An $\mathcal{L}_2$ Disturbance Attenuation Solution to the Nonlinear Benchmark Problem

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Abstract

In this paper, we use the theory of $\mathcal{L}_2$ disturbance attenuation for linear ($\mathcal{H}_\infty$) and nonlinear systems to obtain solutions to the Nonlinear Benchmark Problem (NLBP) proposed in the paper by Bupp *et. al.*[1]. By considering a series expansion solution to the Hamilton-Jacobi-Isaacs Equation associated with the nonlinear disturbance attenuation problem, we obtain a series expansion solution for a nonlinear controller. Numerical simulations compare the performance of the third order approximation of the nonlinear controller with its first order approximation (which is the same as the linear $\mathcal{H}_\infty$ controller obtained from the linearized problem.)

Keywords: Nonlinear Benchmark Problem, Hamilton-Jacobi Equation, Disturbance Attenuation, Series Expansions.

1 Introduction

Control of nonlinear systems has received much attention in recent years and many analysis techniques and design methodologies have been developed[2–11]. It is important to determine the advantages and limitations of these different nonlinear control design methodologies. The Nonlinear Benchmark Problem (NLBP) proposed by Bupp *et. al.*[1] is an initial attempt to achieve this objective.

The NLBP involves a cart of mass M which is constrained to translate along a straight horizontal line. The cart is connected to an inertially fixed point via a linear spring; see Figure 1. Mounted on the cart is a "proof body" actuator of mass m and moment of inertia I. Relative to the cart, the proof body rotates about a vertical line passing through the cart mass center. The horizontal external force F acting on the cart is to be regarded as a disturbance force. A motor on the cart can be used to generate a torque N to control the proof mass in such a way that the force F has minimal effect on the cart's position. In other words, it is desirable to *attenuate* as much as possible the effect of the external (unknown) force F on the cart by appropriate choice of the control input N. The nonlinearity of the problem comes from the interaction between the translational motion of the cart and the rotational motion of the eccentric proof mass.

After suitable normalization[1], the equations of motion for this nonlinear system are

$$\ddot{\xi} + \xi = \varepsilon\,(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + w \tag{1a}$$

$$\ddot{\theta} = -\varepsilon\,\ddot{\xi}\cos\theta + u \tag{1b}$$

where ξ is the (non-dimensionalized) displacement of the cart and θ is the angular position of the proof body. For a complete derivation of the equations of motion, see Ref. [12]. In equations (1), w and u are the (non-dimensionalized) disturbance and control inputs, respectively. The coupling between the translational and rotational motions is captured by the parameter ε which is defined by

$$\varepsilon := \frac{me}{\sqrt{(I + me^2)(M + m)}} \tag{2}$$

where e is the eccentricity of the proof body. Clearly, $0 \leq \varepsilon < 1$ and $\varepsilon = 0$ if and only if $e = 0$; in this case the translational and rotational motions decouple and equations (1) reduce to

$$\begin{aligned} \ddot{\xi} + \xi &= w \\ \ddot{\theta} &= u \end{aligned}$$

This system is clearly not stabilizable from the control input; also, the effect of w is completely decoupled from the effect of u.

Letting $x := [x_1\ x_2\ x_3\ x_4]^T := [\xi\ \dot{\xi}\ \theta\ \dot{\theta}]^T$, system (1) can be written compactly in state-space form as

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} w \tag{3}$$

These equations are well defined since $1 - \varepsilon^2 \cos^2 x_3 \neq 0$ for all x_3 and $\varepsilon < 1$.

2 A Simple Stabilizing Controller

A minimum requirement for any acceptable controller is that it asymptotically stabilizes the system in the absence of external disturbances. In this section we show that the stabilization problem for the Nonlinear Benchmark Problem has a very simple solution. In particular, we show that a simple *linear* controller globally asymptotically stabilizes system (3).

Proposition 2.1 *System (3) (with $w = 0$) is globally asymptotically stabilized using the linear controller*

$$u = -k_1\theta - k_2\dot{\theta} \tag{4}$$

where $k_1 > 0$ and $k_2 > 0$.

Proof. Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}\dot{\xi}^2 + \frac{1}{2}\dot{\theta}^2 + \varepsilon\dot{\xi}\dot{\theta}\cos\theta + \frac{1}{2}\xi^2 + \frac{1}{2}k_1\theta^2 \tag{5}$$

To demonstrate that V is positive definite, note that $V(x) = \frac{1}{2}x^T P(\theta)x$ where

$$P(\theta) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \varepsilon\cos\theta \\ 0 & 0 & k_1 & 0 \\ 0 & \varepsilon\cos\theta & 0 & 1 \end{bmatrix}$$

Since the eigenvalues of $P(\theta)$ are $\{1, k_1, 1 \pm \varepsilon\cos\theta\}$ and $|\cos\theta| \leq 1$, $0 \leq \varepsilon < 1$, we have

$$V(x) \geq \frac{1}{2}\lambda_{\min}(P(\theta))\,||x||^2\,, \qquad \lambda_{\min}(P(\theta)) \geq \min\{k_1,\ 1-\varepsilon\} > 0$$

Hence V is a positive definite function.

Differentiating V along the closed-loop trajectories with controller (4) we get that

$$\frac{dV}{dt} = -k_2\dot{\theta}^2 \leq 0 \tag{6}$$

Thus the closed-loop system is stable about the zero state and all trajectories are bounded. To demonstrate asymptotic stability, consider any solution $x(\cdot)$ of the closed system for which $\dot{V}(x(t)) \equiv 0$. Then $\dot{\theta}(t) \equiv 0$; this implies that $\theta(t) \equiv \theta_0 := \theta(0)$, $\ddot{\theta}(t) \equiv 0$, and (using the closed loop system description)

$$\ddot{\xi} + \xi = 0 \tag{7a}$$

$$\varepsilon\ddot{\xi}\cos\theta_0 + k_1\theta_0 = 0 \tag{7b}$$

From equation (7b) we have that $\cos\theta_0 \neq 0$; hence, $\ddot{\xi}$ is constant. Equation (7a) then implies that ξ is constant as well. Since $\xi(t) \equiv 0$ is the only constant solution to this equation, we have $\xi(t), \dot{\xi}(t), \ddot{\xi}(t) \equiv 0$ and, utilizing (7b), $\theta(t) \equiv 0$; thus $x(t) \equiv 0$. By LaSalle's results[6], system (1) with control law (4) is globally asymptotically stable. ■

We note here that the simple linear controller in Eq. (4) was simultaneously derived and presented for the first time by Tsiotras *et al.*.[13] and Jankovic *et al.*[14] during the invited session devoted to the NLBP in the 1995 American Control Conference. The derivation of the linear controller in Jankovic *et al.*[14] was based on passivity arguments, however.

3 Disturbance Attenuation

The previous section demonstrated that a linear controller globally asymptotically stabilizes system (3) when there is no disturbance acting on the system. Our main objective in this paper is to design a controller that will minimize the effect of the disturbance input w on some pre-specified performance output z given by

$$z = \begin{bmatrix} Cx \\ u \end{bmatrix} \tag{8}$$

where the matrix C can be regarded as a collection of design parameters. We will suppose that w belongs to the set of functions which are square integrable, that is, we assume that $w \in \mathcal{L}_2[0,\infty)$ where $\mathcal{L}_2[0,\infty)$ denotes the set of square-integrable functions with domain $[0,\infty)$.

We propose the following control design problem to address the qualitative design guidelines given in Bupp *et al.*[1].

Disturbance Attenuation Problem (DAP):
Let γ be a specified positive scalar. Obtain a memoryless state-feedback controller

$$u = k(x) \tag{9}$$

for system (3) such that the corresponding closed loop system has the following properties.

(a) When $w(t) = 0$, the closed loop system is asymptotically stable about the zero state.

(b) For zero initial state ($x(0) = 0$) and for every disturbance input $w \in \mathcal{L}_2[0, \infty)$,

$$\int_0^\infty ||z(t)||^2 \, dt \quad \leq \quad \gamma^2 \int_0^\infty ||w(t)||^2 \, dt \tag{10}$$

Note that the second requirement implies that the $\mathcal{L}_2$-gain of the closed loop system from the disturbance input w to the performance output z is less than or equal to γ.

Since the closed loop system is causal, one can readily show that the second requirement above also implies the following property for any $T > 0$. If $x(0) = 0$ and $\int_0^T ||w(t)||^2 \, dt$ is finite, then $\int_0^T ||z(t)||^2 \, dt$ is finite and satisfies

$$\int_0^T ||z(t)||^2 \, dt \quad \leq \quad \gamma^2 \int_0^T ||w(t)||^2 \, dt$$

This observation allows us to extend the class of disturbance inputs to those which are square integrable over a finite interval.

The Disturbance Attenuation Problem (DAP) has been treated for general classes of systems[15–17]. In these references it has been shown that, under mild conditions, the DAP can be solved, provided one has a positive definite solution to the so-called Hamilton-Jacobi-Isaacs Equation. The original idea behind this approach was to formulate the DAP as a differential game in which u and w are two opposing players. The next section reviews the basic results of Isidori[16] and van der Schaft[17] which are used in this paper.

4 The Hamilton-Jacobi-Isaacs Equation (HJIE)

System (3) along with its performance output is described by

$$\dot{x} \;=\; F(x) + G_1(x)u + G_2(x)w \tag{11a}$$

$$z \;=\; \begin{bmatrix} Cx \\ u \end{bmatrix} \tag{11b}$$

where the functions F, G_1, G_2 can be obtained from (3) and $F(0) = 0$. We assume that the system

$$\begin{aligned} \dot{x} &= F(x) \\ z &= Cx \end{aligned}$$

is observable in the sense that, $z(t) = 0$ for all $t \geq 0$ implies $x(t) = 0$ for all $t \geq 0$.

One can readily show[10,14,15] that if there is a continuously differentiable, positive definite, function V which satisfies the following Hamilton-Jacobi-Isaacs Equation

$$V_x(x)F(x) - \frac{1}{4}V_x(x)\left[\,G_1(x)G_1^T(x) - \gamma^{-2}G_2(x)G_2^T(x)\,\right]V_x^T(x) + x^TC^TCx = 0 \tag{12}$$

where V_x is the derivative of V, i.e,

$$V_x(x) = \left[\; \frac{\partial V}{\partial x_1}(x) \quad \cdots \quad \frac{\partial V}{\partial x_n}(x) \;\right]$$

then the feedback controller

$$u = k_*(x) := -\frac{1}{2}G_1^T(x)V_x^T(x) \tag{13}$$

yields a closed loop system with the following property. For every initial condition $x(0) = x_0$ and for every disturbance input $w \in \mathcal{L}_2[0, \infty)$ one has

$$\int_0^\infty ||z(t)||^2 \, dt \quad \leq \quad \gamma^2 \int_0^\infty ||w||^2 \, dt + V(x_0) \tag{14}$$

Also, the "worst case disturbance" is given by

$$w = l_*(x) := \frac{1}{2\gamma^2}G_2^T(x)V_x^T(x) \tag{15}$$

Using V as a Lyapunov function one can show that the undisturbed ($w = 0$) closed loop system corresponding to controller (13) is globally asymptotically stable. Hence, a solution to the DAP is given by controller (13).

The main stumbling block in using the above result is that only rarely is one able to compute a function V satisfying (12) in *closed-form*. So, instead of insisting on closed form solutions, we solve (12) in an iterative fashion based on series expansions. This is the methodology proposed in Al'brekht[18], Lukes[19] (see also Yoshida and Loparo[20]) for the solution of Hamilton-Jacobi equations arising in optimal control problems. We demonstrate here that the same procedure can be applied to nonlinear $\mathcal{L}_2$ disturbance attenuation problems, provided that the linearized version of the problem has a solution. The approach is similar to previous results by van der Schaft[17], Kang *et al.*[21] and Huang and Lin[22]. An alternative iterative solution to the HJIE is given by Wise and Sedwick[23].

First we rewrite system (11) in the form

$$\dot{x} = F(x) + G(x)v \tag{16a}$$

$$z = \begin{bmatrix} Cx \\ u \end{bmatrix} \tag{16b}$$

where

$$G(x) := [G_1(x)\ G_2(x)], \qquad v := \begin{bmatrix} u \\ w \end{bmatrix} \tag{17}$$

Letting

$$Q(x) := x^T C^T C x, \qquad R := \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \tag{18}$$

the Hamilton-Jacobi-Isaacs Equation can be rewritten as

$$V_x(x)F(x) - \frac{1}{4}V_x(x)G(x)R^{-1}G^T(x)V_x^T(x) + Q(x) = 0 \tag{19}$$

and letting

$$v_*(x) := \begin{bmatrix} k_*(x) \\ l_*(x) \end{bmatrix}$$

we have

$$v_*(x) = -\frac{1}{2}R^{-1}G^T(x)V_x^T(x) \tag{20}$$

Note that the matrix R in equation (18) is *not* positive definite. In fact, it is an indefinite matrix.

5 A Series Solution Approach to the HJIE

The approach we follow in solving HJIE (19) is based on a series expansion of the desired solution V. For problems involving a small parameter (as the parameter ε in the NLBP) this methodology typically expands the function V in terms of the parameter. If the zero order problem (setting the parameter to zero) is solvable, then an iterative procedure can be readily devised to generate all the higher order terms in the series expansion. However, according to the discussion at the end of Section 1, the DAP is not solvable for the zero order NLBP; hence a perturbation method based on ε will not work for the NLBP.

Alternatively, one may seek a series expansion of the function V in terms of the state x. Using (13), this will yield a series expansion for the controller k_* which solves the DAP. This is the approach considered here. Note that this approach pre-supposes that the HJIE has a (real) analytic solution, i.e., a solution with convergent Taylor series expansion. This assumption may be restrictive, in general, since it is well known that solutions to Hamilton-Jacobi type equations may have non-differentiable (let alone analytic) solutions even if the system dynamics and the cost function are smooth*.

*See Ref. [24], Example 6.1.8.

5.1 Linearized Problem

In the next section, it will be shown that the first term in the series expansion for controller (13) is the solution to the corresponding linearized problem. Thus, we first consider the linearized DAP.

The linearization of system (11) about $x = 0$ is given by

$$\dot{x} = Ax + B_1 u + B_2 w \tag{21a}$$

$$z = \begin{bmatrix} Cx \\ u \end{bmatrix} \tag{21b}$$

with

$$A = F_x(0), \qquad B_1 = G_1(0), \qquad B_2 = G_2(0)$$

From standard $\mathcal{H}_\infty$ theory, the DAP problem for the above linear system is solvable iff it is solvable via a linear state feedback controller. In addition, this DAP is equivalent to obtaining a stabilizing controller which, for the closed loop system, achieves an $\mathcal{H}_\infty$ norm (for the transfer function from w to z) of magnitude less than or equal γ.

Considering a quadratic form

$$V(x) = x^T P x \tag{22}$$

as a candidate solution to the HJIE associated with the linear DAP we obtain

$$x^T[PA + A^T P - PBR^{-1}B^T P + C^T C]x = 0$$

where $B := [B_1 \; B_2]$. This is satisfied for all x iff the matrix P solves the following Algebraic Riccati Equation (ARE):

$$PA + A^T P - PBR^{-1}B^T P + C^T C = 0 \tag{23}$$

Also, V is positive definite iff the matrix P is positive definite. In this case

$$v_*(x) = -R^{-1}B^T P x \tag{24}$$

and the controller which solves the linear DAP is given by

$$k_*(x) = -B_1^T P x \tag{25}$$

According to standard $\mathcal{H}_\infty$ theory, if the pair (C, A) is observable and the pair (A, B_1) is stabilizable, the existence of a positive definite symmetric solution P to the above ARE, with

$$A_* := A - BR^{-1}B^T P \tag{26}$$

Hurwitz, is a necessary and sufficient condition for the linear DAP to have a solution[25].

5.2 Nonlinear Problem

We seek to obtain a solution V to the HJIE by considering a series expansion of the form

$$V(x) = V^{[2]}(x) + V^{[3]}(x) + \cdots \tag{27}$$

where $V^{[k]}$ is a homogeneous function of order k. A homogeneous function of order k in n scalar variables $x_1, x_2, \ldots, x_n$ is a linear combination of

$$N_k^n := \binom{n+k-1}{k}$$

terms of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, where i_j is a nonnegative integer for $j = 1, ..., n$ and $i_1 + i_2 + \cdots + i_n = k$. The vector whose components consist of these terms is denoted by $x^{[k]}$; for example, with two scalar variables one has

$$x^{[1]} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x^{[2]} = \begin{bmatrix} {x_1}^2 \\ x_1 x_2 \\ {x_2}^2 \end{bmatrix}, \qquad x^{[3]} = \begin{bmatrix} x_1^3 \\ x_1^2 x_2 \\ x_1^2 x_3 \\ x_1 x_2^2 \\ x_1 x_2 x_3 \\ x_1 x_3^2 \\ x_2^3 \\ x_2^2 x_3 \\ x_2 x_3^2 \\ x_3^3 \end{bmatrix}$$

Therefore, a homogeneous function $\psi^{[k]}$ of order k can be written as $\psi^{[k]}(x) = \Psi x^{[k]}$ where $\Psi \in \mathbb{R}^{1\times N_k^n}$.

We assume that $F(x)$ and $G(x)$ have series expansions of the form

$$F(x) = F^{[1]}(x) + F^{[2]}(x) + \cdots \tag{28a}$$

$$G(x) = G^{[0]}(x) + G^{[1]}(x) + \cdots \tag{28b}$$

where each component of $F^{[k]}$ and $G^{[k]}$ are homogeneous functions of order k. Note that

$$F^{[1]}(x) = Ax, \qquad G^{[0]}(x) = B$$

Substituting (27) into (20) one obtains a series expansion for v_* of the form

$$v_* = v_*^{[1]} + v_*^{[2]} + \cdots \tag{29}$$

where $v_*^{[k]}$ is the homogeneous function of order k given by

$$v_*^{[k]} = -\frac{1}{2}R^{-1}\sum_{j=0}^{k-1} G^{[j]\,T} V_x^{[k+1-j]\,T} \tag{30}$$

and where explicit dependence on x has been dropped for notational simplicity. Also, one can obtain a series expansion for the desired controller k_* of the form

$$k_* = k_*^{[1]} + k_*^{[2]} + \cdots \tag{31}$$

where $k_*^{[k]}$ is a homogeneous function of order k consisting of the first p components of $v_*^{[k]}$ and $u \in \mathbb{R}^p$.

To compute the terms in the series expansion for V, first note that HJIE (19) can be written as

$$V_x(x)F(x) - v_*^T(x)R(x)v_*(x) + Q(x) = 0 \tag{32a}$$

$$v_*(x) + \frac{1}{2}R^{-1}(x)G^T(x)V_x^T(x) = 0 \tag{32b}$$

Substitution of the expansions in (27)-(29) into (32a) and equating terms of order $m \geq 2$ to zero yields

$$\sum_{k=0}^{m-2} V_x^{[m-k]} F^{[k+1]} - \sum_{k=1}^{m-1} v_*^{[m-k]\,T} R v_*^{[k]} + Q^{[m]} = 0 \tag{33}$$

For $m = 2$ equation (33) simplifies to

$$V_x^{[2]} F^{[1]} - v_*^{[1]} R v_*^{[1]} + Q^{[2]} = 0$$

Since $F^{[1]}(x) = Ax$ and

$$v_*^{[1]}(x) = -\frac{1}{2}R^{-1}B^T V_x^{[2]\,T}(x), \qquad Q^{[2]}(x) = x^T C^T C x,$$

we obtain

$$V_x^{[2]}(x)Ax - \frac{1}{4}V_x^{[2]\,T}(x)BR^{-1}B^T V_x^{[2]}(x) + x^T C^T C x = 0$$

which is the HJIE for the linearized problem. Hence

$$V^{[2]}(x) = x^T P x$$

where $P^T = P > 0$ solves the ARE with $A_* := A - BR^{-1}B^TP$ Hurwitz; also,

$$v_*^{[1]}(x) = -R^{-1}B^T P x \tag{34}$$

and

$$k_*^{[1]}(x) = -B_1^T P x \tag{35}$$

Consider now any $m \geq 3$ and rewrite (33) as

$$\sum_{k=0}^{m-2} V_x^{[m-k]} F^{[k+1]} - 2 v_*^{[m-1]\,T} R v_*^{[1]} - \sum_{k=2}^{m-2} v_*^{[m-k]\,T} R v_*^{[k]} = 0$$

Note that the last term in the above expression does not depend on $V^{[m]}$. Using

$$v_*^{[m-1]\,T} = -\frac{1}{2}\sum_{k=0}^{m-2} V_x^{[m-k]} G^{[k]} R^{-1}$$

and defining

$$f(x) := F(x) + G(x) v_*^{[1]}(x) \tag{36}$$

the first two terms can be written as

$$\begin{aligned}
\sum_{k=0}^{m-2} V_x^{[m-k]} F^{[k+1]} + \sum_{k=0}^{m-2} V_x^{[m-k]} G^{[k]} v_*^{[1]} &= \sum_{k=0}^{m-2} V_x^{[m-k]} f^{[k+1]} \\
&= V_x^{[m]} f^{[1]} + \sum_{k=1}^{m-2} V_x^{[m-k]} f^{[k+1]}
\end{aligned}$$

where

$$f^{[1]}(x) = A_* x \tag{37}$$

and A_* is given by (26). For $m \geq 3$, equation (33) can now be written as

$$V_x^{[m]} f^{[1]} = -\sum_{k=1}^{m-2} V_x^{[m-k]} f^{[k+1]} + \sum_{k=2}^{m-2} v_*^{[m-k]\,T} R v_*^{[k]} \tag{38}$$

Equation (38) can be solved for $V^{[m]}$ as follows. Consider an expression for $V^{[m]}(x)$ of the form $V^{[m]}(x) = V_m x^{[m]}$, with $V_m \in \mathbb{R}^{1\times N_m^n}$. Substitute this expression for $V^{[m]}(x)$ into (38) and solve the resulting linear system of N_m^n equations for the unknown N_m^n elements of the coefficient vector V_m. It can be shown[21,22] that if the eigenvalues of the matrix A_* in (26) are *nonresonant*† then this linear equation has a unique solution for all $m \geq 3$. In other words, the HJIE can be solved to any order. Moreover, since the eigenvalues of A_* are in the left-half of the complex plane, the solution V of the HJIE is analytic and the series (27) converges.

Thus, starting with $V^{[2]}(x) = x^T P x$ and $v_*^{[1]}(x) = -R^{-1}B^T P x$ one can use equations (38) and (30) to compute consecutively the sequence of terms

$$V^{[3]}(x),\ v_*^{[2]}(x),\ V^{[4]}(x),\ v_*^{[3]}(x),\ \ldots \tag{39}$$

and construct iteratively the solution V of HJIE and the associated v_*. Notice that this procedure generates not only the feedback controller $k_*(x)$ defined in (13) for disturbance attenuation, but also the worst case disturbance strategy $l_*(x)$ given in (15).

†A set of eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called resonant if $\sum_{j=1}^n i_j \lambda_j = 0$ for some nonnegative integers $i_1, i_2, \ldots, i_n$ such that $\sum_{j=1}^n i_j > 0$. Otherwise, it is called nonresonant[21].

6 The Nonlinear Benchmark Problem

According to the specifications set forth by Bupp *et al.*[1] it is desired that the (non-dimensionalized) variables satisfy

$$|\xi| \leq 1.282 \qquad \text{and} \qquad |u| \leq 1.411 \tag{40}$$

These performance specifications suggest that the $(1,1)$ element of the matrix $C^T C$ be equal to $1.282^2/1.411^2 \approx 0.82$). For simplicity, we choose the $(1,1)$ element to be unity. No specific requirements are provided for the other state variables, so the elements penalizing these variables are chosen to be one order of magnitude smaller. Thus, the following matrix is chosen for the performance output z

$$C = diag(1, \sqrt{0.1}, \sqrt{0.1}, \sqrt{0.1}) \tag{41}$$

The eccentricity parameter is given as $\varepsilon = 0.2$.

In order to apply the proposed methodology to the NLBP, we first expand the F and G vector fields in the right hand side of (3) in a series expansion. Noting that $1 - \varepsilon^2 \cos^2 x_3 \neq 0$, these expansions can be readily computed as

$$F(x) = \begin{bmatrix} x_2 \\ -\frac{25}{24}x_1 + \frac{5}{24}{x_4}^2 x_3 + \frac{25}{576}x_1 x_3^2 + \cdots \\ x_4 \\ \frac{5}{24}x_1 - \frac{1}{24}x_4^2 x_3 - \frac{65}{576}x_1 x_3^2 + \cdots \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 & 0 \\ -\frac{5}{24} + \frac{65}{576}x_3^2 + \cdots & \frac{25}{24} - \frac{25}{576}x_3^2 + \cdots \\ 0 & 0 \\ \frac{25}{24} - \frac{25}{576}x_3^2 + \cdots & -\frac{5}{24} + \frac{65}{576}x_3^2 + \cdots \end{bmatrix}$$

6.1 Linear problem

The linearized system is described by (21) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{25}{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{5}{24} & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -\frac{5}{24} \\ 0 \\ \frac{25}{24} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ \frac{25}{24} \\ 0 \\ -\frac{5}{24} \end{bmatrix} \tag{42}$$

Using this data one can show that the linearized DAP has a solution if and only if $\gamma > \gamma_* \approx 5.5$. From the results of van der Schaft[26,24] this value provides a lower bound of the achievable $\mathcal{L}_2$-gain of the nonlinear system. Choosing $\gamma = 6$ one can then solve ARE (23) for $P > 0$ to obtain

$$P = \begin{bmatrix} 19.6283 & -2.8308 & -0.7380 & -2.8876 \\ -2.8308 & 15.5492 & 0.8439 & 1.9915 \\ -0.7380 & 0.8439 & 0.3330 & 0.4967 \\ -2.8876 & 1.9915 & 0.4967 & 1.4415 \end{bmatrix} \tag{43}$$

The matrix A_* in (26) is

$$A_* = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ -1.6134 & 0.2140 & 0.0936 & 0.2777 \\ 0 & 0 & 0 & 1.0000 \\ 2.7408 & 1.1222 & -0.3603 & -1.1422 \end{bmatrix} \tag{44}$$

and has eigenvalues $\{-0.0415 \pm i\,1.0156, -0.4227 \pm i\,0.3684\}$.

The linear term of $v_*(x)$ can be computed from (24) to obtain

$$v_*^{[1]}(x) = \begin{bmatrix} 2.4182\,x_1 + 1.1650\,x_2 - 0.3416\,x_3 - 1.0867\,x_4 \\ -0.0652\,x_1 + 0.4384\,x_2 + 0.0215\,x_3 + 0.0493\,x_4 \end{bmatrix}$$

where the first row is the controller $k_*^{[1]}(x)$ and the second row the disturbance strategy $l_*^{[1]}(x)$.

6.2 Higher order terms

Since A_* is Hurwitz, its set of eigenvalues is nonresonant. Therefore, according to the discussion in Section 5.1 the series solution to the HJIE can be computed to any order and this series converges[21].

The calculations are simplified for the NLBP because, as it is evident from the expressions for $F(x)$ and $G(x)$,

$$F^{[2k]}(x) = 0\,, \qquad G^{[2k-1]}(x) = 0, \qquad k = 1, 2, \dots \tag{45}$$

As a result, $V^{[3]}(x) = 0$ and $v_*^{[2]}(x) = 0$. The first nonzero higher order term for the controller is third order and can be computed from

$$v_*^{[3]}(x) = -\frac{1}{2}R^{-1}\left(B^T V_x^{[4]\,T}(x) + G^{[2]T}(x)V_x^{[2]\,T}(x)\right)$$

$$V_x^{[4]}(x)A_*x = -V_x^{[2]}(x)\left(F^{[3]}(x) + G^{[2]}(x)v_*^{[1]}(x)\right)$$

The solution of these equations yields

$$\begin{aligned}
V^{[4]}(x) \;&=\; 162.1117\,x_1^4 + 91.1375\,x_2^4 - 0.9243\,x_4x_3^2x_1 - 0.4143\,x_4x_3^2x_2 \\
&-\; 0.1911\,x_4x_3^3 + 59.6096\,x_4^2x_1^2 - 66.0818\,x_4^2x_2x_1 + 42.1915\,x_4^2x_2^2 \\
&-\; 8.7947\,x_4^2x_3x_1 + 0.6489\,x_4^4 - 151.8854\,x_4x_1^3 + 235.0386\,x_4x_2x_1^2 \\
&-\; 193.9184\,x_4x_2^2x_1 + 96.2534\,x_4x_2^3 + 43.5234\,x_4x_3x_1^2 - 45.9440\,x_4x_3x_2x_1 \\
&-\; 258.0269\,x_2x_1^3 + 330.6741\,x_2^2x_1^2 - 186.0128\,x_2^3x_1 - 49.3835\,x_3x_1^3 \\
&+\; 92.8910\,x_3\,x_2\,x_1^2 - 70.7044\,x_3x_2^2x_1 - 0.2068\,x_3^3x_2 + 37.1118\,x_4x_3x_2^2 \\
&+\; 9.1538\,x_4^2x_3x_2 - 0.2863\,x_4^2x_3^2 - 9.6367\,x_4^3x_1 + 8.0508\,x_4^3x_2 \\
&+\; 0.7288\,x_4^3x_3 + 46.1405\,x_3x_2^3 + 9.4792\,x_3^2x_1^2 - 6.2154\,x_3^2x_2x_1 \\
&+\; 8.4205\,x_3^2x_2^2 - 0.4965\,x_3^3x_1 - 0.0156\,x_3^4
\end{aligned}$$

and

$$v_*^{[3]}(x) = \begin{bmatrix}
\begin{array}{l}
-0.1853\,x_4^2x_3 + 0.0929\,x_4x_3^2 + 8.1739, x_4^2x_1 - 3.7896\,x_4^2x_2 \\
-0.5132\,x_4^3 - 1.8036\,x_4x_3x_2 - 4.9101\,x_3x_2^2 + 0.3018\,x_3^2x_2 \\
-37.6101\,x_4x_1^2 + 28.4355\,x_4x_2x_1 \\
-13.8702\,x_4x_2^2 + 4.3753\,x_4x_3x_1 + 9.1991\,x_3x_2x_1 + 0.0043\,x_3^3 \\
-53.5255\,x_2x_1^2 + 42.8702\,x_2^2x_1 - 12.9923\,x_3x_1^2 + 52.2292\,x_1^3 \\
-12.1580\,x_2^3 + 0.02812\,x_1x_3^2
\end{array} \\[1ex]
\begin{array}{l}
0.1261\,x_4^2x_3 - 0.0022\,x_4x_3^2 - 0.8724\,x_4^2x_1 + 1.1509\,x_4^2x_2 \\
+0.1089\,x_4^3 + 1.0208\,x_4x_3x_2 + 1.8952\,x_3x_2^2 + 0.2323\,x_3^2x_2 \\
+3.0554\,x_4x_1^2 - 5.2286\,x_4x_2x_1 + 3.9335x_4x_2^2 - 0.6138\,x_4x_3x_1 \\
-1.9129\,x_3x_2x_1 - 0.0018\,x_3^3 + 8.8880\,x_2x_1^2 - 7.5123\,x_2^2x1 \\
+1.2179\,x_3x_1^2 - 3.2935\,x_1^3 + 4.9956\,x_2^3 - 0.0928\,x_1x_3^2
\end{array}
\end{bmatrix} \tag{46}$$

Specifically, the first row in equation (46) yields the controller term $k_*^{[3]}$ and the second row yields the disturbance strategy term $l_*^{[3]}$. In fact, because of (45), one can show that all the even terms of the series expansion for v_* are zero, i.e., $v_*^{[2k]} = 0$ for $k = 1, 2, \dots$.

7 Discussion

As mentioned at the end of Section 4 the HJIE may, in general, fail to have (real) analytic solutions. For the NLBP, nevertheless, the existence of a (convergent) analytic solution is insured by the analyticity of the Hamiltonian of this problem and the fact that the matrix A_* in equation (26) has all its eigenvalues in the left-half of the complex plane (cf. Corollary 4.3 in Kang *et al.*[21]).

The proposed methodology has certainly some limitations. First, in contrast to linear problems, one cannot guarantee that (even global) asymptotic stability of the closed-loop system together with $w \in \mathcal{L}_2[0, \infty)$

implying that $z \in \mathcal{L}_2[0,\infty)$ and $u \in \mathcal{L}_2[0,\infty)$. This technical difficulty limits the attenuation properties of the controller to those disturbance inputs $w \in \mathcal{L}_2[0,\infty)$ which do not drive the state of the closed system outside some neighborhood of the origin; van der Schaft[26], for example, considers only w with compact support in order to ensure that $x(t) \to 0$ as $t \to \infty$.

Another potential problem associated with the series expansion method for solving the HJIE is the lack of efficient algorithms for checking positive definiteness of the solution (27) in the large. By virtue of the positive definiteness of the matrix P in equation (22), we know that V is at least locally positive definite. This ensures local asymptotic stability with the (truncated) optimal controller. Nevertheless, global results are not available, at least as far as the authors know. We mention, however, that a conservative approach for ensuring positive definiteness of the solution of the HJIE has been recently reported by Shue *et al.*[27].

In light of the previous discussion it should be apparent that the truncated controller in equation (29) can only guarantee local asymptotic stability and disturbance attenuation level less than or equal to γ for disturbances which do not "push" the trajectories "too far away" from the origin[17,24]. In that respect, the truncated controller can be naturally viewed as a higher order correction to the linear controller. Both the linear and the nonlinear $\mathcal{H}_\infty$ controllers locally asymptotically stabilize the system. In general, one expects that the region of attraction of the nonlinear controller is larger than the region of attraction of the linear controller[24]. For the NLBP this was checked via numerical simulations. One should be cautioned, however, to the fact that due to the higher order polynomial terms the nonlinear controller may exhibit a more dramatic onset of instability than the linear $\mathcal{H}_\infty$ controller. For the NLBP this could be remedied, for example, by using the simple linear controller of equation (4) far away from the origin and switching to the nonlinear (or even the linear) $\mathcal{H}_\infty$ controller once close to the origin. This, we believe, will offer the most practical solution to the NLBP.

8 Numerical Simulations

Here we present some numerical results for the NLBP. Symbolic calculations for controller design were performed using MAPLE and numerical simulations were carried using the `ode45` command of MATLAB. All the plots correspond to non-dimensionalized variables.

We first demonstrate the global stabilizing properties of the simple linear controller. Simulations were carried out for different initial conditions and asymptotic stability was verified in all cases, as predicted by Proposition 2.1. The results of one of these simulations are illustrated in Figure 2. The initial condition for these simulations is $x(0) = [1\ 0\ 1\ 0]^T$ and the controller parameters were chosen as $k_1 = 1$ and $k_2 = 1$.

Although it has been shown that the simple linear controller in equation (4) is globally asymptotically stabilizing for the nonlinear system, its has no obvious disturbance attenuation properties. On the other hand, the linear $\mathcal{H}_\infty$ controller $k_*^{[1]}$ is only locally asymptotically stabilizing. It is expected, however, that the controller $k_*^{[1]}$ will have superior disturbance attenuation properties than the simple. In order to verify this, we numerically simulated the system in Eqs. (3) both with the simple linear controller and the linear $\mathcal{H}_\infty$ controller $k_*^{[1]}$ subject to zero initial conditions. The disturbance was chosen as $w = 0.1\sin(t)$. This is a sinusoidal disturbance with frequency which corresponds to the peak of the singular value plot (magnitude of transfer function from w to z versus frequency) of the closed-loop linearized systems; see Figure 3. The gains k_1 and k_2 in equation (4) were chosen to be approximately equal to the corresponding terms in the linear $\mathcal{H}_\infty$ gain matrix ($k_1 = 0.35,\ k_2 = 1$). The results of these simulations are shown in Figure 4. The linear controller $k_*^{[1]}$ (although only locally asymptotically stabilizing) has much better disturbance attenuation properties than the linear controller in Eq. (4).

We next compare the linear controller $k_*^{[1]}$ and the nonlinear controller $k_*^{[1]} + k_*^{[3]}$ with respect to the region of attraction they achieve for the closed loop system. This is illustrated in Figures 5-6 which contain the "phase portraits" of the variables ξ and θ. For initial condition $x(0) = [-1\ \ 1\ \ -1\ \ 1]^T$, the dashed lines denote the response due to the linear controller and the solid lines denote the response due to the nonlinear controller. From this simulation, it seems that the region of attraction due to the nonlinear controller is larger than that due to the linear controller: for the chosen initial state, the state trajectory resulting from the nonlinear controller tends asymptotically to the origin, whereas the trajectory resulting from the linear controller tends to a limit cycle. The corresponding histories of the states ξ and θ are shown in Figure 7. Figure 8 depicts the control history.

Finally, we compare the disturbance attenuation properties of the two controllers $k_*^{[1]}$ and $k_*^{[1]} + k_*^{[3]}$. For sinusoidal disturbances of small magnitude the two controllers had almost identical performance. This is to be expected, since in these cases the term $k_*^{[3]}$ is negligible. For larger magnitude of the disturbance the nonlinear controller performed better. Figure 9, for example, shows the case when $w = 0.24\sin(t)$ (zero initial conditions). The linear controller in this case is unable to keep the motion of the system bounded, whereas the nonlinear controller results in a bounded motion. For much larger values of the disturbance, both controllers were unable to keep the motions bounded.

9 Conclusion

We have applied the theory of $\mathcal{L}_2$ disturbance attenuation for nonlinear systems to the recently proposed nonlinear benchmark problem. A nonlinear state-feedback controller is computed recursively by considering a series expansion solution to the associated Hamilton-Jacobi-Isaacs Equation. The procedure is straightforward and can be readily automated in a computer. Numerical simulations indicate that the performance of the third order approximation of the nonlinear controller provides some improvement over its first order approximation (which is the same as a linear $\mathcal{H}_\infty$ controller obtained from the linearized problem). This improvement is however not very significant, thus indicating that higher order terms may be necessary to extend the region of attraction of the closed-loop system, or to enhance the performance of the controller. Future issues should include the choice of weighting filters for shaping the system response, a rather difficult task for nonlinear systems. Also, alternative approaches to the solution of Hamilton-Jacobi type equations, as well as non-conservative algorithms for investigating positive definiteness of the solutions to these equations are highly desirable.

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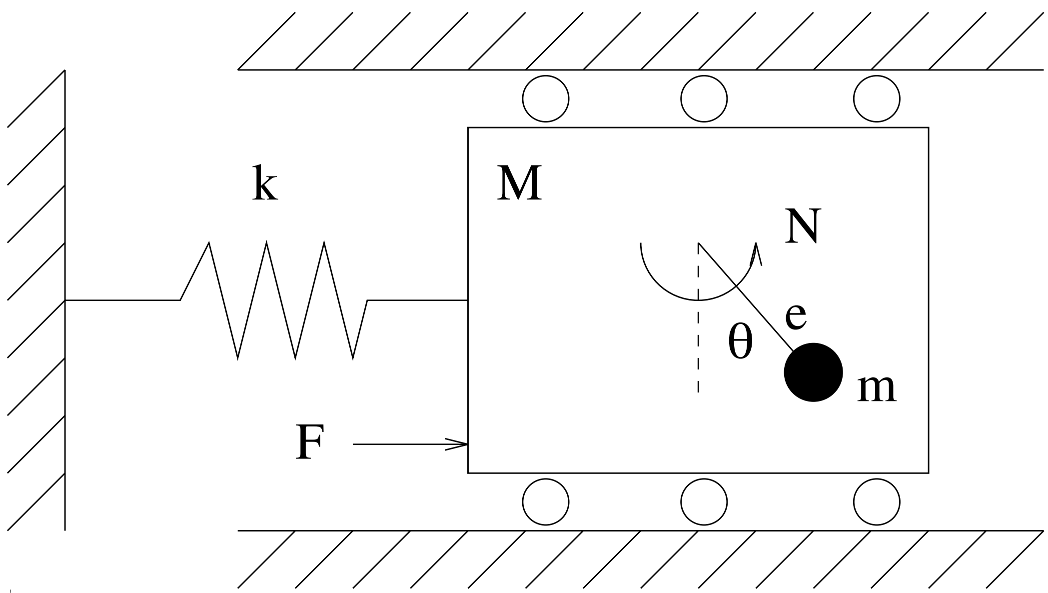

Figure 1: Nonlinear Benchmark Problem.

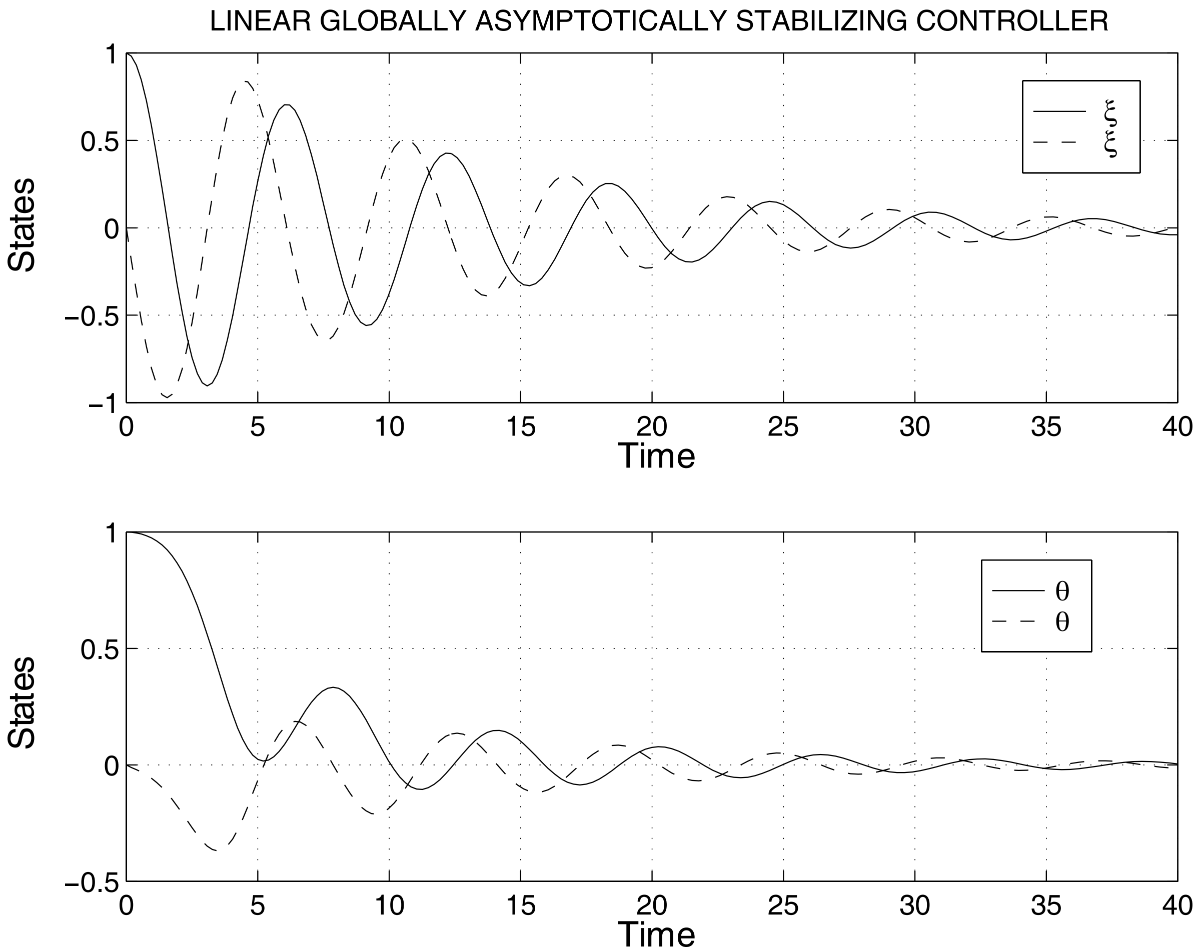

Figure 2: State histories due to the simple linear controller.

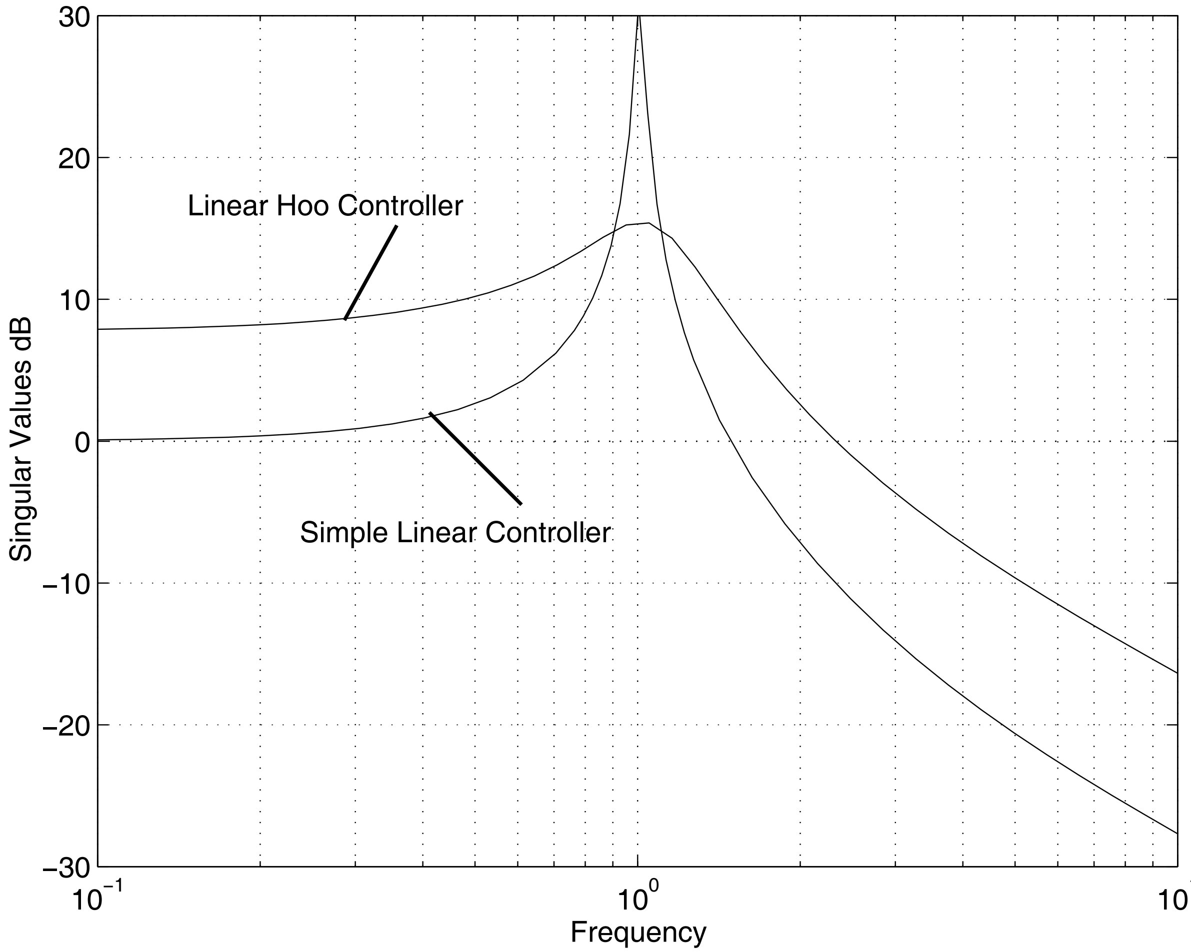

Figure 3: Singular value plots of linear controllers.

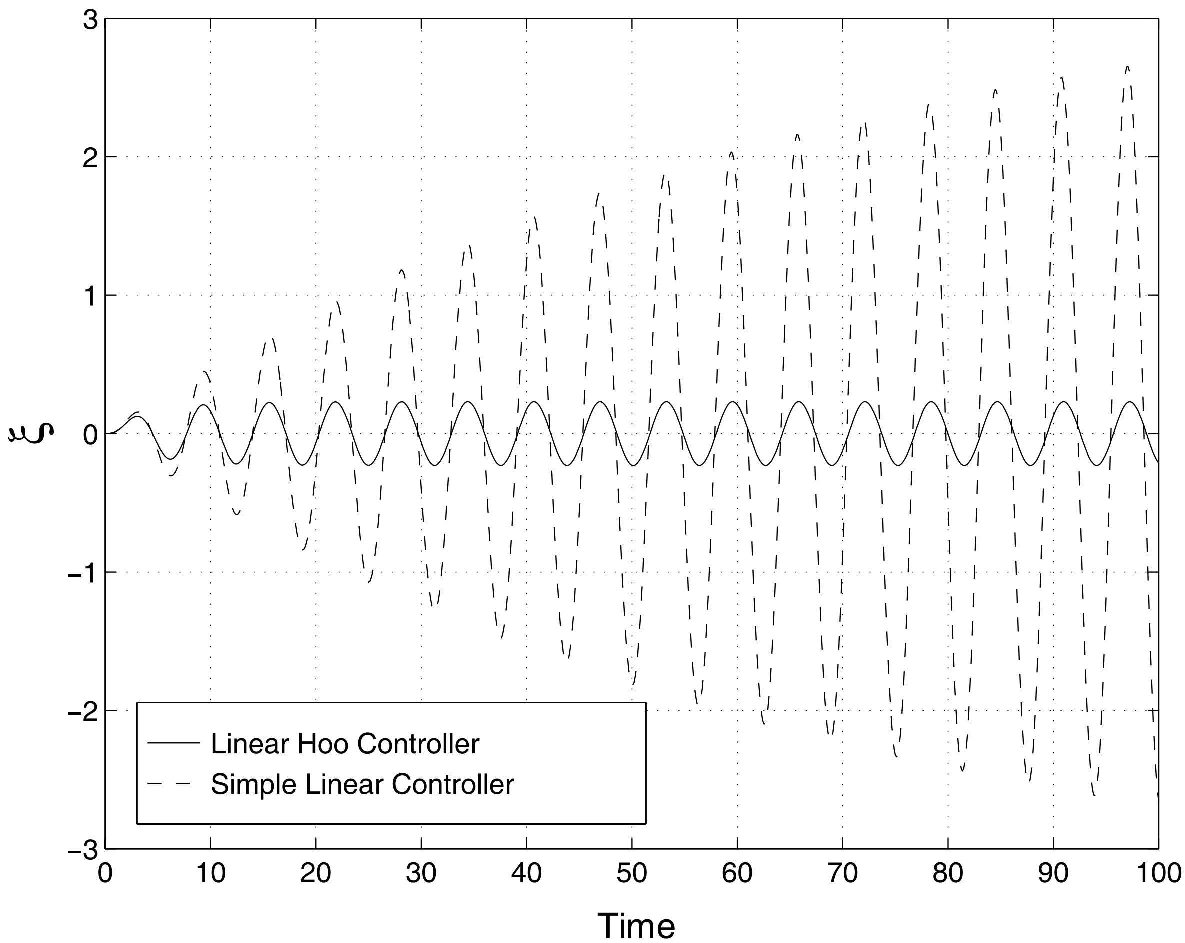

Figure 4: Comparison of disturbance attenuation properties of the linear controllers.

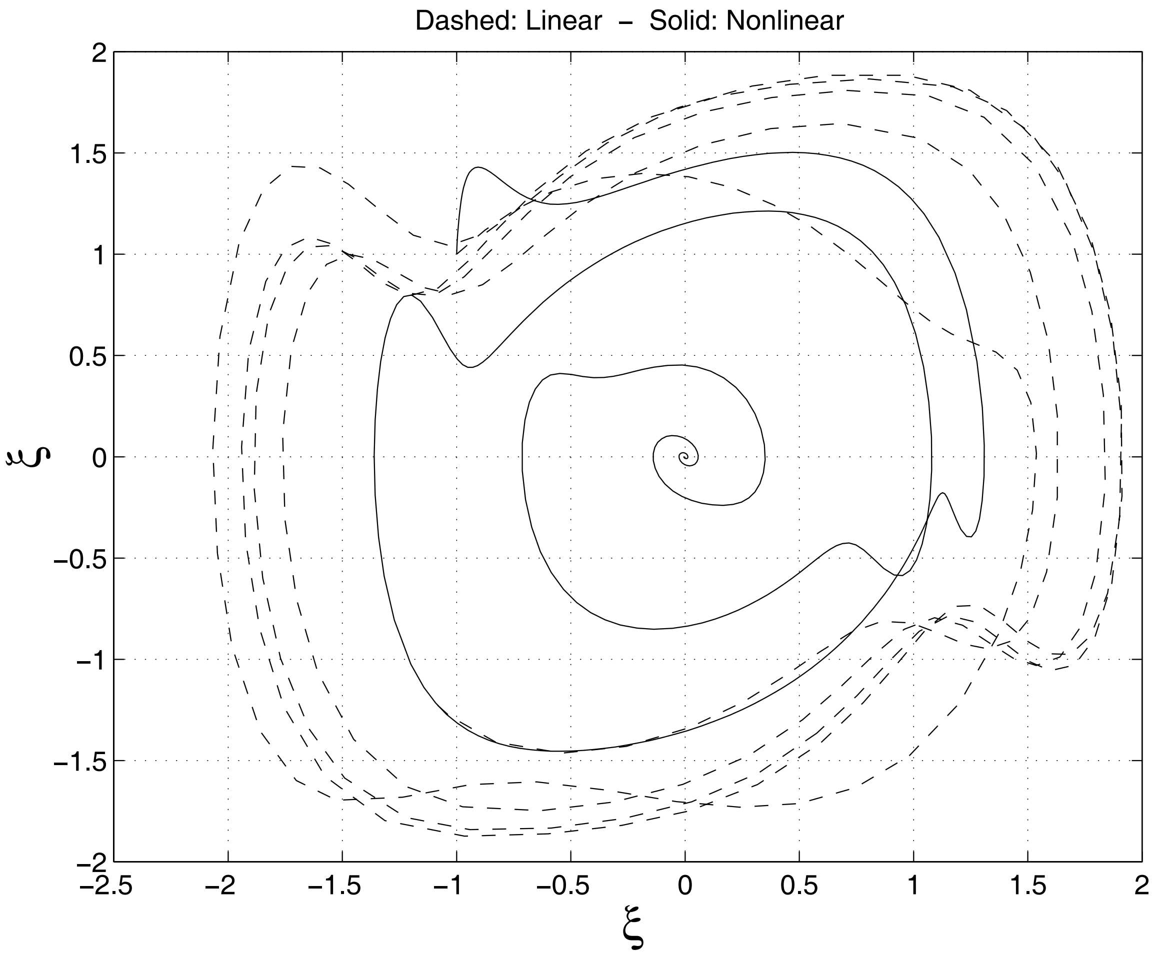

Figure 5: Phase portrait of ξ for linear and nonlinear $\mathcal{H}_\infty$ controllers.

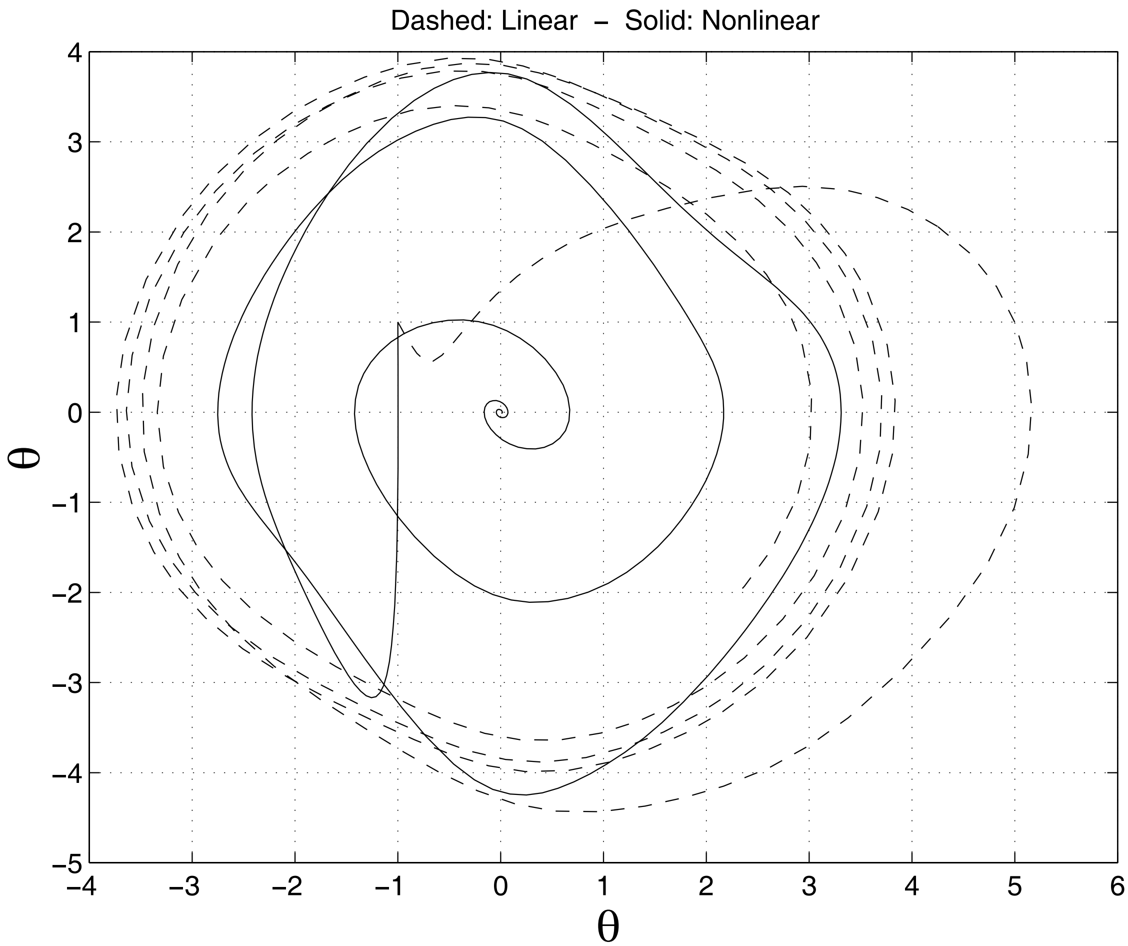

Figure 6: Phase portrait of θ for linear and nonlinear $\mathcal{H}_\infty$ controllers.

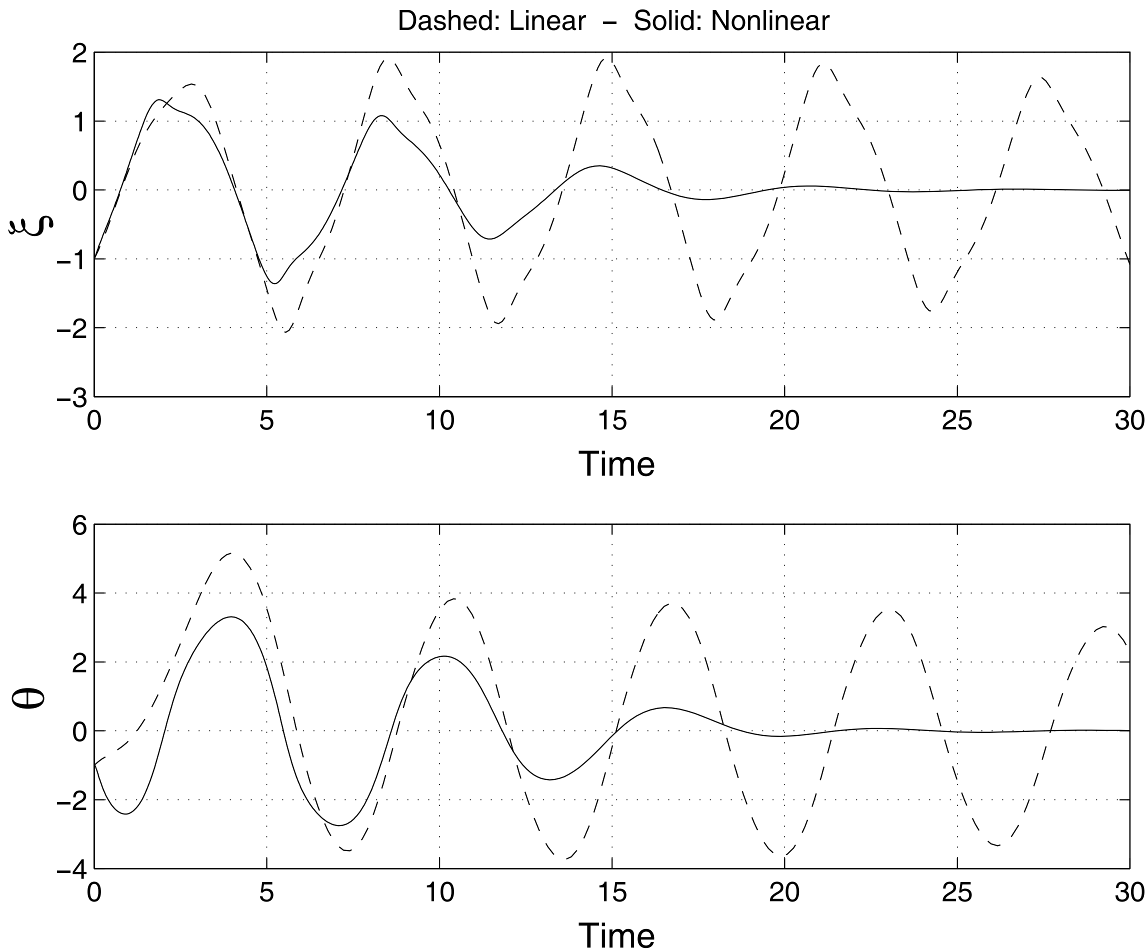

Figure 7: Time history of ξ and θ for linear and nonlinear $\mathcal{H}_\infty$ controllers.

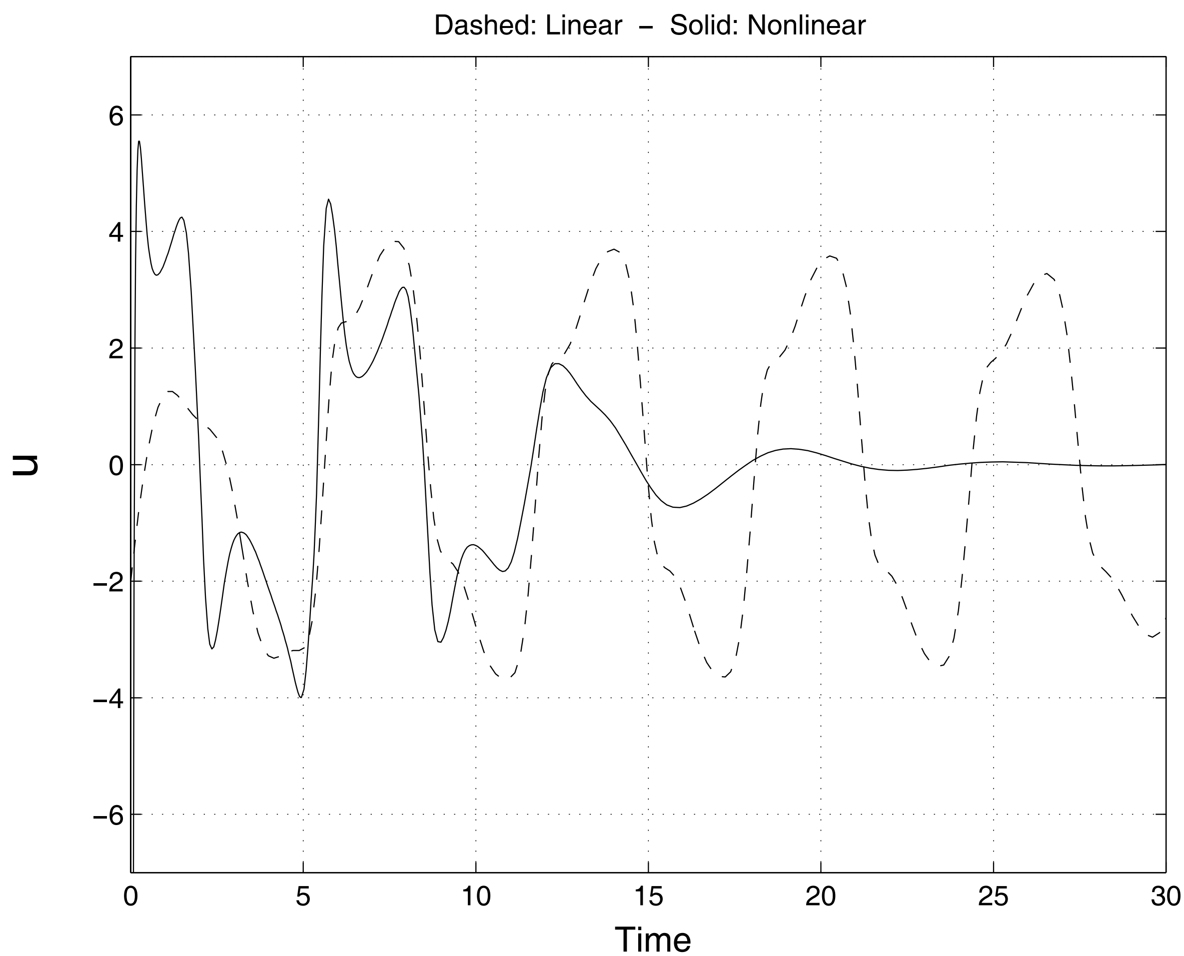

Figure 8: Control history for linear and nonlinear $\mathcal{H}_\infty$ controllers.

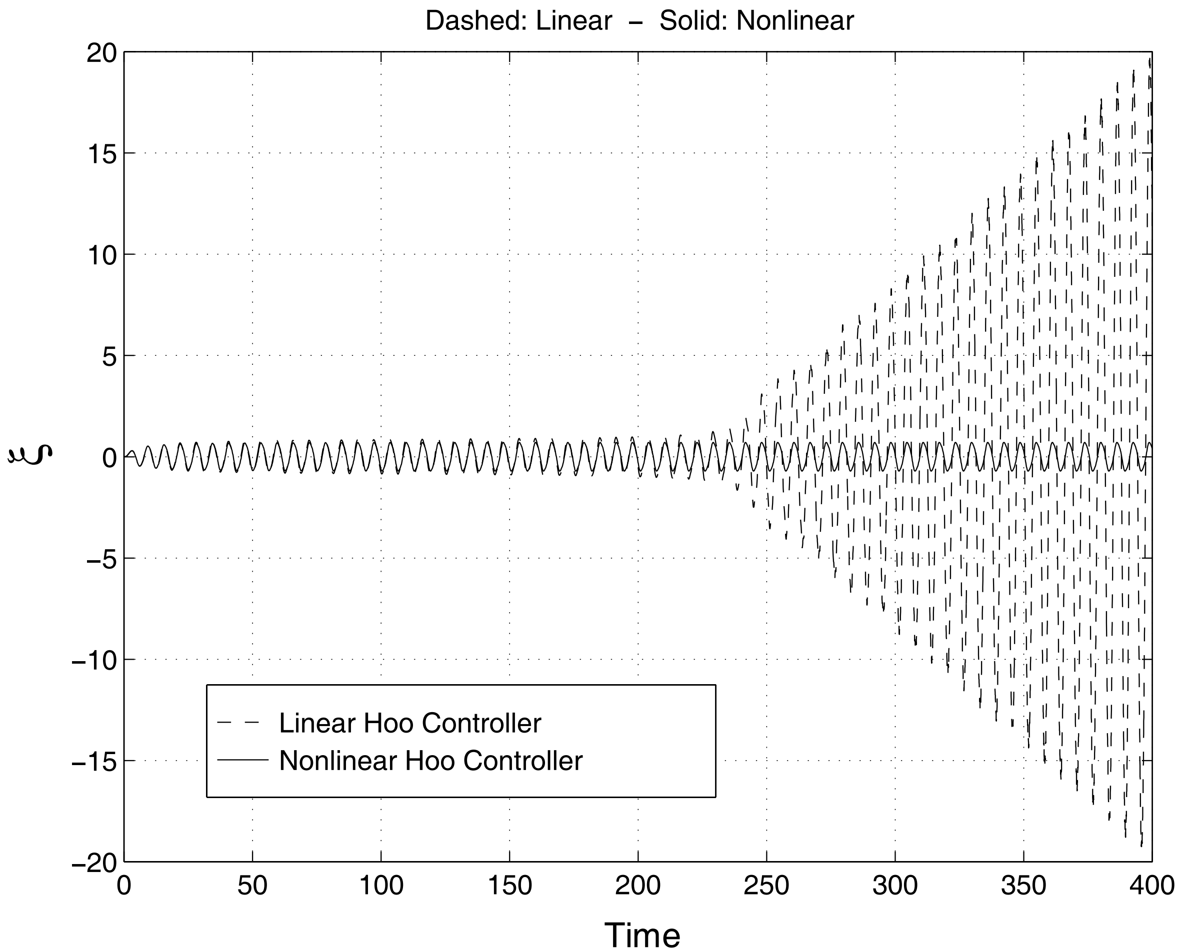

Figure 9: Disturbance attenuation for linear and nonlinear $\mathcal{H}_\infty$ controllers.