram can be interpreted as the region from which a sensor, located at the corresponding Voronoi generator of this cell, can measure accurately the local wind conditions. The wind velocity field is assumed to be constant within each Voronoi cell. The situation is illustrated in Fig. 9(a). In particular, black, grey, and white Voronoi cells correspond to full, artificially full and free cells, respectively. Figure 9(b) illustrates the directed graph $\vec{\mathcal{G}}$ induced by the polygonal decomposition $\mathcal{C}$ after the artificial cells have been characterized and the adjacency relationships between nodes that correspond to neighboring cells have been updated to account for the restriction of the admissible directions motion, which is induced, in turn, by locally strong wings. Finally Fig. 9(c) illustrates the minimum-time path from the starting point A to its "closest", in terms of time of travel, goal destination from the set $\{\mathsf{B}, \mathsf{C}\}$. We observe that the optimal path is a polygonal line,
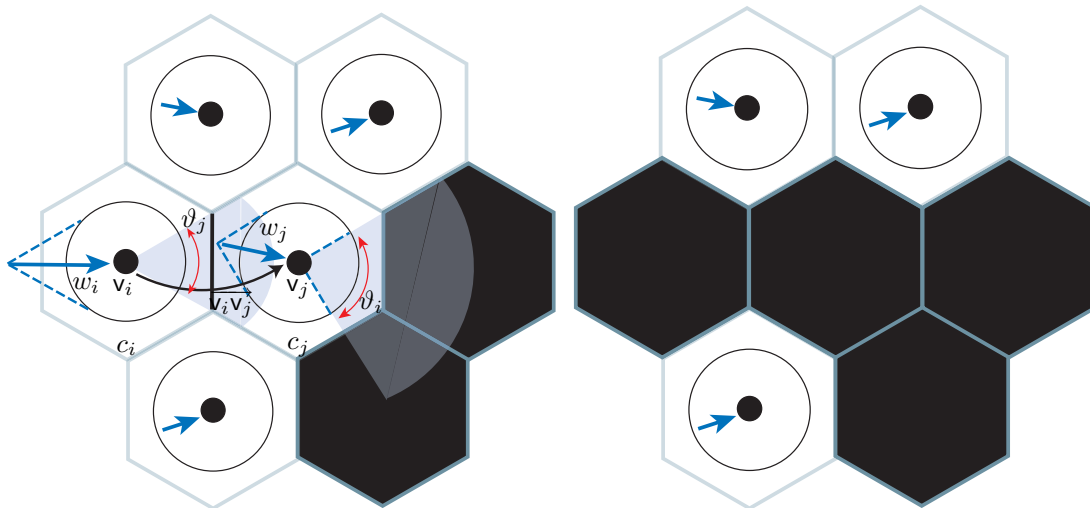
(a) Cells in the vicinity of physical obstacles may con-
tain points from which a collision of the vehicle with an
obstacle may be inevitable.

(b) The obstacle space is expanded in accordance with
condition (24).

Figure 8. The obstacle space of Problem 2 consists of both full and artificially full cells.

where the change of the direction of motion as the vehicle traverses the common face of neighboring free cells follows a pattern which can be interpreted as a non-trivial variation of the Snell's law of refraction through isotropic media (where, in our case, the free cells behave as anisotropic media).

VI.   Conclusions

We have addressed the problem of steering a small lightweight airplane traveling in the presence of obstacles and strong spatially-varying winds to a given set of goal destinations in minimum time. We assume that we are given a polygonal subdivision of the environment such that the wind velocity field is constant within each cell in this subdivision. Due to the discontinuous distribution of the wind velocity field the motion of the vehicle within each cell is described by a different set of equations, where the vehicle's dynamics undergo discontinuous jumps every time the vehicle transverses the common boundary of two neighboring cells. It is shown that the minimum-time paths of our problem are polygonal lines, where the number of curve concatenations equals the number of the cells that the vehicle traverses during its progression towards the goal destination. Additionally, the direction of motion of the vehicle changes every time it transverses the common boundary of two neighboring cells with a pattern similar to Snell's law of refraction from optics with the main difference being that each cell in our problem behaves as an anisotropic medium. Finally, we demonstrate that the multi-regional representation of the vehicle dynamics allow us to reduce the original path-planning problem to a shortest-path problem over a weighted directed graph, for the solution of which efficient algorithms exist in the literature.

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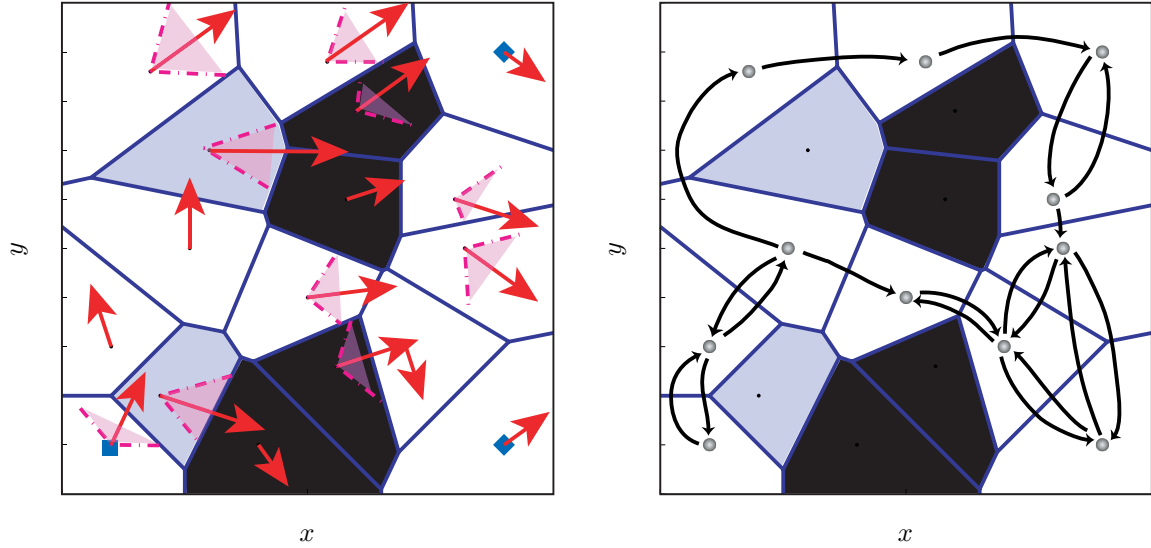
(a) The configuration space is decomposed by means of a Voronoi partition to a collection of free, full and artificially full cells.

(b) The Voronoi diagram is transcribed to a directed graph $\vec{\mathcal{G}}$ in accordance to conditions (23) and (24).
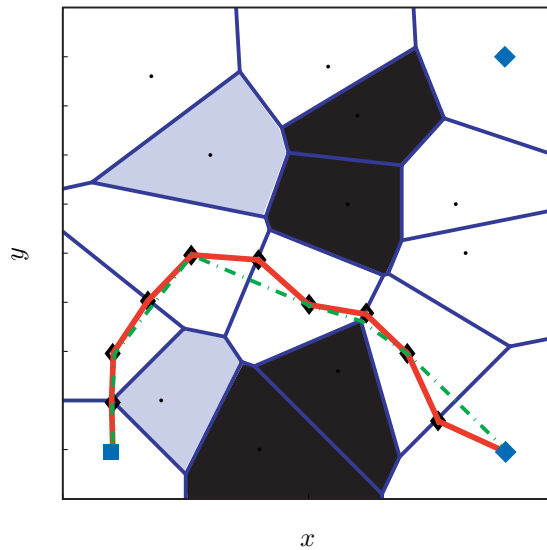
(c) Minimum-time path from A to $\{\mathsf{B},\mathsf{C}\}$.

Figure 9. The solution of Problem 2 can be approximated by the shortest path of a weighted directed graph $\vec{\mathcal{G}}$.

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