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Counterexample to a recent result on the stability of nonlinear systems

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S. P. Banks & K. J. Mhana (1992) claim that a certain condition is sufficient to ensure global asymptotic stability for a broad class of nonlinear systems. We demonstrate, via a counterexample, that satisfaction of this condition does not imply global asymptotic stability.

Counterexample

A recent paper by S. P. Banks & K. J. Mhana (1992) claims that, if a certain condition holds, one can guarantee global asymptotic stability for a nonlinear system of the form

$$\dot{x} = A(x)x, \tag{1}$$

where $t \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$, and A is a continuously differentiable matrix-valued function. Specifically, the following claim is made in a remark in Banks & Mhana (1992: p. 187). If, for each x, all the eigenvalues of the matrix $A(x)$ are in the open left half complex plane, then zero is an asymptotically stable equilibrium solution of (1) for all initial states $x(0)$. We now show via a counterexample that this statement is, in general, *false*.

Consider the two-dimensional nonlinear system

$$\dot{x}_1 = -x_1 + x_1^2 x_2, \qquad \dot{x}_2 = -x_2. \tag{2}$$

Clearly, this system has the form (1) where A is analytic and given by

$$A(x) = \begin{bmatrix} -1 & x_1^2 \\ 0 & -1 \end{bmatrix} \tag{3}$$

For each x, all the eigenvalues of $A(x)$ equal -1.

With initial state

$$x_1(0) = 2, \qquad x_2(0) = 2, \tag{4}$$

a routine calculation shows that for $t \in [0, T_e)$, with $T_e = \ln \sqrt{2}$, the functions

$$x_1(t) = \frac{2x_2(t)}{x_2^2(t) - 2} \quad \text{and} \quad x_2(t) = 2e^{-t} \tag{5}$$

are a solution to (2). As can be easily checked, the solution (5) has a finite escape time at $T_e = \ln \sqrt{2}$; this is because $x_1(t)$ grows without bound as $t \to T_e$. Hence, the system (2) is not asymptotically stable for all initial states $x(0)$.

Moreover, the solution (5) is not even in $L_2(0, T_e)$, i.e. it is not square-integrable. A straight-forward calculation shows that, for $T \in [0, T_e)$,

$$J(T) := \int_0^T [x_1^2(t) + x_2^2(t)]\mathrm{d}t = \frac{1}{2e^{-2T} - 1} - 2e^{-2T} + 1. \tag{6}$$

Clearly, we have $\lim_{T \to T_e} J(T) = \infty$. Therefore, $\boldsymbol{x}(\cdot)$ is not in $L_2(0, T_e)$.

The system (2) also furnishes a counterexample to Lemma 4.1 and Theorem 4.1 of Banks and Mhana (1992). Indeed, $\boldsymbol{Q} = \boldsymbol{I}$, $\boldsymbol{B}(\boldsymbol{x}) = \boldsymbol{O}$, and the matrix $\boldsymbol{A}(\boldsymbol{x})$ of (3) satisfy the assumptions of Lemma 4.1 and Theorem 4.1; however, the conclusions of this lemma and theorem are *false* for the initial state $\boldsymbol{x}(0) = (2, 2)$.

Reference

Banks, S. P., & Mhana, K. J., 1992. Optimal control and stabilization for nonlinear systems. *IMA J. of Mathematical Control & Information* **9**, 179–96.