Principal Rotation Representations of Proper NxN Orthogonal Matrices

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Abstract

Three and four parameter representations of 3x3 orthogonal matrices are extended to the general case of proper NxN orthogonal matrices. These developments generalize the classical Rodrigues parameter, the Euler parameters, and the recently introduced modified Rodrigues parameters to higher dimensions. The developments presented generalize and extend the classical result known as the Cayley transformation.

Introduction

It is well known in rigid body dynamics, and many other areas of Euclidean analysis, that the rotational coordinates associated with Euler's Principal Rotation Theorem [1,2,3] leads to especially attractive descriptions of rotational motion. These parameterizations of proper orthogonal 3x3 matrices include the four-parameters set known widely as the Euler parameters, or the quaternion parameters [1,2,3], as well as the classical three-parameter set known as the Rodrigues parameters, or as the Gibbs vector [1,2,3,4]. Also included is a recent three parameter description known as the modified Rodrigues parameters [4,5,6]. As we review briefly below, these parameterizations are of fundamental significance in the geometry and kinematics of three-dimensional motion. Briefly, their advantages are as follows:

Euler Parameters: This once redundant four-parameter description of three-dimensional rotational motion maps all possible motions into arcs on a four-dimensional unit sphere. This accomplishes a regularization and the representation is universally nonsingular. The kinematic differential equations contain no transcendental functions and are bi-linear without approximation.

Classical Rodrigues Parameters: This three parameter set is proportional to Euler's principal rotation vector. The magnitude is $tan(\phi/2)$, with ϕ being the principal rotation angle. These parameters are singular at $\phi = \pm\pi$ and have elegant, quadratically nonlinear differential kinematic equations.

Modified Rodrigues Parameters: This three parameter set is also proportion to Euler's principal rotation vector, but with a magnitude of $tan(\phi/4)$. The singular orientation is at $\phi = \pm 2\pi$, doubling the principal rotation range over the classical Rodrigues parameters.

The question naturally arises; can these elegant parameterizations be extended to orthogonal projections in higher dimensional spaces? Cayley partially answered this question in the affirmative; his "Cayley Transform" fully extends the classical Rodrigues parameters to higher dimensional spaces [1,2,7]. This paper extends the classical Cayley transform to parameterize a proper NxN orthogonal matrix into a set of higher dimensional modified Rodrigues parameters. Further, a method is shown to parameterize the NxN matrix into a once-redundant set of higher dimensional Euler parameters.

The first section will review the Euler, Rodrigues and the modified Rodrigues parameters used later in this paper to parameterize the NxN orthogonal matrices. The second section will review the classical Cayley transform resulting with the parameterization of an orthogonal matrix into the Rodrigues parameters, followed by the new parameterization into the modified Rodrigues parameters is presented and by the parameterization into higher order Euler parameters.

Review of Rigid Body Rotation Parameterizations

The Direction Cosine Matrix

The 3x3 direction cosine matrix C completely describes any three-dimensional rigid body rotation. The matrix elements are bounded between ±1 and never go singular. The famous Poisson kinematic differential equation for the direction cosine matrix is:

$$\dot{C} = -[\tilde{\omega}]C \tag{1}$$

where the tilde matrix is defined as:

$$[\tilde{\omega}] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{2}$$

The direction cosine matrix C is orthogonal, therefore it satisfies the following constraint.

$$C^T C = C C^T = I \tag{3}$$

This constraint causes the direction cosine matrix representation to be highly redundant. Instead of considering all nine matrix elements, it usually suffices to parameterize the matrix into a set of three or four parameters. However, any minimal set of three parameters will contain singular orientations.

The constraint in equation (3) shows that besides being orthogonal, the direction cosine matrix is also normal [8]. Consequently it has the spectral decomposition of the form

$$C = U \Lambda U^* \tag{4}$$

where U is a unitary matrix containing the orthonormal eigenvectors of C and Λ is a diagonal matrix whose entries are the eigenvalues of C. The * symbol stands for the adjoint operator, which takes the complex conjugate transpose of a matrix. Since C corresponds to rigid body rotations, it always has a real eigenvalue of +1. Having an orthogonal matrix with a negative real eigenvalue means the matrix represents a reflection, not simply a rotation. All matrices with a positive real eigenvalue are labeled as proper matrices.

The Principal Rotation Vector

Euler's principal rotation theorem states that a rigid body (reference frame) can be brought from an arbitrary initial orientation to an arbitrary final orientation by a single principal rotation (ϕ) about a principal line $\hat{e}$ [3]

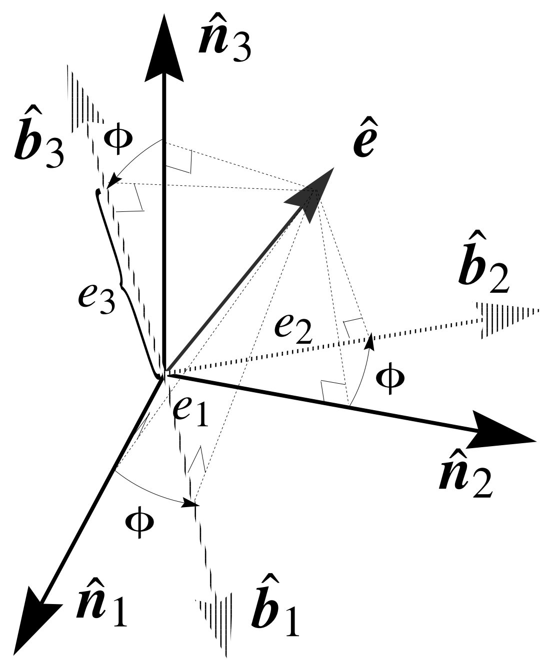

Fig. 1: Euler's Principal Rotation Theorem.

During the principal rotation shown in Figure 1, the body axis components $\hat{b}_i$ of the principal line $\hat{e}$ are identical to the spatial components $\hat{n}_i$.

$$\begin{Bmatrix} e_1 \\ e_2 \\ e_3 \end{Bmatrix} = \hat{e} = C \cdot \hat{e} \tag{5}$$

Therefore $\hat{e}$ must be an eigenvector of C with a corresponding eigenvalue of +1. If C has a eigenvalue of -1, the matrix represents a reflection, not a proper rotation and the concept of the principal rotation theorem does not hold. The principal rotation vector $\vec{\gamma}$ is defined as:

$$\vec{\gamma} = \phi\hat{e} \tag{6}$$

Let us now consider the case where a rigid body performs a pure single-axis rotation. This rotation axis is identical to Euler's principal line of rotation $\hat{e}$. Let the rotation angle be ϕ. The angular velocity vector for this case becomes:

$$\vec{\omega} = \dot{\phi}\hat{e} \tag{7}$$

or in matrix form:

$$[\tilde{\omega}] = \dot{\phi}[\tilde{e}] \tag{8}$$

Substituting equation (8) into (1), one obtains the following development.

$$\frac{dC}{dt} = -\frac{d\phi}{dt}[\tilde{e}]C$$

$$\frac{dC}{d\phi} = -[\tilde{e}]C$$

$$C = e^{-\phi[\tilde{e}]} \tag{9}$$

The last step can be done since the $[\tilde{e}]$ matrix is assumed to be constant during this single axis maneuver. Due to Euler's principal rotation theorem, however, any arbitrary rotation can always be described instantaneously by the equivalent single-axis principal rotation. Hence equation (9) will hold at any instant for an arbitrary direction cosine matrix C. Using the following substitution

$$[\tilde{\gamma}] = \phi[\tilde{e}] \tag{10}$$

equation (9) can be rewritten as [2]:

$$C = e^{-[\tilde{\gamma}]} \tag{11}$$

To find the inverse transformation from the direction cosine matrix C to $[\tilde{\gamma}]$, the matrix logarithm can be taken of equation (11).

$$[\tilde{\gamma}] = -\log C \tag{12}$$

Using the spectral decomposition of C, equation (12) can be rewritten as

$$[\tilde{\gamma}] = -U(\log\Lambda)U^* \tag{13}$$

where calculating the matrix logarithm of a diagonal matrix becomes trivial.

The principal vector representation of C is not unique. Adding or subtracting 2π from the principal rotation angle ϕ describes the same rotation. As expected, equation (11) will always yield the same C matrix for the different principal rotation angles, since all angles correspond to the same physical orientation. However, the inverse transformation given in equation (13) yields only the principal rotation angle which lies between -180° and +180°.

As do all minimal parameters set, the principal rotation vector parameterization has a singular orientation. The vector is not defined for a zero rotation from the reference frame. The differential kinematic equations can be found by substituting equation (11) into (1). However, the result cannot be solved for $\dot{\vec{\gamma}}$ directly.

$$\frac{d}{dt}\left(e^{-[\tilde{\gamma}]}\right) = -[\tilde{\omega}]e^{-[\tilde{\gamma}]} \tag{14}$$

The principal rotation vector parameterization will be convenient later to derive useful relationships. They are unsuitable for kinematic equation, since they cannot be explicitly solved.

The Euler (Quaternion) Parameters

The Euler parameters are a once-redundant set of rotation parameters. They are defined in terms of the principal rotation angle ϕ and the principal line components e_i as follows:

$$\beta_0 = \cos\frac{\phi}{2}\ ,\ \ \beta_i = e_i\sin\frac{\phi}{2} \qquad i = 1, 2, 3 \tag{15}$$

They satisfy the holonomic constraint:

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{16}$$

Equation (16) states that all possible Euler parameter trajectories generate arcs on the surface of a four-dimensional unit sphere. The behavior bounds the parameters to values between ±1.

However, the Euler parameters are not unique. The mirror image trajectories $\beta(t)$ and $-\beta(t)$ both describe the identical physical orientation histories. This is somewhat analogous to saying a body is displaced by +60° or -300°. Both angles correspond to the same physical location. Given a 3x3 orthogonal matrix, there will be two corresponding sets of Euler parameters which differ by a sign. The Euler parameters are the only set of rotation parameters which have a bi-linear system of kinematic differential equations [1], other then the direction cosine matrix itself.

$$\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \beta_0 & -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ \beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ \beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix} \begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{17}$$

It is also of significance that the above 4x4 matrix is orthogonal, so “transportation" between ω_i's and $\dot{\beta}_i$ ‘s is painless. The direction cosine matrix in term of the Euler parameters is [1,3]:

$$[C] = \begin{bmatrix} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{bmatrix} \tag{18}$$

The Euler parameters have several advantages over minimal rotation parameters. Namely, they are bounded between ±1, never encounter a singularity, and have linear kinematic differential equations if the $\omega_i(t)$ are considered known. All of these advantages are slightly offset by the cost of having one extra parameter.

The Classical Rodrigues Parameters

The classical Rodrigues parameter vector $\vec{q}$ is a set of symmetric stereographic parameters [6] with the projection point at the origin and the stereographic mapping hyperplane at $\beta_0 = +1$. Therefore they have their singular orientation at a principal rotation angle from the reference frame of $\phi = \pm 180°$. Their transformation from the Euler parameters is:

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3 \tag{19}$$

Unlike the Euler parameters, the Rodrigues parameters are numerically unique. They uniquely define a rotation on the open range of (-180°,+180°) [6]; as is evident in equation (19), reversing the sign of the Euler parameters has no effect on the q_i. Using equation (15), the classical Rodrigues parameters can also be defined directly in term of the principal rotation angle and the principal axis components.

$$q_i = e_i \tan\frac{\phi}{2} \qquad i = 1, 2, 3 \tag{20}$$

The kinematic differential equation for the Rodrigues parameters contain a low degree polynomial nonlinearity. They can be verified from equations (17,20) to be [1-4]:

$$\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} 1+q_1^2 & q_1q_2-q_3 & q_1q_3+q_2 \\ q_2q_1+q_3 & 1+q_2^2 & q_2q_3-q_1 \\ q_3q_1-q_2 & q_3q_2+q_1 & 1+q_3^2 \end{bmatrix}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{21}$$

Notice that the above coefficient matrix is not orthogonal, although the inverse is well behaved everywhere except when $|\vec{q}| \to \infty$. The direction cosine matrix in terms of the Rodrigues parameters is [1-4]:

$$C(\vec{q}) = \frac{1}{1+q_1^2+q_2^2+q_3^2}\begin{bmatrix} 1+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_3) & 2(q_1q_3-q_2) \\ 2(q_2q_1-q_3) & 1-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_1) \\ 2(q_3q_1+q_2) & 2(q_3q_2-q_1) & 1-q_1^2-q_2^2+q_3^2 \end{bmatrix} \tag{22}$$

The Modified Rodrigues Parameters

The modified Rodrigues parameter vector $\vec{\sigma}$ is also a set of symmetric stereographic parameters, closely related to the classical Rodrigues parameters [2,4-6]. The modified Rodrigues parameters have the projection point at (-1,0,0,0) and the stereographic mapping hyperplane at $\beta_0 = 0$. This projection results in a set of parameters which do not encounter a singularity until a principal rotation from the reference frame of ±360° has been performed. Their transformation from the Euler parameters is:

$$\sigma_i = \frac{\beta_i}{1+\beta_0} \qquad i = 1, 2, 3 \tag{23}$$

Like the Euler parameters, the modified Rodrigues parameters are not unique. They have an associated shadow set found by using -β(t) instead of β(t) in equation (23) [5,6]. The transformation from the original set to the shadow set is [2,5,6]:

$$\sigma_i^S = \frac{-\sigma_i}{\vec{\sigma}^T\vec{\sigma}} \qquad i = 1, 2, 3 \tag{24}$$

The shadow points are denoted with a superscript S. Keep in mind that both $\vec{\sigma}$ and $\vec{\sigma}^S$ describe the same physical orientation, similar and related to the case of the two possible sets of Euler parameters and principal rotation vectors. It turns out that the modified Rodrigues shadow parameters have the opposite singular behavior to the original ones. The original parameters are very linear near a zero rotation and are singular at a ±360° rotation. The shadow parameters are linear near the ±360° rotation and singular at the zero rotation. [6]

Since the modified Rodrigues parameters allow for twice the principal rotation angle compared to the classical Rodrigues parameters, they allow for an *eight times larger* linear domain! Using equation (15), the definition for the modified Rodrigues parameters in equation (23) can be rewritten as [4]:

$$\sigma_i = e_i \tan\frac{\Phi}{4} \tag{25}$$

Equation (25) is very similar to equation (20), except for the scaling factor of the principal rotation angle. The singularity at ±360° is evident in equation (25), and small rotations behave like quarter angles. The differential kinematic equations display a similar degree of nonlinearity as do the corresponding equations in terms of the classical Rodrigues parameters [4-6].

$$\dot{\vec{\sigma}} = \frac{1}{4}\begin{bmatrix} 1+\sigma_1^2-\sigma_2^2-\sigma_3^2 & 2(\sigma_1\sigma_2-\sigma_3) & 2(\sigma_1\sigma_3+\sigma_2) \\ 2(\sigma_2\sigma_1+\sigma_3) & 1-\sigma_1^2+\sigma_2^2-\sigma_3^2 & 2(\sigma_2\sigma_3-\sigma_1) \\ 2(\sigma_3\sigma_1-\sigma_2) & 2(\sigma_3\sigma_2+\sigma_1) & 1-\sigma_1^2-\sigma_2^2+\sigma_3^2 \end{bmatrix}\begin{Bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{Bmatrix} \tag{26}$$

Note that the coefficient matrix of the differential kinematic equation is not orthogonal, but almost. Multiplying it with its transpose yields a *scalar* times the identity matrix. This almost orthogonal behavior allows for a simple transformation between the ω_i and the $\dot{\sigma}_i$.

$$C(\vec{\sigma}) = \frac{1}{\left(1+\vec{\sigma}^T\vec{\sigma}\right)^2}\begin{bmatrix} 4\left(\sigma_1^2-\sigma_2^2-\sigma_3^2\right)+\Sigma^2 & 8\sigma_1\sigma_2+4\sigma_3\Sigma & 8\sigma_1\sigma_3-4\sigma_2\Sigma \\ 8\sigma_2\sigma_1-4\sigma_3\Sigma & 4\left(-\sigma_1^2+\sigma_2^2-\sigma_3^2\right)+\Sigma^2 & 8\sigma_2\sigma_3+4\sigma_1\Sigma \\ 8\sigma_3\sigma_1+4\sigma_2\Sigma & 8\sigma_3\sigma_2-4\sigma_1\Sigma & 4\left(-\sigma_1^2-\sigma_2^2+\sigma_3^2\right)+\Sigma^2 \end{bmatrix} \tag{27}$$

$$\Sigma = 1-\vec{\sigma}^T\vec{\sigma}$$

The direction cosine matrix is shown above. It has a slightly higher level of nonlinearity than the corresponding direction cosine matrix in terms of the classical Rodrigues parameters.

Parameterization of Proper NxN Orthogonal Matrices

A proper orthogonal matrix is an orthogonal matrix whose real eigenvalue is +1. Some aspects of parameterizing NxN orthogonal matrices into N-dimensional Rodrigues parameters have been covered by Junkins and Kim [1] and Shuster [2]. These classical developments date from the work of Cayley [7] and are included here for comparative purposes with the other parameterizations.

Any NxN orthogonal matrix abides by the constraint given in equation (3). Taking the first derivative thereof one obtains:

$$\dot{C}^T C + C^T \dot{C} = 0 \tag{28}$$

The $\dot{C}$ matrix defined in equation (1) can be shown to satisfy this differential equation exactly. Substitute equation (1) into (27) and expand.

$$(-[\tilde{\omega}]C)^T C + C^T(-[\tilde{\omega}]C) = 0$$

$$\left(-C^T[\tilde{\omega}]^T\right)C - C^T[\tilde{\omega}]C = 0$$

$$C^T\left(-[\tilde{\omega}]^T - [\tilde{\omega}]\right)C = 0$$

The above statement is only satisfied if $[\tilde{\omega}]$ is a skew-symmetric matrix satisfying

$$[\tilde{\omega}] = -[\tilde{\omega}]^T$$

Consequently equation (1) will generate an NxN orthogonal matrix, as long as $[\tilde{\omega}]$ is skew-symmetric. This observation allows for the evolution of NxN orthogonal matrices to be viewed as higher order direction cosine matrices, somewhat analogous to a "higher dimensional rigid body rotation," and be parameterized into set of higher dimensional rigid body-motivated rotation parameters.

Higher Dimensional Classical Rodrigues Parameters

Cayley's transformation [7] parameterizes an orthogonal matrix C as a function of a skew-symmetric matrix *Q*.

$$C = (I-Q)(I+Q)^{-1} = (I+Q)^{-1}(I-Q) \tag{29a}$$

$$Q = (I-C)(I+C)^{-1} = (I+C)^{-1}(I-C) \tag{29b}$$

For the 3x3 case, let the Q matrix be defined as the following skew-symmetric matrix:

$$Q = [\tilde{q}] = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{bmatrix} \tag{30}$$

After substituting equation (30) into (29a), it can be verified that resulting *C* matrix is indeed equal to equation (27). Cayley's transformation (29) is a generalization of the classical Rodrigues parameter representation for NxN proper orthogonal matrices [1,2].

Using the $[\tilde{\gamma}]$ matrix defined in equation (13) the Q matrix can be defined as follows [2]:

$$Q = -\tanh\left(\frac{[\tilde{\gamma}]}{2}\right) = -\left(e^{\frac{[\tilde{\gamma}]}{2}} - e^{-\frac{[\tilde{\gamma}]}{2}}\right)\left(e^{\frac{[\tilde{\gamma}]}{2}} + e^{-\frac{[\tilde{\gamma}]}{2}}\right)^{-1} \tag{31}$$

The above transformation can be verified by performing a matrix power series expansion of equation (31) and substituting it into a matrix power series expansion of equation (29a). The result is a matrix power series expansion for the matrix exponential function as expected from equation (11). Note the similarity between equation (31) and (20). Both calculate the Rodrigues parameters in terms of half the principal rotation angle!

The differential kinematic equations of the NxN orthogonal matrix C where shown in equation (1), where the skew-symmetric matrix $[\tilde{\omega}]$ is related to Q and $\dot{Q}$ via the kinematic relationship [1]

$$[\tilde{\omega}] = 2(I+Q)^{-1}\dot{Q}(I-Q)^{-1} \tag{32}$$

or conversely, $\dot{Q}$ can be written as

$$\dot{Q} = \frac{1}{2}(I+Q)[\tilde{\omega}](I-Q) \tag{33}$$

The equations (32-33) are proven for the N-dimensional case in reference 1. For NxN orthogonal matrices, $[\tilde{\omega}] = -[\tilde{\omega}]^T$ represents an analogous "angular velocity" matrix, even though the Gibbsian/Euclidian idea of a cross product does not generalize to higher dimensional spaces.

Higher Dimensional Modified Rodrigues Parameters

As is evident above, the modified Rodrigues parameters have twice the principal rotation range as the classical Rodrigues parameters. Parameterizing the 3x3 direction cosine matrix *C,* the modified Rodrigues parameters have a volume *eight* times the nonsingular range, in comparison to their classical counterparts. For the case of a general NxN orthogonal matrix, the volume increase of the nonsingular domain is more substantial, namely 2^N times.

To find a transformation from the NxN orthogonal matrix C to the modified Rodrigues parameters, let us first examine what happens when taking the matrix square root of C. Let the square root matrix W be defined by:

$$WW = C \tag{34}$$

Let the solutions be restricted to W's that are themselves proper rotation matrices. Therefore their eigenvalues must be either a complex conjugate pair or +1. Obviously, for the general NxN case, there will be many W matrices that satisfy equation (34). Using the spectral decomposition given in equation (4), W can be written as:

$$W = U\sqrt{\Lambda}U^* \tag{35}$$

Keep in mind that the Λ matrix is diagonal and that the matrix square root is trivial to calculate. The W matrix has a principal line and angle associated with it. Multiplying W with itself in equation (34) simply doubles the principal angle, but leaves the principal line unchanged. Therefore W represents a rotation about the same principal line as C, but with half the principal angle.

We note that all odd-dimensioned proper C and W matrices will have only one real (+1) eigenvalue, and all even-dimensioned proper C and W will have only complex conjugate pairs of eigenvalues. Since W must be a proper rotation matrix, the square root of a real eigenvalue must be +1. This still leaves an ambiguity about the sign of the complex eigenvalues in $\sqrt{\Lambda}$. In the 3x3 case there is only one complex conjugate pair of eigenvalues. Hence only two W matrices would satisfy the above conditions. This is to be expected, since any three-dimensional rotation can be described by two principal rotation angles which differ by 2π, one of which is positive and the other is negative. Let us choose to keep the sign of all complex eigenvalues such that their real part is always positive. For three-dimensional rotations, this simple rule can be shown to restrict the principal rotation angle to satisfy $-180° \le \phi \le +180°$. This choice is consistent with many numerical matrix manipulation packages and their way in handling a square root of a matrix. Let the j-th complex conjugate eigenvalue be denoted as $r_j e^{\pm i\theta_j}$, where the magnitude is $r_j \ge 0$ and the phase is $-180° \le \theta_j \le +180°$. If the dimension N is an odd number, W is defined as:

$$W = U \cdot \begin{bmatrix} +\sqrt{r_1}e^{+i\frac{\theta_1}{2}} & 0 & \cdots & 0 & 0 & 0 \\ 0 & +\sqrt{r_1}e^{-i\frac{\theta_1}{2}} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & +\sqrt{r_{N-1}}e^{+i\frac{\theta_{N-1}}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & +\sqrt{r_{N-1}}e^{-i\frac{\theta_{N-1}}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & +1 \end{bmatrix} \cdot U^* \tag{36}$$

If the dimension N is even, then W is defined as:

$$W = U \cdot \begin{bmatrix} +\sqrt{r_1}\,e^{+i\frac{\theta_1}{2}} & 0 & \cdots & 0 & 0 \\ 0 & +\sqrt{r_1}\,e^{-i\frac{\theta_1}{2}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & +\sqrt{r_{N-1}}\,e^{+i\frac{\theta_{N-1}}{2}} & 0 \\ 0 & 0 & 0 & 0 & +\sqrt{r_{N-1}}\,e^{-i\frac{\theta_{N-1}}{2}} \end{bmatrix} \cdot U^* \tag{37}$$

Using the parameterization given in equation (11), the matrix W can also be defined directly in terms of the principal rotation matrix $[\tilde{\gamma}]$.

$$W = e^{\frac{-[\gamma]}{2}} \tag{38}$$

This solution for W can be verified by substituting it back into equation (34). Since, for three-dimensional rotations, there are two possible principal angles for a given attitude, there are two possible solutions for equation (38). Again, by keeping $|\phi| < 180°$, the same W matrix is obtained as with the matrix square root method discussed above.

Remember that the modified Rodrigues parameters have a nonsingular range corresponding to $|\phi| < 360°$. Since W is the direction cosine matrix corresponding to half of the principal rotation angle of C, the resulting nonsingular range of the W matrix has been scaled down to $|\phi| < 180°$. This is the same nonsingular range as the classical Rodrigues parameters. Therefore the Cayley transformations, defined in equations (29a,b), can be applied to W. Let S be the skew-symmetric matrix composed of the modified Rodrigues parameters, similar to the construction of the Q matrix in equation (30). Then the transformation from W to S and its inverse are given as:

$$W = (I - S)(I + S)^{-1} = (I + S)^{-1}(I - S) \tag{39a}$$

$$S = (I - W)(I + W)^{-1} = (I + W)^{-1}(I - W) \tag{39b}$$

Using equation (39a) and (34), a direct transformation from S to C is found.

$$C = (I - S)^2 (I + S)^{-2} = (I + S)^{-2}(I - S)^2 \tag{40}$$

This direct transformation is very similar to the classical Cayley transform, but no elegant direct inverse exists (i.e. we lose the elegance of equation (29b); no analogous equation can be written for S as a function of C). This is due to the overlapping principal rotation angle range of $\pm 360°$. Since the classical Rodrigues parameters are for principal rotations between $(-180°, +180°)$, they have a unique representation and the Cayley transform has the well known elegant inverse.

However, an alternate way to obtain the S matrix from the C matrix is available through the skew-symmetric matrix $[\tilde{\gamma}]$ defined in equation (13).

$$S = -\tanh\left(\frac{[\tilde{\gamma}]}{4}\right) = -\left(e^{\frac{[\tilde{\gamma}]}{4}} - e^{-\frac{[\tilde{\gamma}]}{4}}\right)\left(e^{\frac{[\tilde{\gamma}]}{4}} + e^{-\frac{[\tilde{\gamma}]}{4}}\right)^{-1} \quad (41)$$

The transformations given in equation (41) can be verified by performing a matrix power series expansion and back-substituting it into equation (40). Note again the similarity between equation (41) and equation (25). The principal rotation angle is divided by four in both cases.

Either the W or the $[\tilde{\gamma}]$ matrix can be solved from NxN orthogonal C matrix to obtain the corresponding S. Neither method is as elegant as equation (29b) of the Cayley transformation. The method using the $[\tilde{\gamma}]$ matrix has the advantage that $[\tilde{\gamma}]$ is found by taking the matrix logarithm of the eigenvalues of the C matrix as shown in equation (13). The uniqueness questions do not arise here as in the matrix square root method because solutions are implicitly restricted to proper rotations with $|\phi| < 180°$. Both methods produce the same results. Since each set of modified Rodrigues parameters has its associated shadow set [6], it is usually not important which S parameterization one obtains, as long as at least one valid S matrix is found. Once a parameter set is found, either the original ones or the shadow set, it is trivial to remain with this set during the forward integration of the differential equations governing the evolution of S.

The differential kinematic equations for S are not written directly from C as they were with the classical Cayley transform. Instead W is used to describe the kinematics of the NxN system. The relationship between W and S is the same as between C and Q. Therefore the same equations can be used. The differential kinematic equation for W is:

$$\dot{W} = -[\tilde{\Omega}]W \quad (42)$$

where the skew-symmetric matrix $[\tilde{\omega}]$ is:

$$[\tilde{\Omega}] = 2(I+S)^{-1}\dot{S}(I-S)^{-1} \quad (43)$$

or conversely $\dot{S}$ could be defined as:

$$\dot{S} = \frac{1}{2}(I+S)[\tilde{\Omega}](I-S) \quad (44)$$

Equation (34) can be used during the forward integration to obtain *C(t)*. The time evolution of C in terms of W and $[\tilde{\Omega}]$ is:

$$\dot{C} = -[\tilde{\Omega}]WW - W[\tilde{\Omega}]W = -[\tilde{\Omega}]C - W[\tilde{\Omega}]W \tag{45}$$

Equating equation (45) and (1), the direct transformation from $[\tilde{\Omega}]$ to $[\tilde{\omega}]$ is:

$$[\tilde{\omega}] = [\tilde{\Omega}] + W[\tilde{\Omega}]W^T \tag{46}$$

To verify that equation (46) yields a skew-symmetric matrix $[\tilde{\omega}]$, the definition of a skew-symmetric matrix is used:

$$[\tilde{\omega}] = -[\tilde{\omega}]^T = -\left([\tilde{\Omega}] + W[\tilde{\Omega}]W^T\right)^T$$

$$[\tilde{\omega}] = -[\tilde{\Omega}]^T - \left(W^T\right)^T [\tilde{\Omega}]^T W^T$$

$$[\tilde{\omega}] = [\tilde{\Omega}] + W[\tilde{\Omega}]W^T \quad q.e.d.$$

Although this new parameterization is somewhat more complicated than the classical parameterization into N-dimensional Rodrigues parameters, the complications arise only when setting up the parameterization in terms of *S*. Once a *S* and a corresponding *W* have been found, this method is no different from the classical method. The important improvement is that the nonsingular domain has been expanded by a factor of 2^N!

A Preliminary Investigation of Higher Dimensional Euler Parameters

The classical Euler parameters stood apart from the other parameterizations, because they were bounded, universally nonsingular and had a easy to solve bi-linear differential kinematic equations. All this came at the cost of increasing the dimension of the parameter vector by one. These classical Euler parameters are extended to higher dimensions, where they will retain some, but not all, of the above desirable features.

The Rodrigues parameters and the Euler parameters are very closely related as seen in equation (19). They are identical except for the scaling term of β_0. The classical Rodrigues parameters have been shown to expand to the higher dimensional case where they parameterize a NxN orthogonal matrix *C* [1]. They can always be described as the ratio of two quantities.

$$q_i = \frac{\beta_i}{\beta_0} \qquad i = 1, 2, 3, \ldots, \frac{N(N-1)}{2} \tag{47}$$

The skew-symmetric matrix *Q* can be written as:

$$Q = \frac{1}{\beta_0} B \tag{48}$$

where B is a skew-symmetric matrix containing the numerators of Q. For the three dimensional case, this matrix would be in terms of the Euler parameters β_1, β_2, β_3, and would look like:

$$B = \begin{bmatrix} 0 & -\beta_3 & \beta_2 \\ \beta_3 & 0 & -\beta_1 \\ -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{49}$$

Substituting the new definition of Q given in equation (48) into the Cayley transform in equation (29a) results in the following:

$$C = (\beta_0 I - B)(\beta_0 I + B)^{-1}$$

$$C(\beta_0 I + B) = (\beta_0 I - B)$$

$$(I - C)\beta_0 - (I + C)B = 0 \tag{50}$$

Equation (50) represents a NxN system of linear equations in (β_0, β_1, ..., β_N). Let the $[N^2 x(N+1)]$ matrix A represent the linear relationship between the β_i.

$$A \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \end{bmatrix} = 0 \tag{51}$$

Clearly the set of all possible higher dimensional Euler parameters spans the kernel of A. We know that the ½N(N-1) Rodrigues parameters are a minimal set to parameterize the orthogonal NxN matrix C. By adding the scaling factor β_0, a once redundant set of parameters has been generated. Even though there are N^2 linear equations in equation (50), the dimension of the range of A is only ½N(N-1). The problem is still under determined. The dimension of the kernel of A must be one,since only one additional term was added to a minimal set of rotation parameters. The solution space is a multi-dimensional line through the origin.

After finding the kernel base vector, an infinite number of solutions still exist. Another constraint is needed. Let us set the norm of the higher dimensional Euler parameter vector to be unity. This concept is illustrated in Fig. 2 below.

$$\beta_0^2 + \beta_1^2 + \cdots + \beta_N^2 = 1 \tag{52}$$

Equation (52) is the higher dimensional equivalent of the holonomic constraint of the classical Euler parameters introduced in equation (16).

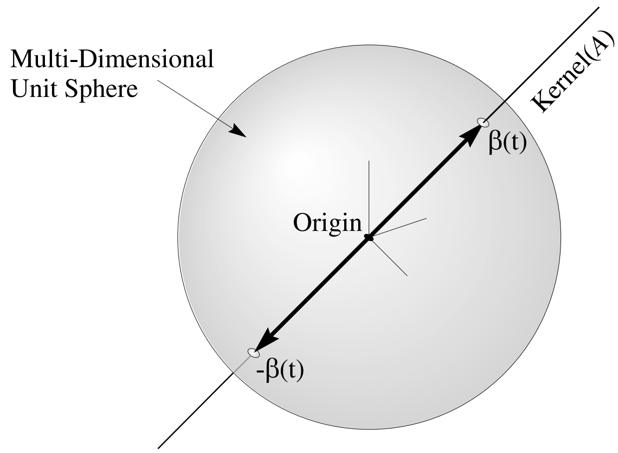

Fig. 2: Solution of the Higher Dimensional Euler Parameters.

Two solutions are found scaling the base vector of the kernel of *A* to unit length. Just as with the classical Euler parameters, any point on the multi-dimensional Euler parameter unit sphere describes the same physical orientation as its counter pole. Therefore the higher order Euler parameters are not unique, but contain a duality. However, this duality should not pose any practical problems, except under one circumstance discussed below.

$$C = (\beta_0 I - B)(\beta_0 I + B)^{-1} = (\beta_0 I + B)^{-1}(\beta_0 I - B) \tag{53}$$

The inverse transformation from higher order Euler parameters to the orthogonal matrix *C* is found by using *Q* from equation (48) in the classical Cayley transform. The result is shown in equation (53). Using a *B,* as shown in equation (49) for the three-dimensional case, in equation (53) results in the same transformation as given in equation (18). Observe that the inverse transformation has a singularity when β_0 is zero. This singularity is a mathematical singularity only. Contrary to the Rodrigues parameters, the higher order Euler parameters are well defined at this orientation. After an appropriate skew-symmetric matrix *B* is constructed and carrying out the algebra in equation (53), a closed form algebraic transformation is found. For the 2x2 case the inverse transformation into the direction cosine matrix is:

$$C_{2x2} = \begin{bmatrix} \beta_0^2 - \beta_1^2 & 2\beta_0\beta_0 \\ -2\beta_0\beta_0 & \beta_0^2 - \beta_1^2 \end{bmatrix} \tag{54}$$

The *C* matrix contains no polynomial fractions and is easy to calculate. To find the direction cosine matrix for the 3x3 case, use the *B* matrix defined in equation (51) in equation (53).

$$C_{3x3} = \frac{1}{\beta_0\left(\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2\right)}\begin{bmatrix} \beta_0\left(\beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2\right) & 2\beta_0(\beta_1\beta_2+\beta_0\beta_3) & 2\beta_0(\beta_1\beta_3-\beta_0\beta_2) \\ 2\beta_0(\beta_1\beta_2-\beta_0\beta_3) & \beta_0\left(\beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2\right) & 2\beta_0(\beta_2\beta_3+\beta_0\beta_1) \\ 2\beta_0(\beta_1\beta_3+\beta_0\beta_2) & 2\beta_0(\beta_2\beta_3-\beta_0\beta_1) & \beta_0\left(\beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2\right) \end{bmatrix}$$

After making some cancellations and enforcing the holonomic constraint equation, the well known result is found which transfers the classical Euler parameters to a 3x3 direction cosine matrix as given in equation (18). This transformation lacks any polynomial fractions and contains no singularities, just as was the case with the 2x2 system.

For dimensions greater than 3x3's, the algebraic transformation contains polynomial fractions. The nice cancelations that occur with a 2x2 and a 3x3 orthogonal matrices do not occur with the higher dimensions. This might have been anticipated, because [2] it is known that quaternion algebra does not generalize to higher-dimensional spaces and the elegant classical Euler parameter results are essentially manifestations of quaternion algebra. To find C_{4x4} in terms of the higher dimensional Euler parameters, define the 4x4 B matrix as:

$$B_{4x4} = \begin{bmatrix} 0 & -\beta_6 & \beta_5 & -\beta_4 \\ \beta_6 & 0 & -\beta_3 & \beta_2 \\ -\beta_5 & \beta_3 & 0 & -\beta_1 \\ \beta_4 & -\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{55}$$

and substitute it into equation (53).

$$C_{4x4} = \frac{1}{\Delta}\begin{bmatrix} \beta_0^2\left(\beta_0^2+\beta_1^2+\beta_2^2+\beta_3^2-\beta_4^2-\beta_5^2-\beta_6^2\right)-\delta^2 & 2\beta_0(\beta_0(\beta_2\beta_4+\beta_3\beta_5+\beta_0\beta_6)+\beta_1\delta) & \cdots \\ 2\beta_0(\beta_0(\beta_2\beta_4+\beta_3\beta_5-\beta_0\beta_6)-\beta_1\delta) & \beta_0^2\left(\beta_0^2+\beta_1^2-\beta_2^2-\beta_3^2+\beta_4^2+\beta_5^2-\beta_6^2\right)-\delta^2 & \cdots \\ 2\beta_0(\beta_0(\beta_0\beta_5+\beta_3\beta_6-\beta_1\beta_4)-\beta_2\delta) & 2\beta_0(\beta_0(\beta_1\beta_2-\beta_0\beta_3+\beta_5\beta_6)-\beta_4\delta) & \cdots \\ 2\beta_0(\beta_0(-\beta_0\beta_4-\beta_1\beta_5-\beta_2\beta_6)-\beta_3\delta) & 2\beta_0(\beta_0(\beta_1\beta_3+\beta_0\beta_2-\beta_4\beta_6)-\beta_5\delta) & \cdots \end{bmatrix}$$

$$\cdots \begin{bmatrix} 2\beta_0(\beta_0(-\beta_0\beta_5+\beta_3\beta_6-\beta_1\beta_4)+\beta_2\delta) & 2\beta_0(\beta_0(\beta_0\beta_4-\beta_1\beta_5-\beta_2\beta_6)+\beta_3\delta) \\ 2\beta_0(\beta_0(\beta_1\beta_2+\beta_0\beta_3+\beta_5\beta_6)+\beta_4\delta) & 2\beta_0(\beta_0(\beta_1\beta_3-\beta_0\beta_2-\beta_4\beta_6)+\beta_5\delta) \\ \beta_0^2\left(\beta_0^2-\beta_1^2+\beta_2^2-\beta_3^2+\beta_4^2-\beta_5^2+\beta_6^2\right)-\delta^2 & 2\beta_0(\beta_0(\beta_0\beta_1+\beta_4\beta_5+\beta_2\beta_3)+\beta_6\delta) \\ 2\beta_0(\beta_0(-\beta_0\beta_1+\beta_4\beta_5+\beta_2\beta_3)-\beta_6\delta) & \beta_0^2\left(\beta_0^2-\beta_1^2-\beta_2^2+\beta_3^2-\beta_4^2+\beta_5^2+\beta_6^2\right)-\delta^2 \end{bmatrix} \tag{56}$$

$$\textit{with} \quad \delta = \beta_3\beta_4+\beta_1\beta_6-\beta_2\beta_5$$
$$\Delta = \beta_0^2+\delta^2$$

This denominator Δ can vanish for several β_i configurations. Observe, however, that whenever Δ is zero, so is the numerator. For each singular case we find a finite limit exists, as was to be expected, since the original orthogonal C matrix was finite. In all cases $\beta_0 = 0$ is a prerequisite for a (0/0) singularity to occur. Finding the transformations for matrices with dimensions greater than 3x3 would show the same behavior. $\beta_0 = 0$ is always a indicator that a mathematical singularity *may* occur. In none of these cases are the higher dimensional Euler parameters themselves going singular. It is always a mathematical singularity of the transformation itself. To circumvent this problem for particular applications, the limit of the fraction can be found as $\beta_0 \to 0$. After substituting $\beta_0 = 0$ into equation (58), for example, most fractions become trivial and the matrix is reduced to:

$$C_{4x4} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -I \tag{57}$$

Substituting $\beta_0 = 0$ into equation (54) yields the same result. Actually, as long as C is of *even* dimension the matrix will be $-I$ if $\beta_0 = 0$. If the dimension is *odd*, as it is for the 3x3 case, the C matrix will be fully populated. With this observation it is easy to circumvent the singular situations if the dimension is even. If the dimension is odd a numerical limit must be found. In either case the transformation will be well behaved everywhere except the $\beta_0 = 0$ surface.

Let us examine the uniqueness of the transformation given in equation (53). Assuming that the transformation is not unique, two possible higher dimensional Euler parameter sets $\acute{\beta}$ and $\H{\beta}$ are chosen.

$$C = \left(\acute{\beta}_0 I - \acute{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)^{-1}$$
$$C = \left(\H{\beta}_0 I + \H{B}\right)^{-1}\left(\H{\beta}_0 I - \H{B}\right)$$

Subtracting one equation from the other the following is obtained:

$$0 = \left(\acute{\beta}_0 I - \acute{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)^{-1} - \left(\H{\beta}_0 I + \H{B}\right)^{-1}\left(\H{\beta}_0 I - \H{B}\right)$$

$$0 = \left(\H{\beta}_0 I + \H{B}\right)\left(\acute{\beta}_0 I - \acute{B}\right) - \left(\H{\beta}_0 I - \H{B}\right)\left(\acute{\beta}_0 I + \acute{B}\right)$$

$$0 = \acute{\beta}_0 \H{B} - \H{\beta}_0 \acute{B}$$

$$\frac{\acute{B}}{\acute{\beta}_0} = \frac{\H{B}}{\H{\beta}_0} \tag{58}$$

Equation (58) is the necessary condition for two higher order Euler parameter sets to yield the same direction cosine matrix C. Obviously, for $\beta_0 \neq 0$ this can only occur when

$$\begin{aligned} \H{B} &= k \cdot \acute{B} \\ \H{\beta}_0 &= k \cdot \acute{\beta}_0 \end{aligned} \tag{59}$$

where k is a scalar. This condition would yield an infinite number of solutions. Since the higher dimensional Euler parameters must satisfy the holonomic constraint given in equation (52), only unit scaling values of k are permissible. Therefore k must be either ±1. This uniqueness study results in the same duality as is observed with the classical Euler parameters. There are always *two* possible sets of classical Euler parameters which describe an orthogonal 3x3 matrix C. This result is now extended to the more general case of NxN orthogonal matrices . This duality was seen earlier when applying the holonomic constraint to the kernel of A.

$$C_{NxN}[\beta(t)] = C_{NxN}[-\beta(t)] \tag{60}$$

If $\beta_0 = 0$ nothing can be said about the transformation uniqueness. As was seen with the 4x4 C matrix, any point on the unit sphere $\sum_{i=1}^{6} \beta_i^2 = 1$ is possible.

After establishing the forward and backward transformations between the NxN orthogonal matrices and the higher order Euler parameters, their kinematic equations need to be studied. To describe the orthogonal matrix C as a rigid body rotation, $\dot{C}$ needs to be of the form given in equation (1). After substituting equation (48) into equation (33) $\dot{Q}$ is:

$$\dot{Q} = \frac{1}{2}\left(I + \frac{B}{\beta_0}\right)[\tilde{\omega}]\left(I - \frac{B}{\beta_0}\right) \tag{61}$$

After differentiating equation (48) directly $\dot{Q}$ is:

$$\dot{Q} = \frac{\beta_0 \dot{B} - \dot{\beta}_0 B}{\beta_0^2} \tag{62}$$

Upon substituting equation (61) into equation (62) and after making some simplifications, the following kinematic relationship is found.

$$\beta_0 \dot{B} - \dot{\beta}_0 B = \frac{1}{2}(\beta_0 I + B)[\tilde{\omega}](\beta_0 I - B) \tag{63}$$

This equation can be solved for the skew-symmetric angular velocity matrix $[\tilde{\omega}]$.

$$[\tilde{\omega}] = 2(\beta_0 I + B)^{-1}\left(\beta_0 \dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1} \tag{64}$$

Note that this equation contains the same *mathematical* singularity at $\beta_0 = 0$ as did equation (53). Carrying out the algebra a closed form algebraic equation is found for the higher order angular velocities.

Let us verify that equation (64) for the angular velocities does indeed return a skew-symmetric matrix. This is accomplished with the definition of a skew-symmetric matrix.

$$[\tilde{\omega}] = -[\tilde{\omega}]^T = -2\left((\beta_0 I + B)^{-1}\left(\beta_0\dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1}\right)^T$$

$$[\tilde{\omega}] = -2(\beta_0 I - B)^{-1\,T}\left(\beta_0\dot{B} - \dot{\beta}_0 B\right)^T(\beta_0 I + B)^{-1\,T}$$

$$[\tilde{\omega}] = -2\left(\beta_0 I^T - B^T\right)^{-1}\left(\beta_0\dot{B}^T - \dot{\beta}_0 B^T\right)\left(\beta_0 I^T + B^T\right)^{-1}$$

Since the matrix B and its derivative are skew-symmetric matrices by definition, further simplifications are possible.

$$[\tilde{\omega}] = -2(\beta_0 I + B)^{-1}\left(-\beta_0\dot{B} + \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1}$$

$$[\tilde{\omega}] = 2(\beta_0 I + B)^{-1}\left(\beta_0\dot{B} - \dot{\beta}_0 B\right)(\beta_0 I - B)^{-1} \quad q.e.d.$$

All higher order Euler parameter differentials must abide by the derivative of the constraint equation (52).

$$2\dot{\beta}_0\beta_0 + 2\dot{\beta}_1\beta_1 + \ldots + 2\dot{\beta}_N\beta_N = 0 \tag{65}$$

After using the B from equation (49) the linear differential kinematic equations of the classical Euler parameters are found. For the 2x2 case, the differential kinematic equation is:

$$\omega_1 = 2[-\beta_1 \;\; \beta_0]\begin{bmatrix}\dot{\beta}_0\\ \dot{\beta}_1\end{bmatrix} \tag{66}$$

Adding the constraint in equation (65), equation (66) can be padded to make it full rank.

$$\begin{bmatrix}0\\ \omega_1\end{bmatrix} = 2\begin{bmatrix}\beta_0 & \beta_1\\ -\beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}\dot{\beta}_0\\ \dot{\beta}_1\end{bmatrix} \tag{67}$$

Note that as with the 3x3 case, the matrix transforming $\dot{\beta}$ to ω is orthogonal for the 2x2 case. Therefore the inverse transformation can be written as:

$$\begin{bmatrix}\dot{\beta}_0\\ \dot{\beta}_1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\beta_0 & -\beta_1\\ \beta_1 & \beta_0\end{bmatrix}\begin{bmatrix}0\\ \omega_1\end{bmatrix} \tag{68}$$

As with the direction cosine matrix, for matrices greater than 3x3 the differential kinematic equations contain polynomial fractions. Using the B matrix from equation (55) in equation (64) results in the differential kinematic equations for a 4x4 system.

$$\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{Bmatrix} = \frac{2}{\Delta} \begin{bmatrix} \Delta\beta_0 & \Delta\beta_1 & \Delta\beta_2 & \Delta\beta_3 & \Delta\beta_4 & \Delta\beta_5 & \Delta\beta_6 \\ \beta_6(\beta_2\beta_5-\beta_3\beta_4)-\beta_1(\beta_0^2+\beta_6^2) & \beta_0(\beta_0^2+\beta_6^2) & \beta_0(\beta_0\beta_3-\beta_5\beta_6) & \beta_0(\beta_4\beta_6-\beta_0\beta_2) & \beta_0(\beta_0\beta_5+\beta_3\beta_6) & -\beta_0(\beta_0\beta_4+\beta_2\beta_6) & \beta_0(\beta_2\beta_5-\beta_3\beta_4) \\ \beta_5(\beta_1\beta_6+\beta_3\beta_4)-\beta_2(\beta_0^2+\beta_5^2) & -\beta_0(\beta_0\beta_3+\beta_5\beta_6) & \beta_0(\beta_0^2+\beta_5^2) & \beta_0(\beta_0\beta_1-\beta_4\beta_5) & \beta_0(\beta_0\beta_6-\beta_5\beta_3) & \beta_0(\beta_3\beta_4+\beta_1\beta_6) & -\beta_0(\beta_0\beta_4+\beta_1\beta_5) \\ \beta_4(\beta_2\beta_5-\beta_1\beta_6)-\beta_3(\beta_0^2+\beta_4^2) & \beta_0(\beta_4\beta_6+\beta_0\beta_2) & -\beta_0(\beta_0\beta_1+\beta_4\beta_5) & \beta_0(\beta_0^2+\beta_4^2) & \beta_0(\beta_2\beta_5-\beta_1\beta_6) & \beta_0(\beta_0\beta_6-\beta_2\beta_4) & \beta_0(\beta_1\beta_4-\beta_0\beta_5) \\ \beta_3(\beta_2\beta_5-\beta_1\beta_6)-\beta_4(\beta_0^2+\beta_3^2) & \beta_0(-\beta_0\beta_5+\beta_3\beta_6) & -\beta_0(\beta_0\beta_6+\beta_5\beta_3) & \beta_0(\beta_2\beta_5-\beta_1\beta_6) & \beta_0(\beta_0^2+\beta_3^2) & \beta_0(\beta_0\beta_1-\beta_2\beta_3) & \beta_0(\beta_0\beta_2+\beta_1\beta_3) \\ \beta_2(\beta_1\beta_6+\beta_3\beta_4)-\beta_5(\beta_0^2+\beta_2^2) & \beta_0(\beta_0\beta_4-\beta_2\beta_6) & \beta_0(\beta_3\beta_4+\beta_1\beta_6) & -\beta_0(\beta_0\beta_6+\beta_2\beta_4) & -\beta_0(\beta_0\beta_1+\beta_2\beta_3) & \beta_0(\beta_0^2+\beta_2^2) & \beta_0(\beta_0\beta_3-\beta_1\beta_2) \\ \beta_1(\beta_2\beta_5-\beta_3\beta_4)-\beta_6(\beta_0^2+\beta_1^2) & \beta_0(\beta_2\beta_5-\beta_3\beta_4) & \beta_0(\beta_0\beta_4-\beta_1\beta_5) & \beta_0(\beta_1\beta_4+\beta_0\beta_5) & \beta_0(-\beta_0\beta_2+\beta_1\beta_3) & -\beta_0(\beta_0\beta_3+\beta_1\beta_2) & \beta_0(\beta_0^2+\beta_1^2) \end{bmatrix} \begin{Bmatrix} \dot\beta_0 \\ \dot\beta_1 \\ \dot\beta_2 \\ \dot\beta_3 \\ \dot\beta_4 \\ \dot\beta_5 \\ \dot\beta_6 \end{Bmatrix} \tag{69}$$

$$\textit{with } \Delta = \beta_0^2 + (\beta_3\beta_4 - \beta_2\beta_5 + \beta_1\beta_6)^2$$

Note that this transformation matrix is no longer orthogonal as was the case for 2x2 and 3x3 systems. Equation (69) has the same denominator as the 4x4 direction cosine matrix did. Hence it contains the identical singular situations. However, if $\beta_0 = 0$ the above transformation matrix is singular and cannot be inverted!

The higher dimensional Euler parameters lose some key properties as they get expanded to higher dimensions. They retain the properties of being bounded and universally nonsingular, but the transformations loose a lot of the elegance of their classical counterparts. Particularly $\beta_0 = 0$ poses several unresolved issues.

Conclusion

The parameterizations presented show great promise as an elegant means for describing the evolution of NxN orthogonal matrices. The modified Rodrigues parameters are only slightly more complicated than their classical counterparts, but provide a nonsingular, near-linear domain which is greatly increased by a factor of 2^N! The higher dimensional Euler parameters retain some of the desirable features of their classical counterparts such as being bounded by ±1 and being universally nonsingular. For orthogonal matrices greater than 3x3 though, their direction cosine matrix and differential kinematic equations contain some mathematical singularities which require taking the limits of polynomial fractions. The computational effort for calculating the higher dimensional Euler parameters grows rapidly when increasing the dimension of the *C* matrix. For higher dimensional rotations, the modified Rodrigues parameters show the greatest promise.

Their gain (increased nonsingular domain) outgrows the extra computation (versus the classical Cayley transformation) with increasing dimension N.

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$$
\begin{Bmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{Bmatrix} = \frac{2}{\Delta}
\begin{bmatrix}
\Delta\beta_0 & \Delta\beta_1 & \Delta\beta_2 & \cdots \\
\beta_6(\beta_2\beta_5 - \beta_3\beta_4) - \beta_1(\beta_0^2 + \beta_6^2) & \beta_0(\beta_0^2 + \beta_6^2) & \beta_0(\beta_0\beta_3 - \beta_5\beta_6) & \cdots \\
\beta_5(\beta_1\beta_6 + \beta_3\beta_4) - \beta_2(\beta_0^2 + \beta_5^2) & -\beta_0(\beta_0\beta_3 + \beta_5\beta_6) & \beta_0(\beta_0^2 + \beta_5^2) & \cdots \\
\beta_4(\beta_2\beta_5 - \beta_1\beta_6) - \beta_3(\beta_0^2 + \beta_4^2) & \beta_0(\beta_4\beta_6 + \beta_0\beta_2) & -\beta_0(\beta_0\beta_1 + \beta_4\beta_5) & \cdots \\
\beta_3(\beta_2\beta_5 - \beta_1\beta_6) - \beta_4(\beta_0^2 + \beta_3^2) & \beta_0(-\beta_0\beta_5 + \beta_3\beta_6) & -\beta_0(\beta_0\beta_6 + \beta_5\beta_3) & \cdots \\
\beta_2(\beta_1\beta_6 + \beta_3\beta_4) - \beta_5(\beta_0^2 + \beta_2^2) & \beta_0(\beta_0\beta_4 - \beta_2\beta_6) & \beta_0(\beta_3\beta_4 + \beta_1\beta_6) & \cdots \\
\beta_1(\beta_2\beta_5 - \beta_3\beta_4) - \beta_6(\beta_0^2 + \beta_1^2) & \beta_0(\beta_2\beta_5 - \beta_3\beta_4) & \beta_0(\beta_0\beta_4 - \beta_1\beta_5) & \cdots
\end{bmatrix}
$$

$$
\begin{bmatrix}
\cdots & \Delta\beta_3 & \Delta\beta_4 & \Delta\beta_5 & \Delta\beta_6 \\
\cdots & \beta_0(\beta_4\beta_6 - \beta_0\beta_2) & \beta_0(\beta_0\beta_5 + \beta_3\beta_6) & -\beta_0(\beta_0\beta_4 + \beta_2\beta_6) & \beta_0(\beta_2\beta_5 - \beta_3\beta_4) \\
\cdots & \beta_0(\beta_0\beta_1 - \beta_4\beta_5) & \beta_0(\beta_0\beta_6 - \beta_5\beta_3) & \beta_0(\beta_3\beta_4 + \beta_1\beta_6) & -\beta_0(\beta_0\beta_4 + \beta_1\beta_5) \\
\cdots & \beta_0(\beta_0^2 + \beta_4^2) & \beta_0(\beta_2\beta_5 - \beta_1\beta_6) & \beta_0(\beta_0\beta_6 - \beta_2\beta_4) & \beta_0(\beta_1\beta_4 - \beta_0\beta_5) \\
\cdots & \beta_0(\beta_2\beta_5 - \beta_1\beta_6) & \beta_0(\beta_0^2 + \beta_3^2) & \beta_0(\beta_0\beta_1 - \beta_2\beta_3) & \beta_0(\beta_0\beta_2 + \beta_1\beta_3) \\
\cdots & -\beta_0(\beta_0\beta_6 + \beta_2\beta_4) & -\beta_0(\beta_0\beta_1 + \beta_2\beta_3) & \beta_0(\beta_0^2 + \beta_2^2) & \beta_0(\beta_0\beta_3 - \beta_1\beta_2) \\
\cdots & \beta_0(\beta_1\beta_4 + \beta_0\beta_5) & \beta_0(-\beta_0\beta_2 + \beta_1\beta_3) & -\beta_0(\beta_0\beta_3 + \beta_1\beta_2) & \beta_0(\beta_0^2 + \beta_1^2)
\end{bmatrix}
\begin{Bmatrix} \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \dot{\beta}_3 \\ \dot{\beta}_4 \\ \dot{\beta}_5 \\ \dot{\beta}_6 \end{Bmatrix}
\tag{69}
$$

$$
\textit{with } \Delta = \beta_0^2 + (\beta_3\beta_4 - \beta_2\beta_5 + \beta_1\beta_6)^2
$$

Note that this transformation matrix is no longer orthogonal as was the case for 2x2 and 3x3 systems. Equation (69) has the same denominator as the 4x4 direction cosine matrix did. Hence it contains the identical singular situations. However, if $\beta_0 = 0$ the above transformation matrix is singular and cannot be inverted!

The higher dimensional Euler parameters lose some key properties as they get expanded to higher dimensions. They retain the properties of being bounded and universally nonsingular, but the transformations loose a lot of the elegance of their classical counterparts. Particularly $\beta_0 = 0$ poses several unresolved issues.

Conclusion

The parameterizations presented show great promise as an elegant means for describing the evolution of NxN orthogonal matrices. The modified Rodrigues parameters are only slightly more complicated than their classical counterparts, but provide a nonsingular, near-linear domain which is greatly increased by a factor of 2^N! The higher dimensional Euler parameters retain some of the desirable features of their classical counterparts such as being bounded by ±1 and being universally nonsingular. For orthogonal matrices greater than 3x3 though, their direction cosine matrix and differential kinematic equations contain some mathematical singularities which require taking the limits of polynomial fractions. The computational effort for calculating the higher dimensional Euler parameters grows rapidly when increasing the dimension of the C matrix. For higher dimensional rotations, the modified Rodrigues parameters show the greatest promise.

Their gain (increased nonsingular domain) outgrows the extra computation (versus the classical Cayley transformation) with increasing dimension N.

Acknowledgments

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