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RESEARCH ARTICLE

Lyapunov-Based Exact Stability Analysis and Synthesis for Linear Single-Parameter Dependent Systems

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We propose a class of polynomially parameter-dependent quadratic (PPDQ) Lyapunov functions for assessing the stability of single-parameter dependent linear, time-invariant, (s-PDLTI) systems in a non-conservative manner. It is shown that stability of s-PDLTI systems is equivalent to the existence of a PPDQ Lyapunov function. A bound on the degree of the polynomial dependence of the Lyapunov function in the system parameter is given explicitly. The resulting stability conditions are expressed in terms of a set of matrix inequalities whose feasibility over a compact and connected set can be cast as a convex, finite-dimensional optimization problem. Extensions of the approach to state-feedback controller synthesis are also provided.

Keywords: Parameter-dependent LTI systems; parameter-dependent Lyapunov functions; linear matrix inequalities.

1 Introduction

Consider the single-parameter dependent linear, time-invariant (s-PDLTI) system

$$\dot{x} = A_\rho x, \quad A_\rho := A(\rho), \quad \rho \in \Omega \subseteq \mathbb{R}, \quad x \in \mathbb{R}^n. \tag{1}$$

The stability of (1) for each parameter $\rho \in \Omega$ can be *characterized* via a parameter-dependent, quadratic Lyapunov function of the form $V(x) = x^\top P(\rho)x := x^\top P_\rho x$. Since the explicit dependence of the matrix P_ρ on the parameter ρ is not a priori evident, one typically starts by *postulating* a convenient functional dependence of P_ρ on ρ, before proceeding to the derivation of the resulting stability conditions. The simplest choice is a constant (that is, parameter-independent) Lyapunov matrix. Using $P_\rho = P$ for all $\rho \in \Omega$, one then obtains the following, infinite-dimensional family of matrix inequalities

$$P > 0, \quad A_\rho P + P A_\rho^\top < 0, \qquad \forall \rho \in \Omega \tag{2}$$

that ensure stability of the system (1).

The type of stability resulting from (2) is the so-called quadratic stability and it ensures more than just Hurwitz stability of the matrix A_ρ for each instance of $\rho \in \Omega$: it actually guards against

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arbitrarily fast time-variations of ρ. If it is known beforehand that ρ is constant, quadratic stability can thus be overly conservative. It is well known that in order to reduce conservatism, and derive conditions that are close to necessary, one needs to resort to the use of parameter-dependent (quadratic) Lyapunov functions.

In light of the previous discussion, given a parameter-dependent matrix P_ρ, robust stability of (1) for all $\rho \in \Omega$ can be established if the following two matrix inequalities are satisfied

$$P_\rho > 0, \qquad A_\rho P_\rho + P_\rho A_\rho^\mathsf{T} < 0, \qquad \rho \in \Omega. \tag{3}$$

In order to obtain exact (that is, *necessary* and sufficient) conditions for robust stability, the "correct" form of P_ρ must be used in (3). Till very recently, even for the simplest case of affine dependence of the system matrix on the parameter ρ, the parameter dependence of the Lyapunov matrix leading to exact stability conditions for (1) has eluded researchers in the field. Notable exceptions (and valid only for the special case when $A_\rho = A_0 + \rho A_1$ with $\operatorname{rank} A_1 = 1$) are (Narendra and Taylor 1973) and (Dasgupta et al. 1994).

Recently, Bliman (Bliman 2004) showed that linear matrix inequalities (LMI's) of the form (3) have solutions that depend polynomially on the parameter ρ, provided they are feasible for each parameter value. The proof of this result hinges on Michael's Selection Theorem (Michael 1956) in order to construct a continuous solution from the pointwise feasible solutions of (3). Polynomial dependence follows from the uniform approximation of continuous functions with polynomials over compact sets via Weirstrass's approximation theorem. Bliman later (Bliman 2004) used this result to show that pointwise Hurwitz stability of the affine-in-ρ system[1]

$$\dot{x} = A_\rho x, \qquad A_\rho := A_0 + \rho A_1, \qquad \forall\, \rho \in \Omega \tag{4}$$

is *equivalent* to the existence of a *polynomially* parameter-dependent quadratic (PPDQ) Lyapunov function of the form $V(x) = x^\mathsf{T} P_\rho x$, where

$$P_\rho = P_0 + \rho P_1 + \rho^2 P_2 + \ldots + \rho^N P_N = \sum_{i=0}^{N} \rho^i P_i \tag{5}$$

that satisfies inequalities (3). Moreover, as it was shown in (Bliman 2004), checking the infinite-dimensional family of LMI's in (3) over a compact set can be cast as a finite-dimensional convex feasibility problem in terms of LMI's. This equivalence of the existence of a PPDQ Lyapunov matrix and the pointwise Hurwitz stability of (4) can be made precise: it means that the Lyapunov matrix depends on the parameter ρ via (5) in such a way that, for those values of the parameter for which the matrix A_ρ is Hurwitz the stability conditions (3) are satisfied, while for the values of the parameter ρ for which the matrix A_ρ is not Hurwitz, the stability conditions (3) fail. This result is in sharp contrast with all prior results, which use parameter-dependent Lyapunov functions to show stability of PDLTI systems (Helmersson 1999, Bernstein and Haddad 1990, Haddad and Bernstein 1995, Apkarian 1998, Ramos and Peres 2002, Henrion et al. 2003, Leite and Peres 2003, Ben-Tal and Nemirovski 2002, Neto 1999), as these only deal with the sufficiency (that is, the "easy") part of the robust stability problem.

The main difficulty with the results in (Bliman 2004) is that the degree N of the polynomial dependence of the matrix P_ρ on ρ in (5) is not known a priori. What is it shown in (Bliman 2004) instead, is that there exists a *sufficiently large* integer N such that (5) will solve (3) if the matrix A_ρ is Hurwitz for each instance of $\rho \in \Omega$. In this paper we remove this restriction. We show that

[1] The results in (Bliman 2004) are, in fact, more general than (4), as they are valid for a matrix A_ρ having multi-affine dependence on a parameter *vector*.

the bound of the polynomial dependence of the Lyapunov matrix can be explicitly computed. We later extend this result to deal with the case of a matrix A_ρ which depends polynomially on ρ, as follows

$$\dot{x} = A_\rho x, \qquad A_\rho := \sum_{i=0}^{\nu_a} \rho^i A_i, \qquad \forall\, \rho \in \Omega. \tag{6}$$

Finally, we illustrate the benefits of the proposed approach for synthesizing parameter-dependent controllers for s-PDLTI systems where both the state and the input matrices depend polynomially on the parameter. Extension of these results to the multi-parameter case is hindered by the computational complexity of the problem. An initial attempt towards this direction is reported in (Zhang et al. 2005).

This paper offers several extensions and improvements over the results in (Bliman 2004) or other similar results in the literature (Apkarian 1998, Leite and Peres 2003, Ramos and Peres 2002, Saydy et al. 1988, Henrion et al. 2004). First, we provide an alternative–much more direct–derivation of the results in (Bliman 2004), at least for the single-parameter case. Our method is quite straightforward and it does not require advanced concepts beyond those of an introductory course in linear algebra. Second, and as an added benefit of this approach, we provide an explicit upper bound on the degree of the polynomial dependence of the quadratic Lyapunov function to ensure the stability of (4). This bound depends on the rank of the matrix A_1. The computational complexity of the conditions is reduced as the rank of A_1 decreases. Finally, we show that checking the conditions for stability over a compact set can be cast as a pair of LMI's *without conservatism*. These results are extended in order to derive necessary and sufficient Lyapunov-based conditions for the stability of polynomial-in-ρ systems as in (6).

We note here that although the stability of s-PDLTI systems can also be checked using the guardian map techniques of (Saydy et al. 1990, Fu and Barmish 1988) (for similar results see also (Zhang et al. 2002)) nonetheless, the Lyapunov-based stability conditions of this paper are also amenable to synthesis (see Section 6). Such an extension to the synthesis problem is not directly evident from the use of guardian maps. Therefore, another major contribution of the paper is an approach for synthesizing parameter-dependent state-feedback controllers for polynomial s-PDLTI systems.

Crucial to our developments is a technical result of independent interest (Proposition 3.2) which is used to test positive-definiteness of a polynomial matrix over a *finite* interval. Although convex characterizations for positive definiteness of polynomial matrices over infinite or semi-infinite intervals are available (Genin et al. 2003, Henrion et al. 2003), the related problem over finite intervals is quite more involved. Only recently such conditions (and for the scalar case only) have been proposed (Nesterov 2000). It is noted that for the multi-variable case, the problem of characterizing the positivity of polynomials is related to the *Positivstellensatz* and is, in general, an NP-hard problem (Parillo 2003).

In order to keep the derivations as terse as possible and avoid cluttering the paper with unnecessary notation we focus on the affine-in-ρ system (4). The results for the polynomial-in-ρ case (6) follow easily from those of (4). The paper is therefore organized as follows. In the first part of the paper we provide the main theorem (Theorem 2.1) that states that Hurwitz stability of the matrix $A_\rho = A_0 + \rho A_1$ for each value of the parameter $\rho \in\ \Omega$ is equivalent to the existence of a polynomially parameter-dependent quadratic Lyapunov function of a known degree. In the second part of the paper we introduce a new lemma for checking the positive definiteness of a polynomial matrix over a finite interval and we then use this lemma to derive convex feasibility conditions for the pointwise Hurwitz stability of the matrix $A_\rho = A_0 + \rho A_1$ over a compact interval. A comparison between the proposed approach and the one given in (Bliman 2004) in terms of the computational complexity of the two is given next. In the third part of the paper we extend these results in order to provide necessary and sufficient conditions for the polynomial s-PDLTI system (6). The paper is concluded with an application of the proposed approach to controller synthesis for polynomial s-PDLTI systems.

For clarity, only the major results are given in the main body of the paper. For the interested reader, most secondary proofs and technical lemmas are given in the Appendix.

2 Main Result

In this section we consider LTI systems which depend affinely on a single real parameter, of the form

$$\dot{x} = A_\rho x, \qquad A_\rho := A_0 + \rho A_1, \qquad \rho \in \Omega \tag{7}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\Omega \subseteq \mathbb{R}$. Our main objective is to find computable, non-conservative conditions for checking the asymptotic stability of system (7). The parameter ρ is assumed to be constant[2] and it is chosen from the set Ω. At this point we make no a priori assumptions on the set Ω (that is, connected, bounded, compact, etc.).

Definition 2.1 ((Mustafa 1995)) Given a symmetric matrix $P = P^\mathsf{T} \in \mathbb{R}^{n\times n}$, define the vector

$$\overline{\mathrm{vec}}(P) := \big(P_{11}, \ldots, P_{n1}, P_{22}, \ldots, P_{n2}, \ldots, P_{nn}\big)^\mathsf{T} \in \mathbb{R}^{\frac{1}{2}n(n+1)}.$$

Note that the usual vec$(\cdot)$ operator (Brewer 1978) that stacks the columns of a matrix P on top of each other consists of all the elements of $\overline{\mathrm{vec}}(P)$ including some repetitions. For every symmetric matrix $P = P^\mathsf{T} \in \mathbb{R}^{n\times n}$, there exists a unique matrix D_n of dimension $n^2 \times \frac{1}{2}n(n+1)$ called *the duplication matrix* (Magnus 1988, Mustafa 1995), which is independent of the matrix P and depends only on the dimension n of the matrix P, which satisfies

$$\mathrm{vec}(P) = D_n \overline{\mathrm{vec}}(P). \tag{8}$$

The Moore-Penrose inverse of D_n satisfies the following properties (Magnus 1988, Mustafa 1995)

$$\overline{\mathrm{vec}}(P) = D_n^+ \mathrm{vec}(P), \quad D_n^+ D_n = I_{\frac{1}{2}n(n+1)}, \quad \mathrm{rank}(D_n) = \mathrm{rank}(D_n^+) = \tfrac{1}{2}n(n+1).$$

Notice, in particular, that D_n is always full column rank. Consequently, $D_n^+ = (D_n^\mathsf{T} D_n)^{-1} D_n^\mathsf{T}$. Thus, the mapping vec$(\cdot)$ establishes a one-to-one correspondence between the symmetric matrices in $\mathbb{R}^{n\times n}$ and vectors in $\mathbb{R}^{\frac{1}{2}n(n+1)}$.

Definition 2.2 ((Mustafa 1995)) Given $A \in \mathbb{R}^{n\times n}$, let $\widehat{A} \in \mathbb{R}^{\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)}$ be defined by

$$\widehat{A} := D_n^+ (A \oplus A) D_n, \tag{9}$$

where $A \oplus A = I_n \otimes A + A \otimes I_n$ is the Kronecker sum of matrix A with itself.

[2]The results also hold when ρ varies very slowly so that a "quasi-static" point of view is valid; see, for example, (Brockett 1970).

It is clear from Definition 2.2 that $\widehat{A}_\rho := \widehat{A_0 + \rho A_1} = \widehat{A}_0 + \rho \widehat{A}_1$.

The first main result of the paper can now be stated as follows:

Theorem 2.1 : *Given the matrices A_0, $A_1 \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank} A_1 = r$, *let $A_\rho := A_0 + \rho A_1$ and let*

$$m_s := \begin{cases} \frac{1}{2}(2nr - r^2 + r), \text{ if } & r < n, \\ \frac{1}{2}n(n+1) - 1, \text{ if } & r = n. \end{cases} \tag{10}$$

Then the following two statements are equivalent:

(i) $A_\rho = A_0 + \rho A_1$ is Hurwitz for all $\rho \in \Omega$.
(ii) There exists a set of $m_s + 1$ real symmetric matrices $\{P_i\}_{0\le i\le m_s}$, such that

$$A_\rho P_\rho + P_\rho A_\rho^\mathsf{T} < 0, \quad \forall \rho \in \Omega \tag{11}$$

$$P_\rho = \sigma_\rho \left(\sum_{i=0}^{m_s} \rho^i P_i \right) > 0, \quad \forall \rho \in \Omega \tag{12}$$

where $\sigma_\rho := -\operatorname{sign}(\det \widehat{A}_\rho)$ with $\det \widehat{A}_\rho \neq 0$ for all $\rho \in \Omega$.

Proof See the Appendix. □

Theorem 2.1 states that checking the stability of (7) is equivalent to the existence of a Lyapunov matrix P_ρ satisfying the two matrix inequalities (11) and (12) (equivalently, (3)). That is, if there exists a positive definite matrix P_ρ for all $\rho \in \Omega$ such that (11) holds, the matrix A_ρ is Hurwitz for each $\rho \in \Omega$. By the same token (and most importantly) if for some $\rho' \in \Omega$ the matrix $A_{\rho'}$ is not Hurwitz, then there exists no positive-definite matrix that satisfies inequality (11).

Remark 2.2 **:** Note that if the domain Ω is connected then σ_ρ is constant via Corollary A.4 and the Lyapunov matrix in (12) reduces to

$$P_\rho = \sum_{i=0}^{m_s} \rho^i P_i, \qquad \forall\, \rho \in \Omega. \tag{13}$$

Example 2.3 Consider the parameter-dependent matrix $A_\rho = A_0 + \rho A_1$, where

$$A_0 = \begin{bmatrix} 0.7493 & -2.4358 & -1.6503 \\ -2.0590 & -3.3003 & -1.4833 \\ -1.5019 & 1.2149 & -4.8737 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1.2149 & 1.6640 & -2.2091 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.1990 & 0.6493 & -0.2214 \end{bmatrix} \tag{14}$$

The exact stability domain for A_ρ is $(-18.3861, -1.2729) \cup (2.1538, 3.7973)$, which is computed with the method presented in (Zhang et al. 2002). In this example the stability domain is composed of two disjoint intervals. Using Theorem 2.1, we seek a polynomial Lyapunov matrix of degree $m_s = \frac{1}{2}n(n+1) - 1 = 5$ since $r = \operatorname{rank}(A_1) = n = 3$. The parameter-dependent Lyapunov function satisfying (A10) is given by

$$P_\rho = \sigma_\rho \Big(P_0 + \rho P_1 + \rho^2 P_2 + \rho^3 P_3 + \rho^4 P_4 + \rho^5 P_5 \Big)$$

where,

$$P_0 = \begin{bmatrix} -7.4544 & 1.3754 & 2.8700 \\ 1.3754 & 3.6225 & -0.7518 \\ 2.8700 & -0.7518 & 1.7334 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 4.1133 & 2.8221 & -7.6811 \\ 2.8221 & -1.0147 & 1.4137 \\ -7.6811 & 1.4137 & 4.7021 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} -5.4508 & 4.2469 & 5.3219 \\ 4.2469 & -2.3203 & -1.3195 \\ 5.3219 & -1.3195 & -6.1676 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1.4747 & -2.7044 & -2.1666 \\ -2.7044 & 1.7968 & 1.9922 \\ -2.1666 & 1.9922 & 3.6788 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} -0.7450 & 0.3384 & -0.0541 \\ 0.3384 & -0.6999 & -0.7148 \\ -0.0541 & -0.7148 & -1.2017 \end{bmatrix}, \quad P_5 = \begin{bmatrix} -0.03532 & -0.01186 & -0.02280 \\ -0.01186 & 0.10092 & 0.05606 \\ -0.02280 & 0.05606 & -0.00667 \end{bmatrix}$$

These values of $P_0, P_1, \ldots, P_5$ were computed from $P_i = \overline{\text{vec}}^{-1}\Big(N_i \overline{\text{vec}}(I_n)\Big)$, $i = 0, 1, \ldots, m_s$ were $\text{Adj}(\widehat{A}_0 + \rho\widehat{A}_1) = \sum_{i=0}^{m_s} \rho^i N_i$, as in the proof of Theorem 2.1. It can be checked numerically that $A_\rho P_\rho + P_\rho A_\rho^\intercal < 0$ for all $\rho \in \mathbb{R}$. However, P_ρ is positive definite only for $\rho \in (-18.3861, -1.2729) \cup (2.1538, 3.7973)$, as required by Theorem 2.1.

Remark 2.4**:** Compared to other existing methods (Barmish 1994, Mustafa and Davidson 1995, Mustafa 1995, Fu and Barmish 1988, Saydy et al. 1988, 1990), which can be only used to check the stability over a connected domain which includes the origin, Theorem 2.1 can be used to determine the whole stability domain of a PDLTI system, even if this domain is composed of several disjoint intervals of $\mathbb{R}$, as in the case of Example 2.3. Moreover, the method of (Barmish 1994, Fu and Barmish 1988, Saydy et al. 1988, 1990) using guardian maps, although it provides necessary and sufficient conditions for stability, it does not seem easily extendable to controller synthesis. The Lyapunov-base result of Theorem 2.1 on the other hand can be used to design parameter-varying controllers for s-PDLTI plants; see Section 6.

3 A Convex Characterization of the Stability Conditions

The analysis of the previous section shows that when the domain Ω is connected, the parameter-dependent matrix $A_\rho = A_0 + \rho A_1$ is Hurwitz for any $\rho \in \Omega$, if and only if there exists a Lyapunov matrix which depends polynomially on the parameter ρ, that is,

$$P_\rho = \sum_{i=0}^{m_s} \rho^i P_i, \tag{15}$$

such that the corresponding two matrix inequalities

$$P_\rho > 0, \qquad A_\rho P_\rho + P_\rho A_\rho^\intercal < 0, \tag{16}$$

are satisfied for each $\rho \in \Omega$. In order to be able to use the stability criterion of Theorem 2.1 in practice, we need a relatively simple method to determine the feasibility of the (infinite) matrix inequalities (16). In the sequel we provide computable, *non-conservative* conditions to test the matrix inequalities (16) over any compact and connected interval Ω. Without loss of generality we take $\Omega = [-1,\, 1]$.

To this end, let the vector $\rho^{[q]} \in \mathbb{R}^q$ be defined by

$$\rho^{[q]} := \begin{pmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^{q-1} \end{pmatrix}, \tag{17}$$

and notice that the parameter-dependent matrix in (15) can be rewritten as

$$P_\rho = \left(\rho^{[k]} \otimes I_n\right)^\mathsf{T} P_\Sigma \left(\rho^{[k]} \otimes I_n\right), \tag{18}$$

where $k := \lceil \frac{m_s}{2} \rceil + 1$ and $P_\Sigma = P_\Sigma^\mathsf{T} \in \mathbb{R}^{nk \times nk}$ is a constant, symmetric matrix (here $\lceil \frac{m_s}{2} \rceil$ denotes the smallest integer which is larger than or equal to $\frac{m_s}{2}$). Note that the matrix P_Σ is not unique. One possible choice is given by

$$P_\Sigma := \frac{1}{2} \begin{bmatrix} 2P_0 & P_1 & & & \\ P_1 & 2P_2 & P_3 & & \\ & P_3 & 2P_4 & \ddots & \\ & & \ddots & \ddots & P_{m-1} \\ & & & P_{m-1} & 2P_m \end{bmatrix}, \qquad \text{if } m_s \text{ is even}, \tag{19}$$

and

$$P_\Sigma := \frac{1}{2} \begin{bmatrix} 2P_0 & P_1 & & & & \\ P_1 & 2P_2 & P_3 & & & \\ & P_3 & 2P_4 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & P_{m-2} & 2P_{m-1} & P_m \\ & & & & P_m & 0 \end{bmatrix}, \qquad \text{if } m_s \text{ is odd}. \tag{20}$$

On the other hand, for any given symmetric matrix P_Σ one can construct a unique polynomial matrix P_ρ of the form (15) using the expression (18). A similar result can be shown for the polynomial matrix in (16). This is shown in the following lemma.

The following notation is adapted from (Bliman 2004)

$$\hat{J}_k := \left[I_k \; 0_{k\times 1}\right], \qquad \check{J}_k := \left[0_{k\times 1} \; I_k\right]. \tag{21}$$

Lemma 3.1 (Bliman 2004) *Given a matrix $A_\rho = A_0 + \rho A_1 \in \mathbb{R}^{n\times n}$ and a symmetric, parameter-dependent matrix $\mathcal{P}_\rho \in \mathbb{R}^{n\times n}$ as*

$$\mathcal{P}_\rho = \left(\rho^{[k]} \otimes I_n\right)^\mathsf{T} P \left(\rho^{[k]} \otimes I_n\right),$$

let $\mathcal{R}_\rho := A_\rho \mathcal{P}_\rho + \mathcal{P}_\rho A_\rho^\mathsf{T}$. Then $\mathcal{R}_\rho$ can be written as

$$\mathcal{R}_\rho = \left(\rho^{[k+1]} \otimes I_n\right)^\mathsf{T} R \left(\rho^{[k+1]} \otimes I_n\right) \tag{22}$$

where,

$$R := H^{\mathsf{T}} P F + F^{\mathsf{T}} P H \tag{23}$$

$$H := \hat{J}_k \otimes I_n \tag{24}$$

$$F := \hat{J}_k \otimes A_0^{\mathsf{T}} + \check{J}_k \otimes A_1^{\mathsf{T}} \tag{25}$$

where $\hat{J}_k$ *and* $\check{J}_k$ *as in (21).*

Notice that the matrix R depends linearly upon each of the matrices P, A_0 and A_1.

3.1 *Exact LMI Conditions for Checking the Stability of PDLTI Systems*

Using (18) the first inequality in (16) can be rewritten as

$$\left(\rho^{[k]} \otimes I_n\right)^{\mathsf{T}} P_\Sigma \left(\rho^{[k]} \otimes I_n\right) > 0, \quad \forall \rho \in \Omega. \tag{26}$$

where P_Σ as in (19) or (20). Moreover, using Lemma 3.1 the second inequality in (16) can be rewritten as

$$\left(\rho^{[k+1]} \otimes I_n\right)^{\mathsf{T}} R_\Sigma \left(\rho^{[k+1]} \otimes I_n\right) < 0, \quad \forall \rho \in \Omega, \tag{27}$$

where,

$$R_\Sigma := H^{\mathsf{T}} P_\Sigma F + F^{\mathsf{T}} P_\Sigma H \tag{28}$$

$$H := \hat{J}_k \otimes I_n \tag{29}$$

$$F := \hat{J}_k \otimes A_0^{\mathsf{T}} + \check{J}_k \otimes A_1^{\mathsf{T}} \tag{30}$$

Proposition 3.2: *Let* $\Theta \in \mathbb{R}^{nk \times nk}$*. Then the matrix inequality*

$$\left(\rho^{[k]} \otimes I_n\right)^{\mathsf{T}} \Theta \left(\rho^{[k]} \otimes I_n\right) < 0 \tag{31}$$

holds for all $\rho \in [-1,\, 1]$ *if and only if there exist matrices* $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$ *and* $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ *such that*

$$D = D^{\mathsf{T}} > 0, \quad G + G^{\mathsf{T}} = 0, \quad \Theta < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -D & G \\ G^{\mathsf{T}} & D \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}. \tag{32}$$

Proof The matrix inequality (31) is equivalent to the condition

$$x^{\mathsf{T}} \left(\rho^{[k]} \otimes I_n\right)^{\mathsf{T}} \Theta \left(\rho^{[k]} \otimes I_n\right) x < 0, \quad \forall \rho \in [-1,\, 1], \quad \forall x \in \mathbb{R}^n \tag{33}$$

Let $\zeta = \left(\rho^{[k]} \otimes I_n\right) x$ and notice that

$$\begin{aligned} (\check{J}_{k-1} \otimes I_n - \rho \hat{J}_{k-1} \otimes I_n)(\rho^{[k]} \otimes I_n) &= \big((\check{J}_{k-1} - \rho \hat{J}_{k-1}) \otimes I_n\big)(\rho^{[k]} \otimes I_n) \\ &= \big((\check{J}_{k-1} - \rho \hat{J}_{k-1})\rho^{[k]}\big) \otimes I_n = 0 \end{aligned}$$

Therefore, ζ satisfies the constraint $\big((\check{J}_{k-1} - \rho\hat{J}_{k-1}) \otimes I_n\big)\zeta = 0$ for all real ρ such that $|\rho| \leq 1$. Let $J = \check{J}_{k-1} \otimes I_n$ and $C = \hat{J}_{k-1} \otimes I_n$. From Lemma A.7 in the Appendix it follows that any ζ that satisfies the constraint $\big((\check{J}_{k-1} - \rho\hat{J}_{k-1}) \otimes I_n\big)\zeta = (J - \rho C)\zeta = 0$ is of the form $\zeta = (\rho^{[k]} \otimes I_n)x$ for some $x \in \mathbb{R}^n$ and $\rho \in [-1, +1]$. Condition (31) is therefore equivalent to the condition $\zeta^\mathsf{T}\Theta\zeta < 0$ with $\zeta = (\rho^{[k]} \otimes I_n)x$, $\forall x \in \mathbb{R}^n$. Applying now Lemma A.8, one has that $\zeta^\mathsf{T}\Theta\zeta < 0$ if and only if there exist matrices $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ such that (32) holds. □

Example 3.3 Let $P_\rho = (1+\epsilon)I_n - \rho^2 I_n$. It is clear that if $\epsilon > 0$, P_ρ is positive definite for all $\rho \in [-1,\, 1]$. If, on the other hand, $\epsilon < 0$, P_ρ is not positive definite for all $\rho \in [-1,\, 1]$. Rewriting P_ρ in the form (18), one obtains

$$\begin{aligned} P_\rho &= \left(\rho^{[2]} \otimes I_n\right)^\mathsf{T} P_\Sigma \left(\rho^{[2]} \otimes I_n\right) \\ &= \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} (1+\epsilon)I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix} \end{aligned} \tag{34}$$

and applying Proposition 3.2, with $k = 2$, the condition $P_\rho > 0$ for all $\rho \in [-1,\, 1]$ is equivalent to the existence of matrices $D = D^\mathsf{T} > 0$ and $G + G^\mathsf{T} = 0$ such that

$$-P_\Sigma < \begin{bmatrix} C \\ J \end{bmatrix}^\mathsf{T} \begin{bmatrix} -D & G \\ G^\mathsf{T} & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} \tag{35}$$

where $J = [0_{n\times n}\ I_n]$ and $C = [I_n\ 0_{n\times n}]$. Using (34) the matrix inequality (35) is equivalent to the existence of matrices $D = D^\mathsf{T} > 0$ and $G + G^\mathsf{T} = 0$ such that

$$\begin{bmatrix} D - (1+\epsilon)I_n & -G \\ -G^\mathsf{T} & I_n - D \end{bmatrix} < 0. \tag{36}$$

A necessary condition for the existence of D in (36) is $I_n < D < (1+\epsilon)I_n$. When $\epsilon > 0$, such a D always exists and choosing $G = 0$ the LMI (35) is feasible. When $\epsilon < 0$, no D can satisfy (36) and the LMI is infeasible. For both cases, the result of Proposition 3.2 agrees with the direct stability analysis.

The following is a direct consequence of Proposition 3.2. It provides convex conditions in terms of LMI's for checking the robust stability of the parameter dependent matrix $A_\rho = A_0 + \rho A_1$ for $\rho \in [-1, +1]$.

Theorem 3.4: *Let the parameter-dependent matrix $A_\rho = A_0 + \rho A_1$, where $A_0, A_1 \in \mathbb{R}^{n\times n}$ with* rank $A_1 = r$ *and let $k = \lceil \frac{m_s}{2} \rceil + 1$ where,*

$$m_s := \begin{cases} \frac{1}{2}(2nr - r^2 + r), & \text{if} \quad r < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if} \quad r = n. \end{cases} \tag{37}$$

Then, A_ρ is Hurwitz for all $|\rho| \leq 1$ if and only if there exist a structured symmetric matrices $P_\Sigma \in \mathbb{R}^{nk\times nk}$ as in (19) or (20), a symmetric matrix $D \in \mathbb{R}^{nk\times nk}$ and a skew-symmetric matrix $G \in \mathbb{R}^{nk\times nk}$, such that

$$P_0 > 0, \quad D = D^\mathsf{T} > 0, \quad G + G^\mathsf{T} = 0, \quad R_\Sigma < \begin{bmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} -D & G \\ G^\mathsf{T} & D \end{bmatrix} \begin{bmatrix} \hat{J}_k \otimes I_n \\ \check{J}_k \otimes I_n \end{bmatrix}, \tag{38}$$

where $R_\Sigma = R_\Sigma(P_\Sigma)$ as in (28)-(30).

Proof According to Theorem 2.1, A_ρ is Hurwitz for all $|\rho| \leq 1$ if and only if there exists a matrix P_ρ which depends polynomially on the parameter ρ of degree m_s, where m_s as in (37), such that the matrix inequalities (16) are satisfied. From Lemma 3.1 these inequalities can be written in the form (26) and (27), and Proposition 3.2 shows that the inequalities (26) and (27) are equivalent to the feasibility of the LMI conditions

$$\tilde{D} = \tilde{D}^\mathsf{T} > 0, \quad \tilde{G} + \tilde{G}^\mathsf{T} = 0, \quad -P_\Sigma < \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} -\tilde{D} & \tilde{G} \\ \tilde{G}^\mathsf{T} & \tilde{D} \end{bmatrix} \begin{bmatrix} \hat{J}_{k-1} \otimes I_n \\ \check{J}_{k-1} \otimes I_n \end{bmatrix}, \tag{39}$$

$$D = D^\mathsf{T} > 0, \quad G + G^\mathsf{T} = 0, \quad R_\Sigma < \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} -D & G \\ G^\mathsf{T} & D \end{bmatrix} \begin{bmatrix} \hat{J}_{k} \otimes I_n \\ \check{J}_{k} \otimes I_n \end{bmatrix}, \tag{40}$$

for some matrices $\tilde{D}, \tilde{G} \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $D, G \in \mathbb{R}^{nk\times nk}$.

Notice now that when A_ρ is nominally stable, that is, when the matrix A_0 is Hurwitz, inequality (39) is not necessary. This is due to the fact that A_0 Hurwitz along with inequality (40) guarantees that $P_0 > 0$. By the same token, the condition $P_0 > 0$ along with (40) implies nominal stability A_0. These two conditions also imply that $P_\rho > 0$ for all $|\rho| \leq 1$ since $A_\rho P_\rho + P_\rho A_\rho^\mathsf{T} < 0$ implies nonsingularity of P_ρ for all $|\rho| \leq 1$; see (Iwasaki and Shibata 2001). Assuming therefore nominal stability, one can replace inequality (39) with the condition $P_0 > 0$, and thus (38) follows. □

Example 3.5 Let $A_\rho = -(1+\epsilon)I_2 + \rho I_2$. Here $A_0 = -(1+\epsilon)I_2$ and $A_1 = I_2$. It is clear that if $\epsilon > 0$, A_ρ is Hurwitz for all $\rho \in [-1, 1]$ whereas if $\epsilon < 0$, A_ρ is not Hurwitz for all $\rho \in [-1, 1]$. Applying Theorem 3.4 with $n = 2$ and $m_s = \frac{1}{2}n(n+1) - 1 = 2$ one has

$$P_\rho = P_0 + \rho P_1 + \rho^2 P_2 = \left(\rho^{[2]} \otimes I_2\right)^\mathsf{T} P_\Sigma \left(\rho^{[2]} \otimes I_2\right)$$
$$R_\rho = A_\rho P_\rho + P_\rho A_\rho^\mathsf{T} = \left(\rho^{[3]} \otimes I_2\right)^\mathsf{T} R_\Sigma \left(\rho^{[3]} \otimes I_2\right)$$

where,

$$P_\Sigma = \begin{bmatrix} P_0 & 0.5P_1 \\ 0.5P_1 & P_2 \end{bmatrix} \tag{41}$$

and

$$R_\Sigma = \begin{bmatrix} A_0P_0 + P_0A_0^\mathsf{T} & \star & \star \\ 0.5(A_0P_1 + P_1A_0^\mathsf{T}) + A_1P_0 & 0.5(A_1P_1 + P_1A_1^\mathsf{T}) + P_2A_0^\mathsf{T} + A_0P_2 & \star \\ 0.5A_1P_1 & A_1P_2 & 0 \end{bmatrix}. \tag{42}$$

Using the MATLAB™ LMI Toolbox (Gahinet et al. 1995) one can solve (38) with a small positive value of ϵ (say, $\epsilon = 0.001$) to obtain the solution

$$P_0 = \begin{bmatrix} 19.4777 & 0 \\ 0 & 319.4777 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 3.4287 & 0 \\ 0 & 3.4287 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 11.9434 & 0 \\ 0 & 11.9434 \end{bmatrix},$$

$$D = \begin{bmatrix} 24.5601 & 0 & 1.6077 & 0 \\ 0 & 24.5601 & 0 & 1.6077 \\ 1.6077 & 0 & 17.1391 & 0 \\ 0 & 1.6077 & 0 & 17.1391 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & -0.1177 & 0 \\ 0 & 0 & 0 & -0.1177 \\ 0.1177 & 0 & 0 & 0 \\ 0 & 0.1177 & 0 & 0 \end{bmatrix}.$$

On the other hand, for any small negative small value of ϵ (say, $\epsilon = -0.001$) no solution to the inequalities (38) exists. Theorem 3.4 thus gives the same results as the direct stability analysis.

Example 3.6 Let $A_\rho = A_0 + \rho A_1$ where

$$A_0 = \begin{bmatrix} 1.1132 & 1.6802 & -1.8252 & -0.5279 \\ 1.2328 & -0.8224 & -0.3503 & -0.8995 \\ 2.8858 & 1.9407 & -3.1417 & -1.1186 \\ 1.5929 & 0.1522 & -0.4807 & -2.0469 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & -7.7372 & 0 & 0 \\ 7.7372 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the method of (Zhang et al. 2002), one can show that the matrix A_ρ is Hurwitz if and only if $\rho \in (-0.9688, 0.5024)$. In this example, $n = 4$, $r = \operatorname{rank}(A_1) = 2$ and $m_s = \frac{1}{2}(2nr - r^2 + r) = 7$. The parameter-dependent Lyapunov matrix $P_\rho = \sum_{i=0}^{7} \rho^i P_i$ where the matrices P_i, $(i = 0, 1, \ldots, 7)$ are given by

$$P_0 = \begin{bmatrix} 0.2259 & 0.1151 & 0.2201 & 0.1185 \\ 0.1151 & 0.0782 & 0.1253 & 0.0589 \\ 0.2201 & 0.1253 & 0.2430 & 0.1199 \\ 0.1185 & 0.0589 & 0.1199 & 0.0779 \end{bmatrix}, \qquad P_1 = \begin{bmatrix} -1.3999 & 0.5126 & -0.7016 & -0.7606 \\ 0.5126 & 0.9097 & 1.0028 & 0.3668 \\ -0.7016 & 1.0028 & 0.0753 & -0.3657 \\ -0.7606 & 0.3668 & -0.3657 & -0.5246 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 25.463 & 6.909 & 19.245 & 11.388 \\ 6.909 & 14.698 & 12.340 & 1.360 \\ 19.245 & 12.340 & 23.776 & 8.885 \\ 11.388 & 1.360 & 8.885 & 9.393 \end{bmatrix}, \qquad P_3 = \begin{bmatrix} -50.590 & 16.443 & -47.871 & -5.090 \\ 16.443 & 25.857 & 39.732 & 34.048 \\ -47.871 & 39.732 & -21.773 & -0.493 \\ -5.090 & 34.048 & -0.493 & -5.181 \end{bmatrix},$$

$$P_4 = \begin{bmatrix} -505.90 & 164.43 & -478.71 & -50.90 \\ 164.43 & 258.57 & 397.32 & 340.48 \\ -478.71 & 397.32 & -217.73 & -4.93 \\ -50.90 & 340.48 & -4.93 & -51.81 \end{bmatrix}, \quad P_5 = \begin{bmatrix} -302.77 & 131.28 & -263.21 & -28.91 \\ 131.28 & 92.38 & 403.28 & 216.23 \\ -263.21 & 403.28 & -15.28 & 5.72 \\ -28.91 & 216.23 & 5.72 & -39.35 \end{bmatrix}$$

$$P_6 = \begin{bmatrix} 1049.6 & 0 & 0 & 0 \\ 0 & 1049.6 & 0 & 0 \\ 0 & 0 & -56.6 & 22.5 \\ 0 & 0 & 22.5 & -79.8 \end{bmatrix}, \qquad P_7 = 0_{4\times 4}, \tag{43}$$

satisfies the matrix inequality $A_\rho P_\rho + P_\rho A_\rho^\top < 0$ for all $\rho \in \mathbb{R}$, but it is positive-definite only when $\rho \in (-0.9688, 0.5024)$. On the other hand, the matrix inequalities (38) are infeasible. This is expected, since $[-1, 1]$ is not a subset of $(-0.9688, 0.5024)$.

Let now $A_\rho = A_0 + \rho A_1'$ where $A_1' = 0.5A_1$. The exact stability domain for this matrix is $(-1.9376, 1.0048)$. Applying the algorithm of Theorem 3.4, and using the MATLAB™ LMI Toolbox (Gahinet et al. 1995), it can be verified that the inequalities (38) are indeed feasible. This result agrees with the direct analysis, since $[-1, 1] \subset (-1.9376, 1.0048)$ and thus $A_0 + \rho A_1'$ is Hurwitz for all $\rho \in [-1, 1]$.

Remark 3.7**:** The results of Example 3.6 indicate that the bound on the degree of the polynomial dependence for the Lyapunov function given in Theorem 2.1 is not tight. This is also evident from the results of (Narendra and Taylor 1973) for the case when $r = 1$. According to (Narendra and Taylor 1973, p. 79) a Lyapunov matrix with a linear dependence characterizes stability when rank $A_1 = 1$. Theorem 2.1 on the other hand requires a Lyapunov matrix of degree n.

Even when $r = \operatorname{rank} A_1 > 1$, the polynomial dependence of the Lyapunov matrix that proves stability may be smaller than the bound given in (37). To see this, let T_ρ be a parameter-dependent transformation such that the matrix $\tilde{A}_\rho := T_\rho^{-1} A_\rho T_\rho$ is in companion form[3], that

[3]The transformation to the companion form imposes the restriction that the matrix A_ρ is nonderogatory for all $\rho \in \Omega$.

is,

$$\tilde{A}_\rho = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n(\rho) & -a_{n-1}(\rho) & -a_{n-2}(\rho) & \cdots & -a_1(\rho) \end{bmatrix}. \tag{44}$$

It can be shown that $a_i(\rho)$ is a polynomial in ρ of degree r for all $i = 1, \ldots, n$. The results of (Parks 1962) and (Mori and Kokame 1995) (see also (Narendra and Taylor 1973, p. 81) as well as (Henrion 2000) and (Henrion et al. 2004)) can be used to show that a Lyapunov matrix, say $\tilde{P}_\rho$, of degree no larger than $2r$ characterizes Hurwitz stability of (44) for each ρ. Notice, however, that this Lyapunov matrix does *not* characterize stability of the original matrix A_ρ. Such a matrix is given by $P_\rho := T_\rho \tilde{P}_\rho T_\rho^\mathsf{T}$ which, in general, may not be polynomial in ρ. Even if the Lyapunov matrix P_ρ is polynomial, its degree will most likely be higher than the degree of $\tilde{P}_\rho$. If, on the other hand, A_ρ is in companion form to begin with, then this approach provides a quadratic-in-ρ Lyapunov matrix (since $r = \operatorname{rank} A_1 = 1$ in this case) although a linear-in-ρ Lyapunov matrix will suffice as mentioned previously. Thus, a general solution for the upper bound on the polynomial degree of the Lyapunov matrix that characterizes stability of the matrix $A_\rho = A_0 + \rho A_1$ seems to remain open.

4 Computational Complexity of the Proposed LMI Conditions

In this section we investigate the computational complexity of the LMI conditions of Theorem 3.4. In particular, we compare the LMI conditions (38) with those of (Bliman 2004) and show that, in general, the LMI conditions proposed herein are more efficient than the ones given in (Bliman 2004). We also propose a modification to the LMI conditions in (Bliman 2004) to make them competitive with those of Theorem 3.4.

Let us consider again the s-PDLTI system (7), rewritten below for convenience

$$\dot{x} = A_\rho x := (A_0 + \rho A_1)x, \qquad |\rho| \le 1. \tag{45}$$

In the context of parameter-dependent Lyapunov function analysis, Bliman (Bliman 2004) proposed a nice method to convert the robust stability problem (45) with real parametric uncertainty to a robust stability problem with a complex uncertainty. The approach of (Bliman 2004) hinges on the observation that

$$\{\ \rho \in \mathbb{R} :\ |\rho| \le 1\ \} = \{\ \frac{z + \bar{z}}{2} :\ z \in \mathbb{C},\ |z| = 1\ \}. \tag{46}$$

Substitution of $\rho = (z + \bar{z})/2$ into (45) yields an n^{th} order auxiliary system of the form

$$\dot{x} = \mathcal{A}x, \quad \mathcal{A} = A_0 + \left(\frac{z + \bar{z}}{2}\right) A_1, \quad z \in \mathbb{C}, \quad |z| = 1, \tag{47}$$

where z is a *complex* parameter. Clearly, system (45) is robustly stable against the real uncertainty $|\rho| \le 1$ if and only if the system (47) is robustly stable with respect to the complex uncertainty $|z| = 1$.

For the single real parameter case the main result in (Bliman 2004) can be briefly summarized as follows. In the following, we assume for simplicity that N is an odd integer and we let

$$q := \frac{N+1}{2}. \tag{48}$$

Also, for a square matrix M, we define $\mathrm{He}(M) := M + M^*$, where M^* is the complex conjugate of M.

Bliman (Bliman 2004) developed a robust stability condition for (47), and hence also for (45), in terms of the following polynomially parameter-dependent quadratic (PPDQ) Lyapunov function

$$V(x) := x^\mathsf{T} \mathcal{P} x, \quad \mathcal{P} := (z^{[N+1]} \otimes I_n)^* P (z^{[N+1]} \otimes I_n) \tag{49}$$

where $P \in \mathbb{R}^{2qn \times 2qn}$ is a constant symmetric matrix. Note that

$$\mathrm{He}(\mathcal{AP}) = \mathrm{He}\left[(A_0 + \left(\frac{z+\bar{z}}{2}\right) A_1)(z^{[2q]} \otimes I_n)^* P (z^{[2q]} \otimes I_n)\right] = (z^{[2q+1]} \otimes I_n)^* R (z^{[2q+1]} \otimes I_n)$$

where,

$$R := \mathrm{He}\left[(\hat{J}_{2q} \otimes I_n)^\mathsf{T} P (\hat{J}_{2q} \otimes A_0^\mathsf{T}) + \frac{1}{2}(\hat{J}_{2q} \otimes I_n)^\mathsf{T} P (\check{J}_{2q} \otimes A_1^\mathsf{T}) + \frac{1}{2}(\check{J}_{2q} \otimes I_n)^\mathsf{T} P (\hat{J}_{2q} \otimes A_1^\mathsf{T})\right] \tag{50}$$

The condition $\mathrm{He}(\mathcal{AP}) < 0$ for all $|z| = 1$ can be converted to an LMI (see, for instance (Iwasaki and Hara 2003)), and a robust stability condition can be given as follows.

Theorem 4.1 : *Consider the uncertain system in (47). Let a positive integer N, and a positive definite matrix $P \in \mathbb{R}^{2qn \times 2qn}$ be given, and define R by (50). The following statements are equivalent.*

(i) The function defined in (49) is a Lyapunov function that proves stability of (47), i.e.,

$$\mathcal{P} > 0, \quad \mathcal{AP} + \mathcal{PA}^\mathsf{T} < 0, \quad \forall\, z \in \mathbb{C}, \quad |z| = 1.$$

(ii) There exists a symmetric matrix $D \in \mathbb{R}^{2qn \times 2qn}$ such that

$$R < \begin{bmatrix} \hat{J}_{2q} \otimes I_n \\ \check{J}_{2q} \otimes I_n \end{bmatrix}^\mathsf{T} \begin{bmatrix} -D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \hat{J}_{2q} \otimes I_n \\ \check{J}_{2q} \otimes I_n \end{bmatrix} \tag{51}$$

Bliman's robust stability condition ((Bliman 2004, Th. 4.3)) for the single parameter case can then be stated as follows.

Corollary 4.2 (Bliman 2004)**:** *If for some non-negative integer N there exist symmetric matrices $P, D \in \mathbb{R}^{2qn \times 2qn}$ satisfying (51) and*

$$P > 0 \tag{52}$$

then the system (47) is robustly stable. Moreover, if the system (47) is robustly stable then there exist symmetric matrices $P, D \in \mathbb{R}^{2qn \times 2qn}$ satisfying (51) and $P > 0$, for sufficiently large N.

Our result of Theorem 3.4, on the other hand, considers the following PPDQ Lyapunov function for the uncertain system in (45):

$$V(x) := x^{\mathsf{T}}\mathcal{P}x, \quad \mathcal{P} := \sum_{i=0}^{N} \rho^i P_i. \tag{53}$$

Theorem 3.4 shows that the satisfaction of the LMI's (38) are sufficient for robust stability of (45) for any non-negative integer N. It is also necessary for N greater than the bound given in (10).

At this point, it is tempting to say that the bound obtained in Theorem 2.1 also provides a bound on the degree N of the PPDQ Lyapunov function (49) such that the condition of Corollary 4.2 is not only sufficient but also necessary for robust stability of (45). However, this does not seem to be the case due to the constraint $P > 0$ in (52). In the sequel, we shall elaborate on this point and show how our result of Theorem 3.4 relates to Corollary 4.2.

By Theorem 3.4, the system (45) is robustly stable if and only if there exists a PPDQ Lyapunov function (53) of degree $N > m_s$ where m_s is a known integer given by (10). With the real/complex conversion $\rho := (z + \bar{z})/2$ discussed earlier, this Lyapunov function can be represented as

$$\mathcal{P} = \sum_{i=0}^{N} \left(\frac{z+\bar{z}}{2}\right)^i P_i = \left(z^{[N+1]} \otimes I_n\right)^* P \left(z^{[N+1]} \otimes I_n\right) \tag{54}$$

for some symmetric matrix $P \in \mathbb{R}^{2qn \times 2qn}$. It is not difficult to give an explicit representation of P in terms of P_i; see, for example, (19) or (20). Note that P is now *structured* and the number of independent variables is the same as that of the P_i, i.e., $(N+1)n(n+1)/2 = qn(n+1)$, which is smaller than the number $qn(2qn+1)$ for the unstructured case of equation (49).

Since the Lyapunov matrix in (53) can be represented as (49) with structured P, after applying Bliman's real/complex conversion, a new robust stability condition is obtained as follows.

Theorem 4.3 : *The system (45) is robustly stable if and only if there exist symmetric matrices* $P_i = P_i^{\mathsf{T}} \in \mathbb{R}^n$ $(i = 0, 1, \ldots, m_s)$ *and* $D = D^{\mathsf{T}} \in \mathbb{R}^{2qn \times 2qn}$ *such that*

$$P_0 > 0, \qquad R < \begin{bmatrix} \hat{J}_{2q} \otimes I_n \\ \check{J}_{2q} \otimes I_n \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -D & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} \hat{J}_{2q} \otimes I_n \\ \check{J}_{2q} \otimes I_n \end{bmatrix} \tag{55}$$

where P is defined by (54) in terms of P_i.

We now have three robust stability conditions: Bliman's original condition (Corollary 4.2), Bliman's structured (Theorem 4.3), and ours (Theorem 3.4). For any nonnegative integer N, each of the latter two is *equivalent* to the existence of a PPDQ Lyapunov function (53) that proves robust stability of (45). Moreover, Theorem 3.4 and Theorem 4.3 give necessary and sufficient conditions for robust stability of (45) if the degree N is greater than the known bound m_s from Theorem 2.1.

However, it is not clear whether Bliman's original condition in Corollary 4.2 is also necessary whenever $N > m_s$, due to the positivity constraint $P > 0$. Without $P > 0$, the condition is weaker than the one of Theorem 4.3 and hence it is necessary, but the additional constraint $P > 0$ may destroy the necessity unless N is chosen sufficiently large as in (Bliman 2004). For the unstructured PPDQ Lyapunov function in (49), it seems difficult to relax the constraint $P > 0$ to be $P_0 > 0$ as in Theorem 3.4 due to the fact that the "nominal" case $z = 0$ is not included in the set of uncertainties characterized by $|z| = 1$. However, one may choose the "nominal" case of (45) to be $\rho = 1$, or $z = 1$, and replace $P > 0$ by

$$\left(1^{[N+1]} \otimes I_n\right)^{\mathsf{T}} P \left(1^{[N+1]} \otimes I_n\right) > 0. \tag{56}$$

With this substitution, the conditions of Corollary 4.2 become necessary and sufficient whenever $N > m_s$.

Assuming the same bound $N = m_s$ for all cases, the computational complexity of the three robust stability conditions can now be assessed in terms of the number of variables and the LMI dimensions. The result is summarized below, where $p := qn = (N+1)n/2$.

Table 1. Comparison of computational complexity of available methods.

Stability Condition	no. of variables	LMI dimensions
Theorem 3.4	$p(n+1) + p^2$	$p+n,\ n,\ p$
Theorem 4.3	$p(n+1) + p(2p+1)$	$2p+n,\ n$
Corollary 4.2	$p(2p+1) + p(2p+1)$	$2p+n,\ 2p$

Thus the LMI conditions (38) of Theorem 3.4 seem computationally most efficient among all the three, as they have the least variables and the smallest LMI dimension (considering two LMI's of dimensions n_1 and n_2 are "smaller" than one LMI of dimension $n_1 + n_2$).

In summary, we have the following observations relating our stability conditions to the ones in (Bliman 2004), for the single real parameter case: (i) our conditions do not seem to provide an explicit lower bound on the degree N of the *unstructured* PPDQ Lyapunov function (49) such that the conditions (51) and (52) are not only sufficient but also necessary for robust stability of (45); (ii) our conditions give a bound on the degree of Bliman's unstructured PPDQ Lyapunov function (49) if $P > 0$ in (52) is replaced by (56); (iii) our conditions give a bound on the degree of Bliman's PPDQ Lyapunov function (49) if P is structured as in (54) and $P > 0$ in (52) is replaced by $P_0 > 0$ as in (55); (iv) finally, our proposed robust stability conditions are of lower computational complexity. The complexity reduction is achieved due to two main factors: one is the degree bound on the PPDQ Lyapunov function, and the other is the direct application of the (D, G)-scaling, rather than the real/complex conversion followed by application of the D-scaling.

5 Stability Conditions for s-PDLTI Systems with Polynomial Parameter Dependence

In this section we deal with a system having a polynomial dependence on a single parameter as follows

$$\dot{x} = \mathcal{A}_\rho x, \quad \mathcal{A}_\rho := \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i, \qquad \rho \in \Omega \tag{57}$$

where $\mathcal{A}_i \in \mathbb{R}^{n\times n}$ for $i = 0, 1, \ldots, \nu_a$. Theorem 2.1 can be extended to the system (57) as follows.

Theorem 5.1 : *Consider the polynomially parameter-dependent matrix (57) and assume that* $\dim[\mathcal{N}(\widehat{\mathcal{A}}_1) \cap \mathcal{N}(\widehat{\mathcal{A}}_2) \cap \cdots \cap \mathcal{N}(\widehat{\mathcal{A}}_{\nu_a})] = \ell$. *Then the following two statements are equivalent:*

(i) $\mathcal{A}_\rho := \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i$ *is Hurwitz for all* $\rho \in \Omega$.

(ii) There exists a set of m_p+1 real symmetric matrices $\{P_i\}_{0\leq i\leq m_p}$, such that

$$\mathcal{A}_\rho P_\rho + P_\rho \mathcal{A}_\rho^\intercal < 0, \quad \forall \rho \in \Omega \tag{58}$$

$$P_\rho = \sigma_\rho \left(\sum_{i=0}^{m_p} \rho^i P_i \right) > 0, \quad \forall \rho \in \Omega \tag{59}$$

where

$$m_p := \nu_a \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \ell\} \tag{60}$$

and where $\sigma_\rho = -\text{sign}(\det \widehat{\mathcal{A}}_\rho)$ with $\det \widehat{\mathcal{A}}_\rho \neq 0$ for all $\rho \in \Omega$. Moreover, if Ω is connected, without loss of generality we can take $\sigma_\rho = +1$ for all $\rho \in \Omega$.

Proof It easy to show that $\widehat{\mathcal{A}}_\rho = \sum_{i=0}^{\nu_a} \rho^i \widehat{\mathcal{A}}_i$. Then using Lemma A.6 in the Appendix one obtains that

$$\text{Adj}\left(\sum_{i=0}^{\nu_a} \rho^i \widehat{\mathcal{A}}_i \right) = \sum_{i=0}^{m_p} \rho^i N_i,$$

for some constant matrices N_i, $(i = 1, 2, \ldots, m_p)$ where m_p as in (60). The rest of the proof now follows as in Theorem 2.1. □

In the sequel we assume that Ω is compact and connected. Without loss of generality we may take $\Omega = [-1, +1]$. Using the fact that the polynomial matrix P_ρ in (59) can be written as $P_\rho = \big(\rho^{[k]} \otimes I_n\big)^\intercal P_\Sigma \big(\rho^{[k]} \otimes I_n\big)$ where $k = \lceil \frac{m_p}{2} \rceil + 1$ for some $P_\Sigma \in \mathbb{R}^{nk\times nk}$, we can easily generalize Lemma 3.1 to show that the matrix in (58) can be written compactly as follows

$$R_\rho := \mathcal{A}_\rho P_\rho + P_\rho \mathcal{A}_\rho^\intercal = \big(\rho^{[k+\nu_a]} \otimes I_n\big)^\intercal R_\Sigma \big(\rho^{[k+\nu_a]} \otimes I_n\big) \tag{61}$$

where,

$$R_\Sigma := H^\intercal P_\Sigma F + F^\intercal P_\Sigma H \tag{62}$$

$$H := (\hat{J}_k \hat{J}_{k+1} \cdots \hat{J}_{k+\nu_a-1}) \otimes I_n \tag{63}$$

$$F := (\hat{J}_k \hat{J}_{k+1} \cdots \hat{J}_{k+\nu_a-1}) \otimes \mathcal{A}_0^\intercal + \sum_{i=1}^{\nu_a} (\check{J}_k \check{J}_k \cdots \check{J}_{k+i-1} \hat{J}_{k+i} \cdots \hat{J}_{k+\nu_a-1}) \otimes \mathcal{A}_1^\intercal \tag{64}$$

We are now ready to provide a necessary and sufficient condition for the stability of (57) for $|\rho| \leq 1$ in terms of LMIs.

Corollary 5.2: *Let the parameter-dependent matrix $\mathcal{A}_\rho = \sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i$, where $\mathcal{A}_i \in \mathbb{R}^{n\times n}$ with* $\dim[\mathcal{N}(\widehat{\mathcal{A}}_1) \cap \mathcal{N}(\widehat{\mathcal{A}}_2) \cap \cdots \cap \mathcal{N}(\widehat{\mathcal{A}}_{\nu_a})] = \ell$ *and let $\kappa := \lceil \frac{m_p}{2} \rceil + \nu_a$ where,*

$$m_p := \nu_a \min\{\frac{1}{2}n(n+1) - 1, \frac{1}{2}n(n+1) - \ell\} \tag{65}$$

Then, $\mathcal{A}_\rho$ is Hurwitz for all $|\rho| \leq 1$ if and only if there exist symmetric matrices $P_\Sigma \in$

$\mathbb{R}^{n(\kappa-\nu_a+1)\times n(\kappa-\nu_a+1)}$, $D \in \mathbb{R}^{n\kappa\times n\kappa}$ *and a skew-symmetric matrix* $G \in \mathbb{R}^{n\kappa\times n\kappa}$, *such that*

$$P_0 > 0, \quad D = D^{\mathsf{T}} > 0, \quad G + G^{\mathsf{T}} = 0, \quad R_\Sigma < \begin{bmatrix} \hat{J}_\kappa \otimes I_n \\ \check{J}_\kappa \otimes I_n \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -D & G \\ G^{\mathsf{T}} & D \end{bmatrix} \begin{bmatrix} \hat{J}_\kappa \otimes I_n \\ \check{J}_\kappa \otimes I_n \end{bmatrix}, \tag{66}$$

where $R_\Sigma = R_\Sigma(P_\Sigma)$ *as in (62)-(64).*

6 State-Feedback Controller Synthesis

The results of the previous section can be used to design *parameter-dependent* state-feedback controllers for the controlled s-PDLTI system

$$\dot{x} = A_\rho x + B_\rho u, \qquad \rho \in \Omega \tag{67}$$

where $A_\rho \in \mathbb{R}^{n\times n}$ and $B_\rho \in \mathbb{R}^{n\times m}$ are parameter-dependent matrices

$$A_\rho := \sum_{i=0}^{n_a} \rho^i A_i, \qquad B_\rho := \sum_{i=0}^{n_b} \rho^i B_i \tag{68}$$

and Ω is a compact and connected set. This is shown in the next theorem.

Theorem 6.1 : *The following statements are equivalent:*

(i) There exists a polynomially parameter-dependent state feedback controller $u = K_\rho x$ *that stabilizes the system (67)-(68) for all* $\rho \in \Omega$.
(ii) There exist an integer m_p *and symmetric matrices* P_i, $(i = 1, \ldots, m_p)$ *such that*

$$P_\rho := \sum_{i=0}^{m_p} \rho^i P_i, \quad P_\rho > 0, \quad A_\rho P_\rho + P_\rho A_\rho^{\mathsf{T}} < B_\rho B_\rho^{\mathsf{T}}, \qquad \forall \rho \in \Omega. \tag{69}$$

If statement (i) holds for a gain matrix $K_\rho \in \mathbb{R}^{m\times n}$ *of degree* n_k, *then statement (ii) holds with* $m_p \le \nu_a\big(\frac{1}{2}n(n+1)-1\big)$ *where* $\nu_a := \max\{n_a, n_b+n_k\}$. *Conversely, if (ii) holds, then a stabilizing state feedback gain in (i) can be given by*

$$K_\rho = -\mu_\rho B_\rho^{\mathsf{T}} P_\rho^{-1}, \quad \mu_\rho := \det(P_\rho)/\epsilon, \quad \epsilon := \min_{\rho\in\Omega} \det(P_\rho) \tag{70}$$

which is a polynomial matrix of degree $n_k \le n_b + m_p(n_a - 1)$.

Proof Suppose (i) holds for a matrix K_ρ of degree n_k. Then the closed-loop system is described by (57) with $\mathcal{A}_\rho := A_\rho + B_\rho K_\rho$. The matrix $\mathcal{A}_\rho$ has degree $\nu_a := \max\{n_a, n_b + n_k\}$. Then Theorem 5.1 implies satisfaction of (58)-(59) for some P_ρ of degree $m_p \le \nu_a\big(\frac{1}{2}n(n+1)-1\big)$. Let now N_ρ be the null space of B_ρ^{T}. Then multiplying the Lyapunov inequality in (58) by N_ρ from the right and by N_ρ^{T} from the left, one obtains

$$N_\rho^{\mathsf{T}}(A_\rho P_\rho + P_\rho A_\rho^{\mathsf{T}})N_\rho < 0.$$

By Finsler's theorem (Skelton et al. 1997), there exists $\tau_\rho > 0$ such that

$$A_\rho P_\rho + P_\rho A_\rho^{\mathsf{T}} < \tau_\rho B_\rho B\rho^{\mathsf{T}}$$

for each $\rho \in \Omega$. Since Ω is compact and τ_ρ is a continuous function of ρ (see Lemma A.9 in the Appendix), $\tau_{\max} := \max_{\rho\in\Omega} \tau_\rho > 0$ is well defined. Then, redefining P_ρ to be $P_\rho/\tau_{\max}$, we have (69). Thus we have (i) $\Rightarrow$ (ii). The converse can be proved by direct substitution. Specifically, with the control gain matrix (70) one obtains

$$(A_\rho + B_\rho K_\rho)P_\rho + P_\rho(A_\rho + B_\rho K_\rho^\mathsf{T}) = A_\rho P_\rho + P_\rho A_\rho^\mathsf{T} - 2\mu_\rho B_\rho B_\rho^\mathsf{T} < 0$$

for all $\rho \in \Omega$, where it is noted that $\mu_\rho \geq 1$ by definition. This proves that K_ρ is stabilizing. Since $\deg \mu_\rho \leq n m_p$ (see (A16)) from the expression $K_\rho P_\rho = -\mu_\rho B_\rho^\mathsf{T}$ it follows that $n_k + m_p \leq n_a m_p + n_b$. Therefore, $n_k \leq n_b + m_p(n-1)$. □

The synthesis condition (69) is a pair of polynomially-parameterized LMI's and hence it can be converted exactly to finite dimensional LMI's by eliminating the parameter ρ using Proposition 3.2. The details are left to the reader.

Example 6.2 Consider the following controlled polynomial s-PDLTI system

$$A_\rho = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} + \rho \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_\rho = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{71}$$

The uncontrolled system is stable for $\rho < -2$ and unstable otherwise. Using the Theorem 6.1 we can design the following parameter-dependent feedback gain that will ensure that the closed-loop system is stable for all $\rho \in [-1, +1]$

$$K_\rho = \begin{bmatrix} -89.579 & -69.882 \end{bmatrix} + \rho \begin{bmatrix} 4.4698 & -35.807 \end{bmatrix} + \rho^2 \begin{bmatrix} 70.575 & 18.009 \end{bmatrix}$$

The polynomial Lyapunov matrix that ensures the stability of the closed-loop system is computed by the solution of the LMI problem (69) and is given by

$$P = \begin{bmatrix} 0.49120 & -0.29686 \\ -0.29686 & 0.38053 \end{bmatrix} + \rho \begin{bmatrix} 0.088923 & 0.24193 \\ 0.24193 & -0.35769 \end{bmatrix}$$

for this example $\epsilon = 0.008496$. The closed-loop system matrix is third order and is given by

$$A_{\mathrm{cl}} = \begin{bmatrix} -177.16 & -138.76 \\ 2 & 1 \end{bmatrix} + \rho \begin{bmatrix} -78.640 & -141.50 \\ -89.579 & -68.882 \end{bmatrix} + \rho^2 \begin{bmatrix} 145.62 & 0.21074 \\ 4.4698 & -35.807 \end{bmatrix} + \rho^3 \begin{bmatrix} 70.575 & 18.009 \\ 70.575 & 18.009 \end{bmatrix}$$

It can be easily verified that this matrix is Hurwitz for all $\rho \in [-1, +1]$.

7 Conclusions

In this paper we propose a class of parameter-dependent Lyapunov matrices $P_\rho = P(\rho)$, which can be used to test the stability of single parameter-dependent linear, time-invariant, (s-PDLTI) systems of the form $\dot{x} = (\sum_{i=0}^{\nu_a} \rho^i \mathcal{A}_i)x$ where $\rho \in \Omega$. The proposed Lyapunov matrix has polynomial dependence on the parameter ρ of a known degree and can be used to derive exact (that is, necessary and sufficient) conditions for the stability of s-PDLTI systems. We show that checking these conditions over a compact interval can be cast as a finite-dimensional convex programming problem in terms of linear matrix inequalities without conservatism. Finally, using these results we provide an extension to state-feedback control design for s-PDLTI systems using parameter-varying gains. Specifically, our synthesis result (Theorem 6.1) can be useful for designing gain-scheduled controllers for LPV systems with a slowly varying parameter. This is because conditions (58)-(59) for each frozen parameter $\rho \in \Omega$ are necessary and sufficient for robust stability of (57) with respect to an arbitrarily slowly varying parameter $\rho(t) \in \Omega$. Furthermore, the result of Theorem 6.1 can be used for on-site parameter tuning for plants containing a constant parameter whose value is unknown but its range Ω is known at the time of control design. Then one may design a class of controllers parameterized by $\rho \in \Omega$ and then tune the parameter ρ on site when the controller is actually implemented.

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Appendix A: Proofs

Before providing the proof of Theorem 2.1, we need to first establish a series of technical lemmas.

The following lemma is a direct consequence of the properties of the determinant of a matrix. It can be found, for instance in (Kailath 1980, p. 384).

Lemma A.1: *Let matrices $A, B \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank} B = r$ *and let $\rho \in \mathbb{R}$. Then* $\deg(\det(A + \rho B)) \leq r$.

The following lemma states that the adjoint of the parameter-dependent matrix $A + \rho B$ is a matrix polynomial in ρ of a certain maximal degree which depends on the rank of the matrix B. Recall that given an invertible matrix $A \in \mathbb{R}^{n\times n}$, its inverse can be calculated from $A^{-1} = \operatorname{Adj} A/\det(A)$ where $\operatorname{Adj} A$ is the adjoint of A.

Lemma A.2: *Given matrices $A, B \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank} B = r$ *and $\rho \in \mathbb{R}$, the adjoint of the matrix $A + \rho B$ is a matrix polynomial in ρ of degree at most* $\min\{r, n-1\}$, *i.e.,*

$$\operatorname{Adj}(A + \rho B) = \sum_{i=0}^{\min\{r,n-1\}} \rho^i N_i. \tag{A1}$$

Proof From the definition of the adjoint of a matrix (Horn and Johnson 1991) it follows that

$$\left[\operatorname{Adj}(A + \rho B)\right]_{ij} = (-1)^{i+j}\det(A + \rho B)_{[ji]}, \quad 1 \leq i, j \leq n \tag{A2}$$

where $(\cdot)_{[ji]}$ is the $(n-1)\times(n-1)$ submatrix of $(\cdot)$ in which the j-th row and the i-th column are eliminated and $\left[\,\cdot\,\right]_{ij}$ is the ij–th element of the matrix $\left[\,\cdot\,\right]$. Since $\operatorname{rank}\left(B_{[ji]}\right) \leq \operatorname{rank} B = r$ it follows that

$$\operatorname{rank} B_{[ji]} \leq \min\{r, n-1\}, \quad 1 \leq i, j \leq n$$

From Lemma A.1, and since $(A + \rho B)_{[ji]} \in \mathbb{R}^{(n-1)\times(n-1)}$, it follows that

$$\deg\left(\det(A + \rho B)_{[ji]}\right) = \deg\left(\det(A_{[ji]} + \rho B_{[ji]}\right) \leq \operatorname{rank} B_{[ji]} \leq \min\{r, n-1\}, \quad 1 \leq i, j \leq n \tag{A3}$$

From (A2) and (A3) it follows that $\operatorname{Adj}(A + \rho B)$ is a matrix-valued polynomial of degree $\min\{r, n-1\}$ and hence, there exist constant matrices $\{N_i\}_{i=0,1,\ldots,\min\{r,n-1\}}$ such that (A1) holds. □

We also note that the matrices N_i in (A1) can be calculated explicitly from the matrices A and B. The details are left to the reader.

Lemma A.3 (Magnus 1988, Mustafa 1995)**:** *Given $A \in \mathbb{R}^{n\times n}$ and $\widehat{A}$ as in Definition 2.2, the eigenvalues of $\widehat{A}$ are the $\frac{1}{2}n(n+1)$ numbers $\lambda_i + \lambda_j$, $(1 \leq j \leq i \leq n)$ where λ_i, λ_j are the eigenvalues of A.*

The following is immediate from Lemma A.3.

Corollary A.4: *Suppose the parameter-dependent matrix $A_0 + \rho A_1 \in \mathbb{R}^{n\times n}$ is Hurwitz for all $\rho \in \Omega$. Then*

$$\det(\widehat{A_0 + \rho A_1}) = \det(\widehat{A}_0 + \rho \widehat{A}_1) \neq 0, \qquad \forall \rho \in \Omega \tag{A4}$$

Lemma A.5: *Given a matrix $A \in \mathbb{R}^{n\times n}$ with* $\operatorname{rank} A = r$*, then* $\operatorname{rank} \widehat{A} \leq \frac{1}{2}(2nr - r^2 + r)$.

Proof

Since $\operatorname{rank} A = r$, it follows that there exist two nonsingular matrices $U, V \in \mathbb{R}^{n\times n}$ such that

$$UAV = \begin{bmatrix} M & 0_{r\times(n-r)} \\ 0_{(n-r)\times r} & 0_{(n-r)} \end{bmatrix} \tag{A5}$$

where $M \in \mathbb{R}^{r\times r}$. Since $D_n^+(U \otimes U)D_n$ and $D_n^+(V \otimes V)D_n$ are nonsingular we have that

$$\operatorname{rank} \widehat{A} = \operatorname{rank}\Big(D_n^+(U \otimes U)D_n \widehat{A} D_n^+(V \otimes V)D_n\Big), \tag{A6}$$

Using the properties of the duplication matrix (Magnus 1988) the following series of equalities are easy to show

$$\begin{aligned} D_n^+(U \otimes U)D_n \widehat{A} D_n^+(V \otimes V)D_n &= D_n^+(U \otimes U)D_n D_n^+(I_n \otimes A + A \otimes I_n)D_n D_n^+(V \otimes V)D_n \\ &= D_n^+(U \otimes U)(I_n \otimes A + A \otimes I_n)D_n D_n^+(V \otimes V)D_n \\ &= D_n^+(U \otimes U)(I_n \otimes A + A \otimes I_n)(V \otimes V)D_n \\ &= D_n^+\Big(U \otimes (UA) + (UA) \otimes U\Big)(V \otimes V)D_n \\ &= D_n^+\Big((UV) \otimes (UAV) + (UAV) \otimes (UV)\Big)D_n \\ &= 2D_n^+\Big((UAV) \otimes (UV)\Big)D_n \end{aligned}$$

Using (A5), one obtains

$$\begin{aligned} D_n^+(U \otimes U)D_n \widehat{A} D_n^+(V \otimes V)D_n &= 2D_n^+ \begin{bmatrix} M \otimes (UV) & 0_{r\times(n-r)} \otimes (UV) \\ 0_{(n-r)\times r} \otimes (UV) & 0_{(n-r)} \otimes (UV) \end{bmatrix} D_n \\ &= 2D_n^+ \begin{bmatrix} L & 0_{rn\times n(n-r)} \\ 0_{n(n-r)\times rn} & 0_{n(n-r)} \end{bmatrix} D_n \end{aligned}$$

where $L := M \otimes (UV) \in \mathbb{R}^{rn\times rn}$. Using now Definition 2.1 one can show that the duplication matrix D_n has the structure

$$D_n = \begin{bmatrix} D_{11} & 0_{rn\times \frac{1}{2}(n-r)(n-r+1)} \\ D_{21} & D_{22} \end{bmatrix} \tag{A7}$$

where r is any integer such that $r < n$. Using therefore (A7) one obtains

$$\operatorname{rank} \widehat{A} = \operatorname{rank}\left(D_n^+ \begin{bmatrix} LD_{11} & 0_{rn\times\frac{1}{2}(n-r)(n-r+1)} \\ 0_{n(n-r)\times\frac{1}{2}(2nr-r^2+r)} & 0_{n(n-r)\times\frac{1}{2}(n-r)(n-r+1)} \end{bmatrix}\right) \leq \operatorname{rank} D_{11} \tag{A8}$$

Since $D_{11} \in \mathbb{R}^{rn\times\frac{1}{2}(2nr-r^2+r)}$, and since $rn \geq \frac{1}{2}(2nr - r^2 + r)$ for all $r \geq 1$, it follows that $\operatorname{rank} D_{11} \leq \frac{1}{2}(2nr - r^2 + r)$ and finally, $\operatorname{rank} \widehat{A} \leq \frac{1}{2}(2nr - r^2 + r)$. □

We are now ready to give the proof of Theorem 2.1.

Proof [Of Theorem 2.1] $(ii) \Rightarrow (i)$: This is obvious.

$(i) \Rightarrow (ii)$: Since A_ρ is Hurwitz for all $\rho \in \Omega$, from Corollary A.4 it follows that $\det(\widehat{A}_0+\rho\widehat{A}_1) \neq 0$. Let the parameter-dependent matrix

$$Q_\rho := |\det \widehat{A}_\rho|\, I_n > 0, \quad \rho \in \Omega. \tag{A9}$$

Note that Q_ρ is positive definite for all $\rho \in \Omega$. Since $A_0 + \rho A_1$ is Hurwitz for all $\rho \in \Omega$, the following Lyapunov equation has a unique, positive definite-solution $P_\rho > 0$ for each $\rho \in \Omega$

$$A_\rho P_\rho + P_\rho A_\rho^\intercal + |\det \widehat{A}_\rho|\, I_n = 0. \tag{A10}$$

Solving this equation for P_ρ one obtains

$$\begin{aligned} \big(A_\rho \oplus A_\rho\big)\mathrm{vec}(P) &= -|\det \widehat{A}_\rho|\,\mathrm{vec}(I_n) \\ D_n^+\big(A_\rho \oplus A_\rho\big)D_n\overline{\mathrm{vec}}(P) &= -|\det \widehat{A}_\rho|\,\overline{\mathrm{vec}}(I_n) \\ \overline{\mathrm{vec}}(P) &= -|\det \widehat{A}_\rho|\,\widehat{A}_\rho^{-1}\,\overline{\mathrm{vec}}(I_n) \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\mathrm{vec}}(P) &= -|\det \widehat{A}_\rho|\frac{1}{\det \widehat{A}_\rho}\mathrm{Adj}(\widehat{A}_\rho)\,\overline{\mathrm{vec}}(I_n) \\ &= \sigma_\rho \mathrm{Adj}(\widehat{A}_0 + \rho\widehat{A}_1)\,\overline{\mathrm{vec}}(I_n) \end{aligned} \tag{A11}$$

where $\sigma_\rho := -\mathrm{sign}(\det(\widehat{A}_0 + \rho\widehat{A}_1))$.

Let $\hat{r} := \operatorname{rank} \widehat{A}_1$. According to Lemma A.5 we have that $\hat{r} \leq \frac{1}{2}(2nr-r^2+r)$. Moreover, according to Lemma A.2 there exist constant matrices N_i such that $\mathrm{Adj}(\widehat{A}_0 + \rho\widehat{A}_1) = \sum_{i=0}^{m_s} \rho^i N_i$ where $m_s = \min\{\hat{r}, \frac{1}{2}n(n+1) - 1\} \leq \min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\}$. Notice, in particular, that

$$\min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\} = \begin{cases} \frac{1}{2}(2nr - r^2 + r) & \text{if} \quad r < n, \\ \frac{1}{2}n(n+1) - 1 & \text{if} \quad r = n. \end{cases}$$

Finally, since the mapping $\overline{\mathrm{vec}}(\cdot)$ is one-to-one, its inverse mapping $\overline{\mathrm{vec}}^{-1}(\cdot)$ exists. It follows from (A11) that

$$P_\rho = \sigma_\rho \left(\sum_{i=0}^{m_s} \rho^i P_i\right), \tag{A12}$$

where $P_i \in \mathbb{R}^{n\times n}$, $0 \leq i \leq m_s$, are constant matrices, given by $P_i = \overline{\mathrm{vec}}^{-1}\Big(N_i\,\overline{\mathrm{vec}}(I_n)\Big)$. □

The following lemma deals with the degree of the adjoint of the polynomial matrix $A_\rho = \sum_{i=0}^{p} \rho^i A_i$ and it is used in the proof of Theorem 5.1. Below $\mathcal{N}(M)$ denotes the nullspace of the matrix M.

Lemma A.6: *Consider matrices* $A_i \in \mathbb{R}^{n\times n}$, $(i = 0, 1, \ldots, p)$ *with* $\dim[\mathcal{N}(A_1)\cap\cdots\cap\mathcal{N}(A_p)] = q$. *Then*

$$\mathrm{Adj}\,(A_\rho) := \mathrm{Adj}\left(\sum_{i=0}^{p}\rho^i A_i\right) = \sum_{i=0}^{\mu}\rho^i N_i \tag{A13}$$

for some constant matrices N_i, $(i = 1, 2, \ldots, \mu)$, *where*

$$\mu \le p\ \min\{n-1, n-q\}. \tag{A14}$$

Proof Recall that the determinant of a matrix $F \in \mathbb{R}^{n\times n}$ can be computed from (Horn and Johnson 1991)

$$\det F = \sum_{a\in\mathbf{A}} \mathrm{sign}(a)\prod_{i=1}^{n} F_{i,a_i}, \tag{A15}$$

where $a := (a_1, a_2, \ldots, a_n)$, $\mathbf{A}$ is the set of permutations of $\{1, 2, \ldots, n\}$, and $\mathrm{sign}(a)$ is the signature of the permutation a taking the values of either $+1$ or -1. The determinant of $A_\rho = \sum_{i=0}^{p}\rho^i A_i$ is thus a sum of $n!$ terms, each term being the product of n elements. Moreover, each of these elements is chosen from a different row and column of the matrix A_ρ. Therefore, for every possible permutation $(a_1, a_2, \ldots, a_n)$,

$$\deg\prod_{i=1}^{n} F_{i,a_i} \le np,$$

which together with (A15), yields that

$$\deg\left(\det\sum_{i=0}^{p}\rho^i A_i\right) \le np \tag{A16}$$

Assume now that $\dim[\mathcal{N}(A_1)\cap\mathcal{N}(A_2)\cap\cdots\cap\mathcal{N}(A_p)] = q$. Then there exist q of linearly independent constant vectors $v_1, v_2, \ldots, v_q \in \mathbb{R}^n$ such that

$$A_i v_j = 0, \quad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q.$$

Choose now $n-q$ linearly independent constant vectors $u_1, u_2, \ldots, u_{n-q} \in \mathbb{R}^n$ such that the matrix

$$T = \Big[u_1, u_2, \ldots, u_{n-q}, v_1, v_2, \ldots, v_q\Big] \tag{A17}$$

is invertible. Furthermore,

$$\begin{aligned}\det\sum_{i=0}^{p}\rho^i A_i &= \det\left(T^{-1}\big(\sum_{i=0}^{p}\rho^i A_i\big)T\right)\\ &= \det T^{-1}\det\Big(\sum_{i=0}^{p}\rho^i A_i T\Big)\\ &= \det T^{-1}\det\Big[\bar{u}_1, \cdots, \bar{u}_{n-q}, \bar{v}_1, \cdots, \bar{v}_q\Big]\end{aligned}$$

where $\bar{u}_i = \sum_{j=0}^{p} \rho^j A_j u_i$, $i = 1, 2, \ldots, n-q$ and $\bar{v}_i = A_0 v_i$, $i = 1, 2, \ldots, q$. Since $\bar{v}_i$ are constant vectors, together with the determinant formula (A15), one has

$$\begin{aligned} &\det \left[\bar{u}_1, \ldots, \bar{u}_{n-q}, \bar{v}_1, \ldots, \bar{v}_q\right] \\ &= \sum_{a_1 \neq a_2 \neq \ldots \neq a_n} \pm(\bar{u}_{1,a_1} \bar{u}_{2,a_2} \ldots \bar{u}_{(n-q),a_{(n-q)}} \bar{v}_{1,a_{(n-q+1)}} \ldots \bar{v}_{q,a_n}) \end{aligned}$$

For every possible permutation $(a_1, a_2, \ldots, a_n)$, we have that $\deg \bar{u}_{1,a_1} \cdots \bar{u}_{(n-q)a_{(n-q)}} \bar{v}_{1,a_{(n-q+1)}} \cdots \bar{v}_{q,a_n} = \deg \bar{u}_{1,a_1} \cdots \bar{u}_{(n-q)a_{(n-q)}} \leq p(n-q)$. It follows that $\deg \left(\det [\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{n-q}, \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_q]\right) \leq p(n-q)$ and hence

$$\deg \left(\det \sum_{i=0}^{p} \rho^i A_i \right) \leq p\,(n-q).$$

The result now follows from the fact that

$$\left[\mathrm{Adj}\,(A_\rho)\right]_{ij} = (-1)^{i+j} \det(A_\rho)_{[ji]}, \qquad 1 \leq j, i \leq n$$

$\square$

The following result is need for the proof of Proposition 3.2.

Lemma A.7: *Let the matrices* $J = \check{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1)\times nk}$ *and* $C = \hat{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1)\times nk}$. *Consider the sets*

$$\mathcal{C}_1 := \{\zeta \in \mathbb{R}^{nk} :\ (J - \delta C)\zeta = 0,\ \text{some } \delta \in [-1, +1]\},$$

and

$$\mathcal{C}_2 := \{\zeta \in \mathbb{R}^{nk} :\ \zeta = (\rho^{[k]} \otimes I_n)x,\ \rho \in [-1, +1],\ x \in \mathbb{R}^n\}. \tag{A18}$$

Then $\mathcal{C}_1 = \mathcal{C}_2$.

Proof Since $(\check{J}_{k-1} - \rho \hat{J}_{k-1})\rho^{[k]} = 0$ and

$$\begin{aligned} (J - \rho C)(\rho^{[k]} \otimes I_n)x &= \left(\check{J}_{k-1} \otimes I_n - \rho \hat{J}_{k-1} \otimes I_n\right)(\rho^{[k]} \otimes I_n)x \\ &= \left((\check{J}_{k-1} - \rho \hat{J}_{k-1}) \otimes I_n\right)(\rho^{[k]} \otimes I_n)x \\ &= \left(\left((\check{J}_{k-1} - \rho \hat{J}_{k-1})\rho^{[k]}\right) \otimes I_n\right)x \\ &= (0 \otimes I_n)x = 0\ , \end{aligned}$$

it follows immediately that $\mathcal{C}_2 \subseteq \mathcal{C}_1$.

Conversely, let $\zeta \in \mathcal{C}_1$. Since $\zeta \in \mathbb{R}^{nk}$, we may write

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix}, \quad \text{where} \quad \zeta_i \in \mathbb{R}^n, \quad i = 1, 2, \ldots, k.$$

Since $(J - \delta C)\zeta = 0$ for some $\delta \in [-1, +1]$, it follows that

$$\Big[\big[0_{(k-1),1} \otimes I_n,\, I_{k-1} \otimes I_n\big] - \delta \big[I_{k-1} \otimes I_n,\, 0_{(k-1),1} \otimes I_n\big] \Big] \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix} = 0$$

for some $\delta \in [-1, +1]$. Expanding the previous expression componentwise, one obtains

$$\begin{aligned} \zeta_2 - \delta\zeta_1 &= 0 \\ \zeta_3 - \delta\zeta_2 &= 0 \\ &\vdots \\ \zeta_k - \delta\zeta_{k-1} &= 0 \end{aligned}$$

or

$$\zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_k \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \delta\zeta_1 \\ \delta^2\zeta_1 \\ \vdots \\ \delta^{k-1}\zeta_1 \end{bmatrix} = \delta^{[k]} \otimes \zeta_1 \ .$$

Let now $x := \zeta_1 \in \mathbb{R}^n$, and $\rho := \delta \in [-1, +1]$. It follows that $\zeta = (\rho^{[k]} \otimes I_n)x$ and thus $\zeta \in \mathcal{C}_2$. Therefore, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ and the claim is shown. □

The following lemma is instrumental in casting the matrix feasibility problem (16) to a finite-dimensional convex optimization problem. It is an extension of a result given in (Iwasaki et al. 2000).

Lemma A.8: *Let the matrices $\Theta = \Theta^{\mathsf{T}} \in \mathbb{R}^{n \times n}$ and $J, C \in \mathbb{R}^{k \times n}$ be given. The following statements are equivalent.*

(i) The inequality $\zeta^{\mathsf{T}}\Theta\zeta < 0$ holds for all nonzero vectors $\zeta \in \mathbb{R}^n$ which satisfy $(J - \delta C)\zeta = 0$, for some real scalar δ such that $|\delta| \leq 1$.
(ii) There exist matrices $D \in \mathbb{R}^{k \times k}$ and $G \in \mathbb{R}^{k \times k}$ such that

$$D = D^{\mathsf{T}} > 0, \quad G + G^{\mathsf{T}} = 0, \quad \Theta < \begin{bmatrix} C \\ J \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -D & G \\ G^{\mathsf{T}} & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}. \tag{A19}$$

Proof From Lemma 2 of (Iwasaki et al. 2000), a vector $\zeta \in \mathbb{C}^n$ satisfies $\big(J - j\delta(jC)\big)\zeta = 0$ for some real scalar δ such that $|\delta| \leq 1$ if and only if

$$\zeta^* \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix} \zeta \leq 0$$

holds for all $U = U^* > 0$ and $V = V^*$. Then applying the generalized $\mathcal{S}$-procedure ((Iwasaki et al. 2000), Theorem 1) and noting the losslessness of the following set of Hermitian matrices

$$\mathcal{S} := \left\{ \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix} \ : \ V = V^*, \ \ U = U^* > 0 \right\}$$

we see ((Iwasaki et al. 2000), Lemma 3) that statement (i) is equivalent to the existence of a matrix $V = V^*$ and $U = U^* > 0$ such that

$$\Theta < \begin{bmatrix} J \\ jC \end{bmatrix}^* \begin{bmatrix} U & V \\ V & -U \end{bmatrix} \begin{bmatrix} J \\ jC \end{bmatrix}, \tag{A20}$$

The result then follows by letting $D := U$ and $G := -jV$, and taking the real part of the equation. □

The following is an extension of the well-known Finsler's Lemma to the case of continuously parameter-varying matrices.

Lemma A.9 Parameter-dependent Finsler's Lemma *Let the parameter-dependent matrices* $B_\rho \in \mathbb{R}^{n\times m}$ *and* $P_\rho \in \mathbb{R}^{n\times n}$. *Suppose that* $\operatorname{rank} B_\rho < n$ *and* $P_\rho = P_\rho^\mathsf{T}$ *for all* $\rho \in \Omega$ *and suppose that*

$$N_\rho^\mathsf{T} P_\rho N_\rho < 0, \qquad \forall \rho \in \Omega \tag{A21}$$

where N_ρ *is the null space of* B_ρ. *Then there exists a* $\tau_\rho > 0$ *such that*

$$P_\rho < \tau_\rho B_\rho B_\rho^\mathsf{T}, \qquad \forall \rho \in \Omega \tag{A22}$$

Moreover, τ_ρ *can be chosen to depend continuously on* ρ.

Proof The existence of a $\tau_\rho > 0$ satisfying (A22) for each $\rho \in \Omega$ follows from the standard Finsler's Theorem (Skelton et al. 1997). It remains to show that τ_ρ can be selected to depend continuously on ρ. To this end, let $\phi(\rho, \tau)$ denote the minimum eigenvalue of $\tau B_\rho^\mathsf{T} B_\rho - P_\rho$. Clearly, ϕ is continuous in both its arguments and defines a continuous surface Φ in the (ρ, τ, ϕ)-space. Moreover, $\phi(\rho, \tau_2) \geq \phi(\rho, \tau_1)$ whenever $\tau_2 > \tau_1$. From the proof of Finsler's Theorem (Skelton et al. 1997) it follows that for each $\rho \in \Omega$ there exists a unique $\hat{\tau}_\rho$ such that $\phi(\rho, \hat{\tau}_\rho) = 0$. Furthermore, $\phi(\rho, \tau) > 0$ for all $\tau > \hat{\tau}_\rho$ and $\phi(\rho, \tau) < 0$ for all $\tau < \hat{\tau}_\rho$. This means that the surface Φ intersects transversally the level set $\phi = 0$. Moreover, this intersection defines a continuous curve $\tau = \hat{\tau}_\rho$ on the (ρ, τ)-plane. Then $\tau_\rho := \hat{\tau}_\rho + 1$ is continuous in ρ and satisfies (A22). □