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An approach for computing the exact stability domain for a class of LTI parameter dependent systems

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In this paper, the complete stability domain for both single- and multi-parameter dependent LTI systems is synthesized by extending existing results in the literature. This domain is calculated through a guardian map which involves the determinant of the Kronecker sum of a matrix with itself. The single parameter case is easily computable, whereas the multi-parameter case is more involved. The determinant of the bialternate sum of a matrix with itself is also exploited to reduce the computational complexity of the results presented in this paper.

1. Introduction

In this paper we study the stability of linear time invariant parameter-dependent (LTIPD) systems. The need to determine the bounds on the parameters that guarantee stability of a system perturbed by these parameters has been the subject of intensive research in the past. Methods based on Lyapunov function theory have been proposed. Specifically, several parameter-dependent Lyapunov functions have been suggested in the literature to find such bounds (Khargonekar and Rotea 1988, Bernstein and Haddad 1990, Haddad and Bernstein 1995, Helmersson 1999, Iwasaki and Shibata 1999, Neto 1999). The use of Lyapunov function methods typically gives rise to stability conditions that are sufficient but not necessary. Chilali and Gahinet (1996) and Chilali *et al*. (1999) studied quadratic δ-Hurwitz and $\mathcal{D}$-stability and gave robust stability conditions for parametric uncertainty. For quadratic stability, Amato *et al*. (1996) and Lee *et al*. (1996) gave necessary and sufficient conditions, which are valid even for time-varying linear systems. However, quadratic stability is, in general, more conservative than robust stability (Rern *et al*. 1994, Chilali *et al*. 1999). Saydy *et al*. (1988, 1990) defined a particular guardian map and used it to study the stability of LTIPD systems of the form:

$$\dot{x} = A(\rho)x, \quad A(\rho) = \sum_{i=0}^{m} \rho^i A_i \tag{1}$$

and

$$\dot{x} = A(\rho_1, \rho_2)x, \quad A(\rho_1, \rho_2) = \sum_{i_1, i_2=0}^{i_1+i_2=m} \rho_1^{i_1} \rho_2^{i_2} A_{i_1, i_2}. \tag{2}$$

The guardian map in Saydy *et al*. (1988) is the determinant of the Kronecker sum of a matrix with itself. Using this guardian map, they gave necessary and sufficient stability conditions with respect to a given parameter domain for the particular LTIPD systems in (1) and (2). This method was later extended in Barmish (1994) and Rern *et al*. (1994) to LTI systems with many parameters of the form:

$$\dot{x} = A(\rho_1, \rho_2, \ldots, \rho_m)x,$$
$$A(\rho_1, \rho_2, \ldots, \rho_m) = A_0 + \sum_{i=1}^{m} \rho_i A_i. \tag{3}$$

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However, the stability conditions in Barmish (1994) and Rern *et al.* (1994) are only sufficient. Fu and Barmish (1988) gave the maximal stability interval around the origin for LTIPD systems of the form (3) with $m = 1$ and A_0 Hurwitz. Mustafa and Davidson (1995) studied the robust stability problem of LTIPD systems using the bialternate sum of matrices. The determinant of the bialternate sum of a matrix $A \in \mathbb{R}^{n\times n}$ with itself is not a guardian map. Nonetheless, it can be used in a similar way (allowing for some minor changes) as the Kronecker sum to guard Hurwitz matrices. The advantage of the bialternate sum used by Mustafa and Davidson is that it involves fewer calculations than the Kronecker sum. This property is also explored in this paper to reduce the computations required for the derived stability tests.

The existing results, for example, Saydy *et al.* (1988, 1990) and Barmish (1994) give necessary and sufficient stability conditions for an *a priori* given single- or multi-parameter interval set. Furthermore, Rern *et al.* (1994) provides a bounded interval, which is only sufficient in guaranteeing the stability of LTIPD systems. A question which arises naturally from this research is how to find the entire stability domain for single- or multi-parameter dependent systems. In many cases, the complete stability domain may be composed of one or several pieces of connected sets.

In this paper, we extend existing results in the literature to give the entire stability domain for single-parameter dependent LTI systems. We then generalize this result to multi-parameter dependent LTI systems. In order to reduce the computational complexity of the derived stability conditions, the guardian map which involves the determinant of the Kronecker sum of an $n \times n$ matrix with itself is replaced by the determinant of the bialternate sum of an $n \times n$ matrix with itself. Specifically, the stability test requires the computation of the eigenvalues of the inverse of an $n^2 \times n^2$ matrix if the Kronecker sum is used. This reduces to computing the eigenvalues of the inverse of an $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix if the bialternate sum is used.

It should be pointed out that the derived conditions in this paper can also be used to determine the stability of "slow" linear parameter varying (LPV) systems. As shown in Guo and Rugh (1995), given the system

$$\dot{x} = (A_0 + \rho(t)A_g)x, \tag{4}$$

where $A_0, A_g \in \mathbb{R}^{n\times n}$, $\rho(t) \in [\underline{\rho}, \bar{\rho}], \dot{\rho}(t) \in [\underline{\dot{\rho}}, \bar{\dot{\rho}}]$, and $\ddot{\rho}(t) \in [\underline{\ddot{\rho}}, \bar{\ddot{\rho}}]$ for all $t \geq 0$, with $\underline{\dot{\rho}}, \bar{\dot{\rho}}, \underline{\ddot{\rho}}, \bar{\ddot{\rho}}$ being sufficient small, then the following conditions are equivalent:

(i) The system (4) is asymptotically stable.
(ii) $\mathrm{Re}[\lambda_i(A_0 + \rho A_g)] < 0, \quad \forall \rho \in [\underline{\rho}, \bar{\rho}], \quad i = 1, 2, \ldots n.$

This implies that stability of the "slowly-varying" LPV system in (4) can be inferred from the stability of the LTIPD system $\dot{x} = (A_0 + \rho A_g)x$ where ρ is unknown but constant in the interval $[\underline{\rho}, \bar{\rho}]$.

The paper has eight sections and is arranged as follows: §2 gives some preliminaries and the main mathematical tools used in this paper. Section 3 introduces two methods for computing the maximal open stability interval on $\mathbb{R}$ which includes zero, such that the single parameter-dependent system matrix will be Hurwitz if the parameter is within this interval. This result is the same as the one in Fu and Barmish (1988) and is included here for completeness, albeit with an alternate proof. The methods in §3 have the limitation that the system matrix must be Hurwitz when the parameter is zero. The guardian map induced by the Kronecker sum and the map induced by the bialternate sum are then exploited to compute the maximal interval of stability. Section 4 extends the results of §3 and gives two algorithms for computing the complete stability domain for a single parameter-dependent system matrix. This domain may be an open interval or a union of several open intervals. When the parameter is zero, the system matrix is not required to be Hurwitz in order to apply these two algorithms. Section 5 generalizes these results to multi-parameter dependent LTI systems. Section 6 gives some numerical examples. Some comments on the numerical complexity of the proposed algorithms are provided in §7, while §8 presents the conclusions.

The notation used in this paper is as follows:

$\otimes \quad \oplus$	Kronecker product and sum
$\star$	Bialternate product
$\lambda_i(A)$	ith eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$
I_n	Identity matrix of dimension $n \times n$ (also denoted I when the dimension is clear from the context)
$\mathrm{int}(\mathcal{D})$	Interior of the set $\mathcal{D}$
$\partial\mathcal{D}$	Boundary of the set $\mathcal{D}$
$\mathcal{A}$	Set of Hurwitz matrices $A \in \mathbb{R}^{n\times n}$
$\bar{A}$	$A \oplus A$, $A \in \mathbb{R}^{n\times n}$
$\tilde{A}$	$A \star I_n + I_n \star A = 2A \star I_n$, $A \in \mathbb{R}^{n\times n}$
$\mathrm{mspec}(A)$	Multispectrum of matrix $A \in \mathbb{R}^{n\times n}$, i.e. the set consisting of all the eigenvalues of A, including repeated eigenvalues
$\mathcal{I}_n$	Index set $\{1, 2, \ldots, n\}$
$\mathcal{I}_n^0$	Index set $\{0, 1, 2, \ldots, n\}$
$\cup$	Ordered union of two sets, taking only one occurrence of repeated members
$\mathcal{D}^{\#}$	Cardinality of the set $\mathcal{D}$
$\det(A)$ or $\lvert A\rvert$	Determinant of the matrix A

2. Preliminaries

2.1 *The guardian map*

Our results rely heavily on the concept of a guardian map for the set of Hurwitz matrices (Barmish 1994, Rern *et al.* 1994, Saydy *et al.* 1988, 1990). A guardian map transforms a matrix stability problem to a non-singularity problem of an associated matrix. The most common guardian map is the one that involves the Kronecker sum of a matrix with itself. The definitions of the Kronecker product and Kronecker sum of two matrices may be found in several standard references (see Brewer (1978) for example).

The following mathematical results will be used in this paper.

Lemma 1 (Zhou *et al.* 1996)**:** *Let* $A \in \mathbb{R}^{n\times n}$ *and* $B \in \mathbb{R}^{m\times m}$. *Then* $\mathrm{mspec}(A \oplus B) = \{\lambda_i + \mu_j\colon \lambda_i \in \mathrm{mspec}(A), \mu_j \in \mathrm{mspec}(B),\ i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m\}$.

Corollary 1: *Given a matrix* $A \in \mathbb{R}^{n\times n}$, *define* $\bar{A} := A \oplus A$. *Assume that A is Hurwitz. Then*

(i) $\bar{A}$ *is Hurwitz.*
(ii) $\det\bar{A} \neq 0$.

Proof: Follows directly from the definition of $\bar{A}$ and Lemma 1. □

The following definition is taken from Barmish (1994).

Definition 1 (Guardian Map)**:** Let $\mathcal{S} \subseteq \mathbb{R}^{n\times n}$ an open set. The map $\nu\colon \mathbb{R}^{n\times n} \to \mathbb{R}$ is said to guard the set $\mathcal{S}$ if $\nu(A) \neq 0$ for all $A \in \mathcal{S}$ and $\nu(A) = 0$ for all $A \in \partial\mathcal{S}$. The map ν is called a guardian map for $\mathcal{S}$.

Remark 1: The Kronecker sum induces the guardian map $\nu_1\colon \mathbb{R}^{n\times n} \to \mathbb{R}$,

$$\nu_1(A) := \det(A \oplus A), \tag{5}$$

which guards the set $\mathcal{A}$ of Hurwitz matrices (Barmish 1994).

2.2 *Bialternate sum*

For $A, B \in \mathbb{R}^{n\times n}$ with elements a_{ij} and b_{ij}, let the index function $\tilde{m}$ be defined by:

$$\tilde{m}(n, i, j) := (j-1)n + i - \tfrac{1}{2}j(j+1). \tag{6}$$

Then, the bialternate product of A and B is the matrix $F = A \star B$ of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$, with elements as follows (Jury 1974, Magnus 1988, Mustafa and Davidson 1995)

$$f_{\tilde{m}(n,p,q),\tilde{m}(n,r,s)} := \frac{1}{2}\left(\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}\right),$$

where $p, r = 2, 3, \ldots, n; q = 1, 2, \ldots, p-1$ and $s = 1, 2, \ldots, r-1$. From this definition it is clear that $A \star B = B \star A$. For example, if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{7}$$

then

$$A \star B = \frac{1}{2}\begin{bmatrix} \begin{pmatrix} a_{22}b_{11} + a_{11}b_{22} \\ -a_{12}b_{21} - a_{21}b_{12} \end{pmatrix} & \begin{pmatrix} a_{11}b_{23} + a_{23}b_{11} \\ -a_{21}b_{13} - a_{13}b_{21} \end{pmatrix} & \begin{pmatrix} a_{12}b_{23} + a_{23}b_{12} \\ -a_{22}b_{13} - a_{13}b_{22} \end{pmatrix} \\ \begin{pmatrix} a_{11}b_{32} + a_{32}b_{11} \\ -a_{12}b_{31} - a_{31}b_{12} \end{pmatrix} & \begin{pmatrix} a_{11}b_{33} + a_{33}b_{11} \\ -a_{13}b_{31} - a_{31}b_{13} \end{pmatrix} & \begin{pmatrix} a_{12}b_{33} + a_{33}b_{12} \\ -a_{13}b_{32} - a_{32}b_{13} \end{pmatrix} \\ \begin{pmatrix} a_{21}b_{32} + a_{32}b_{21} \\ -a_{22}b_{31} - a_{31}b_{22} \end{pmatrix} & \begin{pmatrix} a_{21}b_{33} + a_{33}b_{21} \\ -a_{23}b_{31} - a_{31}b_{23} \end{pmatrix} & \begin{pmatrix} a_{22}b_{33} + a_{33}b_{22} \\ -a_{23}b_{32} - a_{32}b_{23} \end{pmatrix} \end{bmatrix}. \tag{8}$$

The bialternate sum $\tilde{A}$ of matrix A with itself is defined as (Fuller 1968, Jury 1974, Mustafa and Davidson 1995)

$$\tilde{A} = A \star I_n + I_n \star A = 2A \star I_n. \tag{9}$$

If $\tilde{a}_{ij}$ denotes the ijth element of $\tilde{A}$ then, clearly,

$$\tilde{a}_{\tilde{m}(n,p,q),\tilde{m}(n,r,s)} = \begin{vmatrix} a_{pr} & a_{ps} \\ \delta_{qr} & \delta_{qs} \end{vmatrix} + \begin{vmatrix} \delta_{pr} & \delta_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}, \tag{10}$$

where δ_{ij} is the Kronecker delta $\left(\delta_{ij} = 1, \text{ if } i = j, \delta_{ij} = 0, \text{ if } i \neq j\right)$ and $p, r = 2, 3, \ldots, n$; $q = 1, 2, \ldots, p-1$ and $s = 1, 2, \ldots, r-1$. Clearly, if $A \in \mathbb{R}^{n\times n}$, then $\tilde{A} \in \mathbb{R}^{\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)}$. For example, using (10), we have

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \tilde{A}_1 = a_{11} + a_{22} \tag{11}$$

and

$$A_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$
$$\tilde{A}_2 = \begin{bmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{bmatrix}. \tag{12}$$

From the definition of the bialternate sum of a matrix with itself, it immediately follows that

$$A_0 \widetilde{+\rho} A_g = \tilde{A}_0 + \rho \tilde{A}_g, \tag{13}$$

where $A_0, A_g \in \mathbb{R}^{n\times n}$ and $\rho \in \mathbb{R}$.

LEMMA 2 (Jury 1974): *Let $A \in \mathbb{R}^{n\times n}$. Then* $\mathrm{mspec}(\tilde{A}) = \{\lambda_i(A) + \lambda_j(A),\ i = 2, 3, \ldots, n,\ j = 1, 2, \ldots, i-1\}$.

The next Corollary follows immediately from Lemma 2.

Corollary 2: *Let $A \in \mathbb{R}^{n\times n}$, let $\tilde{A}$ as in* (9) *and assume that A is Hurwitz. Then*

(i) *$\tilde{A}$ is Hurwitz.*
(ii) $\det\tilde{A} \neq 0$.

Remark 2: The determinant of the bialternate sum of a matrix with itself cannot be used as a guardian map of $\mathcal{A}$. To see this, let a matrix $A \in \mathbb{R}^{n\times n}$ having only one eigenvalue zero and all other eigenvalues in the open left half complex plane. In this case, $A \in \partial\mathcal{A}$, but $\det\tilde{A} \neq 0$. However, the map

$$\nu_2(A) = \det A \ \det\tilde{A} \tag{14}$$

is a guardian map which guards the set $\mathcal{A}$. First, it is easy to see that $\nu_2(A) \neq 0$ if $A \in \mathcal{A}$. Moreover, if $A \in \partial\mathcal{A}$, some eigenvalues of the matrix A are on the $j\omega$-axis and all the others are in the open left half plane of $\mathbb{C}$. Let $\mathcal{F}$ be the set of matrices in $\partial\mathcal{A}$ with at most one eigenvalue at the origin

$$\mathcal{F} := \big\{A \in \partial\mathcal{A}\colon \lambda_i(A) = 0 \quad \text{and} \quad \lambda_j(A) \neq 0 \ \text{ for all } j \neq i, \quad i, j \in \mathcal{I}_n\big\}. \tag{15}$$

If $A \in \mathcal{F}$ then $\det A = 0$ and if $A \in \partial\mathcal{A} \setminus \mathcal{F}$ then $\det\tilde{A} = 0$. In either case, $\nu_2(A) = 0$. Hence, $\nu_2(A)$ is a guardian map for the set $\mathcal{A}$ according to Definition 1. Moreover, $\nu_2(A)$ is easier to compute than $\nu_1(A)$ since the dimension of $\tilde{A}$ is $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ whereas that of $\bar{A}$ is $n^2 \times n^2$.

The following definitions will be used in the sequel.

Definition 2: Given $M \in \mathbb{R}^{n\times n}$, let $\tilde{\lambda}_i(M)$, $i = 1, \ldots, p$ denote the real non-zero eigenvalues of M, where repeated eigenvalues are counted only once. If $p = 0$, let $\tilde{\lambda}_0(M) = 0$. The open interval $\mathcal{N}(M)$ is defined as follows:

$$\mathcal{N}(M) := \left(-\frac{1}{\max\limits_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M)},\ -\frac{1}{\min\limits_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M)} \right), \tag{16}$$

where

$$-\frac{1}{\max\limits_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M)} = -\infty \quad \text{if } \max_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M) = 0,$$
$$-\frac{1}{\min\limits_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M)} = +\infty \quad \text{if } \min_{i\in\mathcal{I}_p^0} \tilde{\lambda}_i(M) = 0. \tag{17}$$

The following Corollary is a direct consequence of Definition 2.

Corollary 3: *For any $M \in \mathbb{R}^{n\times n}$,*

(i) $0 \in \mathcal{N}(M)$
(ii) $\det(I + rM) \neq 0$, *for all $r \in \mathcal{N}(M)$*

Definition 3: Given $M \in \mathbb{R}^{n\times n}$, let $\tilde{\lambda}_i(M)$, $i = 1, \ldots, p$ denote the real non-zero eigenvalues of M taking only once occurrence of repeated eigenvalues. Let $r_0 = -\infty$, $r_{p+1} = +\infty$ and $r_i = -1/\tilde{\lambda}_i(M)$, $i = 1, 2, \ldots, p$. Define the ordered set (after, perhaps, a relabeling of the indices) $\mathcal{B}(M) := \{r_0, r_1, r_2, \ldots, r_p, r_{p+1}\}$ such that $r_i < r_{i+1}$.

Remark 3: From the definition of $\mathcal{B}(M)$, it follows that, for any $r \in \mathbb{R}$, $\det(I + rM) = 0$ if and only if $r \in \mathcal{B}(M)$.

3. Maximal stability domain of single parameter-dependent LTI systems

3.1 *Maximum stability interval about the origin*

Saydy *et al.* (1990) and Barmish (1994) derived a stability condition for a family of $n \times n$ parameter-dependent matrices given by $A(\rho) = \sum_{i=0}^{m} \rho^i A_i$. Their result tests whether $A(\rho)$ is robustly stable for all $\rho \in [0, 1]$. In Rern *et al.* (1994) the authors construct an interval which guarantees robust stability for single and multi-parameter dependent LTI systems. However, this interval is derived from sufficient conditions and hence it is not the maximal robust stability interval. Fu and Barmish (1988) presented a method to synthesize the maximal stability interval containing the origin for single parameter-dependent LTI systems. Next, we re-state the theorem in Fu and Barmish (1988) but we give an alternate proof that will provide an insight for the extensions proposed in the subsequent sections.

Theorem 1: *Given an open interval $\Omega \subseteq \mathbb{R}$ and A_0, $A_g \in \mathbb{R}^{n\times n}$, define $\bar{A}_0 := A_0 \oplus A_0$ and $\bar{A}_g := A_g \oplus A_g$. Then, the following two statements are equivalent*

(i) $0 \in \Omega$ *and* $A(\rho) := A_0 + \rho A_g$ *is Hurwitz for all* $\rho \in \Omega$.
(ii) A_0 *is Hurwitz and* $0 \in \Omega \subseteq \mathcal{N}(\bar{A}_0^{-1}\bar{A}_g)$.

Proof: First note that if $A(\rho) = A_0 + \rho A_g$ we can write

$$\begin{aligned}\bar{A}(\rho) &:= A(\rho) \oplus A(\rho) = A(\rho) \otimes I_n + I_n \otimes A(\rho) \\ &= (A_0 \otimes I_n + I_n \otimes A_0) + \rho(A_g \otimes I_n + I_n \otimes A_g) \\ &= \bar{A}_0 + \rho \bar{A}_g.\end{aligned}$$

We can now proceed as follows:

(i) $\Rightarrow$ (ii): If $A(\rho)$ is Hurwitz for all $\rho \in \Omega$ and $0 \in \Omega$, then A_0 is Hurwitz. Then from Corollary 1 it follows that $\det \bar{A}_0 \neq 0$ and $\bar{A}_0^{-1}$ exists. Furthermore, since $A(\rho)$ is Hurwitz for all $\rho \in \Omega$ then also from Corollary 1, $\det \bar{A}(\rho) \neq 0$ for all $\rho \in \Omega$. Therefore,

$$\begin{aligned}0 \neq \det(\bar{A}_0 + \rho \bar{A}_g) &= \det[\bar{A}_0(I + \rho \bar{A}_0^{-1} \bar{A}_g)] \\ &= \det \bar{A}_0 \det(I + \rho \bar{A}_0^{-1} \bar{A}_g) \quad \forall \rho \in \Omega.\end{aligned}$$

Hence $0 \neq \det(I + \rho \bar{A}_0^{-1} \bar{A}_g)$ for all $\rho \in \Omega$. This, in turn, implies that

$$\rho \lambda_i(\bar{A}_0^{-1} \bar{A}_g) \neq -1, \quad \forall i \in \mathcal{I}_{n^2}, \ \forall \rho \in \Omega. \tag{18}$$

If $\bar{A}_0^{-1} \bar{A}_g$ has no real eigenvalues or if the only real eigenvalues lie at the origin then $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g) = (-\infty, \infty)$ and trivially $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$. If $\bar{A}_0^{-1} \bar{A}_g$ has some non-zero real eigenvalues, then the largest interval which includes $\rho = 0$ such that $\rho \lambda_i(\bar{A}_0^{-1} \bar{A}_g) \neq -1$, $i = 1, 2, \ldots, n^2$ is given by the definition of $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$. Hence from (18), $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$.

(ii) $\Rightarrow$ (i): The proof follows by contradiction. Assume A_0 is Hurwitz and $0 \in \Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ but suppose $A(\rho)$ is not Hurwitz for all $\rho \in \Omega$. Then, there exists a $\rho_1 \in \Omega$ such that $\mathrm{Re}[\lambda_k(A(\rho_1))] \geq 0$ for some $k \in \mathcal{I}_n$. If $\rho_1 = 0$, the proof is complete since A_0 is assumed Hurwitz. Consequently, and without loss of generality, we may assume that $\rho_1 > 0$ (the case for $\rho_1 < 0$ being identical). Because $\mathrm{Re}[\lambda_i(A_0)] < 0$ for every $i \in \mathcal{I}_n$ and the eigenvalues of $A(\rho)$ change continuously with ρ (see Horn and Johnson (1991), Appendix D), there exists $\rho_2 \in (0, \rho_1] \subseteq \Omega$ such that $\mathrm{Re}[\lambda_k(A(\rho_2))] = 0$ for some $k \in \mathcal{I}_n$. There are two possibilities:

First, $\lambda_k(A(\rho_2)) = 0$. Then by Lemma 1, there exists $m \in \mathcal{I}_{n^2}$ such that $\lambda_m(\bar{A}(\rho_2)) = \lambda_k(A(\rho_2)) + \lambda_k(A(\rho_2)) = 0$. Second, $\lambda_k(A(\rho_2)) = j\omega$ and $\omega \neq 0$. Since $A(\rho_2) \in \mathbb{R}^{n \times n}$, there exists $k' \in \mathcal{I}_n$ such that $\lambda_{k'}(A(\rho_2)) = -j\omega$ and hence by Lemma 1, there exists $m \in \mathcal{I}_{n^2}$ such that $\lambda_m(\bar{A}(\rho_2)) = \lambda_k(A(\rho_2)) + \lambda_{k'}(A(\rho_2)) = 0$.

Consequently, in either case, there exists $m \in \mathcal{I}_{n^2}$ such that $\lambda_m(\bar{A}(\rho_2)) = 0$ with $\rho_2 \in \Omega$ and $\det \bar{A}(\rho_2) = 0$. However, since A_0 is Hurwitz, $\bar{A}_0^{-1}$ exists (by Corollary 1) and we can write:

$$\begin{aligned}0 = \det \bar{A}(\rho_2) &= \det(\bar{A}_0 + \rho_2 \bar{A}_g) \\ &= \det[\bar{A}_0(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g)] \\ &= \det \bar{A}_0 \det(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g).\end{aligned}$$

Since $\det \bar{A}_0 \neq 0$ (A_0 is Hurwitz), it follows necessarily that $\det(I + \rho_2 \bar{A}_0^{-1} \bar{A}_g) = 0$. This contradicts the fact that $\rho_2 \in \Omega$ and $\Omega \subseteq \mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ (see Corollary 3), thus completing the proof. □

Corollary 4: *Given $A_0, A_g \in \mathbb{R}^{n \times n}$ such that A_0 is Hurwitz, the interval $\mathcal{N}(\bar{A}_0^{-1} \bar{A}_g)$ is the largest continuous interval of $\mathbb{R}$ containing the origin for which the matrix $A_0 + \rho A_g$ is Hurwitz.*

In the next section, Theorem 1 is extended so as to reduce the computations involved through the use of the bialternate sum of matrices.

3.2 *Improved stability condition for single-parameter dependent LTI systems*

The application of the stability condition of Theorem 1 is limited owing to the large number of computations required to calculate the inverse of the $n^2 \times n^2$ matrix $\bar{A}_0$, especially when the system is of high order. This limitation can be overcome somewhat by using the guardian map of Remark 2, which involves the determinant of the bialternate sum of a matrix with itself. The resulting improved stability condition requires the calculation of the inverses of an $n \times n$ and an $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ matrix. Using the map induced by the bialternate sum, one can easily obtain the following robust stability condition, which can also be used to synthesize the maximal continuous robust stability interval that includes the origin.

Theorem 2: *Given an open interval Ω in $\mathbb{R}$, and A_0, $A_g \in \mathbb{R}^{n \times n}$, define $\tilde{A}_0 := 2A_0 \star I$ and $\tilde{A}_g := 2A_g \star I$. Then, the following two statements are equivalent*

(i) $0 \in \Omega$ *and* $A(\rho) := A_0 + \rho A_g$ *is Hurwitz for all* $\rho \in \Omega$
(ii) A_0 *is Hurwitz and* $0 \in \Omega \subseteq \mathcal{N}(A_0^{-1} A_g) \cap \mathcal{N}(\tilde{A}_0^{-1} \tilde{A}_g)$.

Proof: First, recall from the definition of the bialternate sum of the matrix $A(\rho)$, that

$$\begin{aligned}\tilde{A}(\rho) &:= 2A(\rho) \star I = (2A_0 \star I) + \rho(2A_g \star I) \\ &= \tilde{A}_0 + \rho \tilde{A}_g.\end{aligned} \tag{19}$$

We can then proceed as follows.

(i) $\Rightarrow$ (ii): If $A(\rho)$ is Hurwitz for all $\rho \in \Omega$ and $0 \in \Omega$, then A_0 is Hurwitz. Then, from Corollary 2, $\det \tilde{A}_0 \neq 0$ and $\tilde{A}_0^{-1}$ exists. Furthermore, since $A(\rho)$ is Hurwitz for all $\rho \in \Omega$ and, using Corollary 2 again, $\det \tilde{A}(\rho) \neq 0$ for all $\rho \in \Omega$. Therefore,

$$\begin{aligned}0 \neq \det(\tilde{A}_0 + \rho \tilde{A}_g) &= \det[\tilde{A}_0(I + \rho \tilde{A}_0^{-1} \tilde{A}_g)] \\ &= \det \tilde{A}_0 \det(I + \rho \tilde{A}_0^{-1} \tilde{A}_g) \quad \forall \rho \in \Omega.\end{aligned}$$

Hence, $\det(I+\rho\tilde{A}_0^{-1}\tilde{A}_g)\neq 0$ for all $\rho\in\Omega$. It follows that

$$\rho\lambda_i(\tilde{A}_0^{-1}\tilde{A}_g)\neq -1 \quad \forall i\in\mathcal{I}_{(1/2)n(n-1)}, \quad \forall\rho\in\Omega. \tag{20}$$

If $\tilde{A}_0^{-1}\tilde{A}_g$ has no real eigenvalues or the only real eigenvalues lie at the origin then $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)=(-\infty,\infty)$ and, trivially, $\Omega\subseteq\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$. If $\tilde{A}_0^{-1}\tilde{A}_g$ has some non-zero real eigenvalues, then the largest continuous interval which includes $\rho=0$ such that $\rho\lambda_i(\tilde{A}_0^{-1}\tilde{A}_g)\neq -1$ for all $i=1,2,\ldots,\frac{1}{2}n(n-1)$ is given by $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$. Hence from (20), $\Omega\subseteq\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$. Furthermore, since $A(\rho)=A_0+\rho A_g$ is Hurwitz for all $\rho\in\Omega$, $\det(A_0+\rho A_g)\neq 0$. This implies that $\det(I+\rho A_0^{-1}A_g)\neq 0$ for all $\rho\in\Omega$, and hence the largest continuous interval which includes $\rho=0$ for which $\rho\lambda_i(A_0^{-1}A_g)\neq -1$ for all $i=1,2,\ldots,n$ is given by $\mathcal{N}(A_0^{-1}A_g)$. Hence $\Omega\subseteq\mathcal{N}(A_0^{-1}A_g)$ and thus finally $\Omega\subseteq\mathcal{N}(A_0^{-1}A_g)\cap\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$.

(ii) ⇒ (i): The proof follows by contradiction. Assume A_0 is Hurwitz and $0\in\Omega\subseteq\mathcal{N}(A_0^{-1}A_g)\cap\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ but suppose $A(\rho)$ is not Hurwitz for all $\rho\in\Omega$. Then, there exists a $\rho_1\in\Omega$ such that $\mathrm{Re}[\lambda_i(A(\rho_1)]\geq 0$ for some $i\in\mathcal{I}_n$. If $\rho_1=0$, the proof is complete since A_0 is assumed Hurwitz. Consequently, and without loss of generality, let us assume that $\rho_1>0$ (the case $\rho_1<0$ being identical). Since $\mathrm{Re}[\lambda_i(A_0)]<0$ for all $i\in\mathcal{I}_n$ and the eigenvalues of $A(\rho)$ change continuously with ρ (see Horn and Johnson (1991), Appendix D), there exists $\rho_2\in(0,\rho_1]\subseteq\Omega$ such that $\mathrm{Re}[\lambda_k(A(\rho_2)]=0$ for some $k\in\mathcal{I}_n$. There are two possibilities:

First, $\lambda_k(A(\rho_2))=0$. This implies that $\det(A_0+\rho_2A_g)=0$ which, in turn, implies that $\det(I+\rho_2A_0^{-1}A_g)=0$. This cannot be satisfied when $\rho_2\in\Omega$ since $\Omega\subseteq\mathcal{N}(A_0^{-1}A_g)$, hence we get a contradiction. Second, $\lambda_k(A(\rho_2))=j\omega$ and $\omega\neq 0$. Since $A(\rho_2)\in\mathbb{R}^{n\times n}$, there exists $k'\in\mathcal{I}_n$ such that $\lambda_{k'}(A(\rho_2))=-j\omega$ and hence by Lemma 2, there exists $m\in\mathcal{I}_{\frac{1}{2}n(n-1)}$ such that $\lambda_m(\tilde{A}(\rho_2))=\lambda_k(A(\rho_2))+\lambda_{k'}(A(\rho_2))=0$. Consequently, $\det\tilde{A}(\rho_2)=0$ with $\rho_2\in\Omega$. However, since A_0 is Hurwitz, by Corollary 2, $\tilde{A}_0^{-1}$ exists and we can write

$$\begin{aligned}0&=\det\tilde{A}(\rho_2)=\det(\tilde{A}_0+\rho_2\tilde{A}_g)\\&=\det\tilde{A}_0\det(I+\rho_2\tilde{A}_0^{-1}\tilde{A}_g)\quad\text{for some}\quad\rho_2\in\Omega.\end{aligned}$$

This implies that $\rho_2\lambda_i(\tilde{A}_0^{-1}\tilde{A}_g)=-1$ for some $i\in\mathcal{I}_{(1/2)n(n-1)}$. This condition cannot be satisfied when $\rho_2\in\Omega$ and $\Omega\subseteq\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$, thus completing the proof. □

Corollary 5: *Let $A_0, A_g\in\mathbb{R}^{n\times n}$ with A_0 Hurwitz. Define $\tilde{A}_0:=2A_0\star I$ and $\tilde{A}_g:=2A_g\star I$. Then $\mathcal{N}(A_0^{-1}A_g)\cap\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g)$ is the largest continuous interval of $\mathbb{R}$ containing the origin for which the matrix $A_0+\rho A_g$ is Hurwitz.*

The following result follows immediately from Corollary 4 and Corollary 5.

Corollary 6: *Let $A_0, A_g\in\mathbb{R}^{n\times n}$ with A_0 Hurwitz. Then,*

$$\mathcal{N}(\bar{A}_0^{-1}\bar{A}_g)=\mathcal{N}(A_0^{-1}A_g)\cap\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g). \tag{21}$$

4. Complete stability domain of single parameter-dependent LTI systems

4.1 *Stability condition using the kronecker sum*

Theorems 1 and 2 give the maximal continuous stability interval in $\mathbb{R}$ which includes the origin. These two theorems, nonetheless, provide only sufficient conditions for a single-parameter dependent matrix to be Hurwitz, because in many cases the maximal stability interval around the origin is not the complete stability domain. Additionally, the requirement that A_0 is Hurwitz limits the applicability of Theorems 1 and 2. In this section, we provide a methodology for computing the complete stability domain without requiring A_0 to be Hurwitz.

Theorem 3: *Let $A_0, A_g\in\mathbb{R}^{n\times n}$ with $\det(A_0\oplus A_0)\neq 0$. If there exists a stability domain $\Omega\subseteq\mathbb{R}$ such that $A_0+\rho A_g$ is Hurwitz for all $\rho\in\Omega$, then this domain Ω is an open interval or a union of disjoint open intervals of $\mathbb{R}$ and the number of such intervals is finite. Furthermore, this number is no larger than (n^2+1).*

Proof: Since the eigenvalues $\lambda_j(A_0+\rho A_g)$, $j=1,2,\ldots,n$ vary continuously with the parameter ρ, if $A_0+\rho_iA_g$ is Hurwitz for some $\rho_i\in\Omega$, then there exists $\delta>0$ such that $A_0+\rho A_g$ is Hurwitz for all $\rho\in(\rho_i-\delta,\rho_i+\delta)$. Therefore, if Ω exists, it must be an open interval or a disjoined union of open intervals. Let Ω be expressed as $\Omega=\bigcup_{i=1}^m(\underline{\rho}_i,\bar{\rho}_i)$ (with the possibility that $\rho_{-1}=-\infty$ and $\bar{\rho}_m=+\infty$), where $\underline{\rho}_i<\bar{\rho}_i$ and m is the (perhaps infinite) number of the disjoint open intervals composing Ω. Since Ω is the entire stability region of ρ, it follows that for each $\underline{\rho}_i\in\mathbb{R}$, $i\in\mathcal{I}_m$, $\mathrm{Re}[\lambda_k(A_0+\underline{\rho}_iA_g)]=0$ for some $k\in\mathcal{I}_n$. Hence, by Lemma 2, $\det(\bar{A}_0+\underline{\rho}_i\bar{A}_g)=0$. Since $\det(A_0\oplus A_0)=\det\bar{A}_0\neq 0$, $\bar{A}_0^{-1}$ exists. Thus,

$$\det(I+\underline{\rho}_i\bar{A}_0^{-1}\bar{A}_g)=0,\quad\forall i\in\mathcal{I}_m \\ (\text{excluding } i=1 \text{ if } \underline{\rho}_1=-\infty). \tag{22}$$

Since this equation has a finite number of solutions, $m<\infty$. By Definition 3 and equation (22), it follows that $\underline{\rho}_i\in\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, $i\in\mathcal{I}_m$. Similarly, one can show that $\bar{\rho}_i\in\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, $i\in\mathcal{I}_m$. Therefore,

$$m\leq\mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g)-1. \tag{23}$$

From the definition of the set $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ it is clear that $\mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g)\leq n^2+2$. Using (23) it follows that $m\leq n^2+1$. □

Theorem 4: *Let $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$. Define $\bar{A}_0 := A_0 \oplus A_0$ and $\bar{A}_g := A_g \oplus A_g$ and let $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g) - 2$. Suppose there exists a real number $\rho_i \in (r_i, r_{i+1})$, where r_i, r_{i+1} are consecutive members of $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, $i \in \mathcal{I}_p^0$ such that $A_0 + \rho_i A_g$ is Hurwitz. Then $A_0 + \rho A_g$ is Hurwitz for all $\rho \in (r_i, r_{i+1})$.*

Proof: The map ν_1: $\mathbb{R}^{n\times n} \to \mathbb{R}$ given in (5) is a guardian map for the set $\mathcal{A}$ of stable $n \times n$ matrices (see Barmish (1994), page 303). Let $A(\rho) := A_0 + \rho A_g$. According to the definition of $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, if r_i, r_{i+1} are consecutive members of $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$, such that $r_i, r_{i+1} \neq \pm\infty$, then $\nu_1(A(r_i)) = 0$ and $\nu_1(A(r_{i+1})) = 0$. Furthermore, $\nu_1(A(\rho)) \neq 0$ for all $r_i < \rho < r_{i+1}$. Let now some $\rho_i \in (r_i, r_{i+1})$ such that $A_0 + \rho_i A_g$ is Hurwitz. Since ν_1 is a guardian map, it follows that $A(\rho)$ is Hurwitz for all $\rho \in (r_i,\ r_{i+1})$.

Theorem 5: *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$, let $\bar{A}_0 := A_0 \oplus A_0$, $\bar{A}_g := A_g \oplus A_g$ and let $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g) - 2$. Define the open set*

$$\Omega_\epsilon := \bigcup_{i\in\mathcal{I}} (r_i,\ r_{i+1}), \tag{24}$$

where r_i, r_{i+1} are consecutive members of $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ and the index set $\mathcal{I}$ is given by

$$\mathcal{I} := \{ i \in \mathcal{I}_p^0 :\ A_0 + \rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i,\ r_{i+1}),\ r_i, r_{i+1} \text{ consecutive members of } \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g) \}. \tag{25}$$

Then, $A_0 + \rho A_g$ is Hurwitz if and only if $\rho \in \Omega_\epsilon$.

Proof: To prove sufficiency, choose $\rho \in \Omega_\epsilon$ and let $\rho \in (r_i, r_{i+1})$ for some $i \in \mathcal{I}$. From Theorem 4 and the fact that $A_0 + \rho_i A_g$ is Hurwitz for $\rho_i \in (r_i, r_{i+1})$, it follows that $A_0 + \rho A_g$ is Hurwitz. To prove necessity, assume that $A_0 + \rho A_g$ is Hurwitz. It follows that $\rho \notin \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$. Therefore, there exists $i \in \mathcal{I}_p^0$ such that $r_i < \rho < r_{i+1}$. Since $A_0 + \rho A_g$ is Hurwitz, it follows that $i \in \mathcal{I}$. Hence, $\rho \in (r_i, r_{i+1}) \subseteq \Omega_\epsilon$. □

Remark 4: Theorem 5 can be used to find the exact stability domain Ω_ϵ for a parameter-dependent matrix $A(\rho) = A_0 + \rho A_g$ where $\rho \in \Omega$ and $A_0, A_g \in \mathbb{R}^{n\times n}$. The procedure involves four steps.

1. Calculate $\bar{A}_0, \bar{A}_g$ and the eigenvalues of the matrix $\bar{A}_0^{-1}\bar{A}_g$.
2. Choose the real non-zero eigenvalues (ignoring repetitions) of the matrix $\bar{A}_0^{-1}\bar{A}_g$ and construct the set $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g)$ according to Definition 3.
3. Check whether the matrix $A_0 + \rho_i A_g$ is Hurwitz for some $\rho_i \in (r_i, r_{i+1})$, $i \in \mathcal{I}_p^0, p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g) - 2$, and construct the index set $\mathcal{I}$.
4. Let Ω_ϵ as in (24).

4.2 *Stability condition using the bialternate sum*

The need for intensive numerical calculations in order to calculate the inverse and the eigenvalues of the $n^2 \times n^2$ matrix $\bar{A}_0 = \det(A_0 \oplus A_0)$ (Step 1 in the algorithm of Remark 4) limits the applicability of Theorem 5. This limitation can be overcome somewhat using a map induced by the bialternate sum of a matrix with itself (see (14) and Remark 2).

Theorem 6: *Let $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$. If there exists a stability domain $\Omega \subseteq \mathbb{R}$ such that $A_0 + \rho A_g$ is Hurwitz for all $\rho \in \Omega$, then this domain Ω is an open interval or a union of disjointed open intervals of $\mathbb{R}$, and the number of such intervals is finite. Furthermore, this number is no larger than $\frac{1}{2}(n^2 + n + 2)$.*

Proof: The fact that the stability domain is an open interval or a union of disjoined open intervals follows from the proof of Theorem 3. Ω can therefore be expressed, as explained before, as $\Omega = \bigcup_{i=1}^{m} (\underline{\rho}_i, \bar{\rho}_i)$, where $\underline{\rho}_i < \bar{\rho}_i$ and m is the number of the disjointed open intervals composing Ω. Since Ω is the entire stability region of ρ, it follows that for every $\underline{\rho}_i \in \mathbb{R}$, $i \in \mathcal{I}_m$, $\mathrm{Re}[\lambda_k(A_0 + \underline{\rho}_i A_g)] = 0$ for some $k \in \mathcal{I}_n$. Next, notice that the condition $\det(A_0 \oplus A_0) \neq 0$ implies that $\det A_0 \neq 0$ and $\det\tilde{A}_0 \neq 0$. Following now an argument similar to the one in the proof of Theorem 2, one can show that $\underline{\rho}_i \in \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ for all $i \in \mathcal{I}_m$. Similarly, one can show that for every $\bar{\rho}_i \in \mathbb{R}, i \in \mathcal{I}_m$, $\mathrm{Re}[\lambda_k(A_0 + \bar{\rho}_i A_g)] = 0$ for some $k \in \mathcal{I}_n$. Then $\bar{\rho}_i \in \mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ for all $i \in \mathcal{I}_m$. Therefore,

$$m \leq (\mathcal{B}(A_0^{-1}A_g) \cup \mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g))^{\#} - 1. \tag{26}$$

From the definition of the sets $\mathcal{B}(A_0^{-1}A_g)$ and $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$, it is clear that $\mathcal{B}^{\#}(A_0^{-1}A_g) \leq n + 2$ and $\mathcal{B}^{\#}(\tilde{A}_0^{-1}\tilde{A}_g) \leq \frac{1}{2}n(n-1) + 2$. Using (26) and the fact that $\{-\infty, +\infty\}$ belongs to both sets, it follows that $m \leq \frac{1}{2}(n^2 + n + 2)$.

Remark 5: Since $\frac{1}{2}(n^2 + n + 2) \leq (n^2 + 1)$ for all $n \geq 1$, Theorem 6 gives a better estimate for the number of stability intervals than Theorem 3.

Theorem 7: *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$ let $\tilde{A}_0 := 2A_0 \star I$ and $\tilde{A}_g := 2A_g \star I$. Let $p = (\mathcal{B}\times(\tilde{A}_0^{-1}\tilde{A}_g)\bar{\cup}\mathcal{B}(A_0^{-1}A_g))^{\#} - 2$. Suppose there exists a real number $\rho_i \in (r_i, r_{i+1})$, where r_i, r_{i+1} are consecutive members of $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)\bar{\cup}\mathcal{B}(A_0^{-1}A_g)$, $i \in \mathcal{I}_p^0$, such that $A_0 + \rho_i A_g$ is Hurwitz. Then $A_0 + \rho A_g$ is Hurwitz for all $\rho \in (r_i, r_{i+1})$.*

Proof : The map ν_2: $\mathbb{R}^{n\times n} \to \mathbb{R}$ given in (14) is a guardian map for the set $\mathcal{A}$ of stable $n \times n$ matrices (see Remark 2). Let $A(\rho) := A_0 + \rho A_g$. Then,

$$\begin{aligned}\nu_2(A(\rho)) &= \det A(\rho)\det\tilde{A}(\rho)\\ &= \det(A_0+\rho A_g)\det(\tilde{A}_0+\rho\tilde{A}_g)\\ &= \det A_0 \det\tilde{A}_0 \det(I+\rho A_0^{-1}A_g)\det(I+\rho\tilde{A}_0^{-1}\tilde{A}_g).\end{aligned}$$

According to the definition of $\mathcal{B}(A_0^{-1}A_g)$ and $\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$, if $r_i, r_{i+1} \in \mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ and $r_i, r_{i+1} \neq \pm\infty$, $\nu_2(A(r_i)) = 0$ and $\nu_2(A(r_{i+1})) = 0$. Furthermore, $\nu_2(A(\rho)) \neq 0$ if $r_i < \rho < r_{i+1}$. Let now some $\rho_i \in (r_i, r_{i+1})$ be such that $A_0 + \rho_i A_g$ is Hurwitz. Since ν_2 is a guardian map, it follows that $A(\rho)$ is Hurwitz for all $\rho \in (r_i, r_{i+1})$. □

Theorem 8: *Given $A_0, A_g \in \mathbb{R}^{n\times n}$ with $\det(A_0 \oplus A_0) \neq 0$, let $\tilde{A}_0 := 2A_0 \star I$, $\tilde{A}_g := 2A_g \star I$ and $p = (\mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g))^{\#} - 2$. Define the open set*

$$\Omega_\epsilon := \bigcup_{i\in\mathcal{I}} (r_i, r_{i+1}), \tag{27}$$

where r_i, r_{i+1} are consecutive members of $\mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$ and the index set $\mathcal{I}$ is given by

$$\mathcal{I} := \Big\{ i \in \mathcal{I}_p^0 \colon A_0 + \rho_i A_g \text{ is Hurwitz for some } \rho_i \in (r_i, r_{i+1}),\ r_i, r_{i+1} \text{ consecutive members of } \mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g) \Big\}. \tag{28}$$

Then, $A_0 + \rho A_g$ is Hurwitz if and only if $\rho \in \Omega_\epsilon$.

Proof: To prove sufficiency, choose $\rho \in \Omega_\epsilon$ and let $\rho \in (r_i, r_{i+1})$ for some $i \in \mathcal{I}$. From Theorem 7 and the fact that $A_0 + \rho_i A_g$ is Hurwitz for $\rho_i \in (r_i, r_{i+1})$, it follows that $A_0 + \rho A_g$ is Hurwitz. To prove necessity, assume that $A_0 + \rho A_g$ is Hurwitz. Since $\nu_2(A(\rho)) = \det A_0 \det\tilde{A}_0 \det(I+\rho A_0^{-1}A_g)\det(I+\rho\tilde{A}_0^{-1}\tilde{A}_g) \neq 0$, it follows that $\rho \notin \mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$. Therefore there must exist $i \in \mathcal{I}_p^0$ such that $r_i < \rho < r_{i+1}$ with r_i, r_{i+1} being consecutive members of $\mathcal{B}(A_0^{-1}A_g)\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g)$. Since $A_0 + \rho A_g$ is Hurwitz, it follows that $i \in \mathcal{I}$ and hence $\rho \in (r_i, r_{i+1}) \subseteq \Omega_\epsilon$.

Remark 6: The computational bottleneck in Theorem 7 is due to the computation of the inverse and the eigenvalues of the matrix $\tilde{A}_0$ of dimension $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$. On the other hand, Theorem 5 requires the inverse and the eigenvalues of the $n^2 \times n^2$ matrix $\bar{A}_0$. Recall that the number of computations required for calculating the inverse of an $n\times n$ matrix is of the order $O(n^3)$, in general, and of order $O(n^{\log 7/\log 2}) = O(n^{2.81})$ at best, using Strassen's method (Strassen, 1969, Bailey *et al.* 1990, Dumitrescu, 1998). Calculation of the eigenvalues of an $n\times n$ matrix, using the QR algorithm (Golub and Van Loan 1989, Geist and Davis 1990) is of complexity $O(n^3)$. Therefore, the complexity of Theorem 5 is of order $O(n^{2^{2.81}}) + O(n^{2^{2.81}}) + O(n^{2^3}) = O(C_1 n^6)$. By the same token, the complexity of Theorem 5 is of order $O(C_2 n^6)$. The gains from the use of Theorem 7 in lieu of Theorem 5 is hidden in the constant C_2. One can easily show that $C_1/C_2 \sim 8/1$.

Alternatively, one can compute the sets $\mathrm{mspec}(\bar{A}_0^{-1}\bar{A}_g)$ or $\mathrm{mspec}(\tilde{A}_0^{-1}\tilde{A}_g)$ in Theorems 5 and 7, while avoiding the computation of the inverse of the matrices $\bar{A}_0$ or $\tilde{A}_0$ by solving a generalized eigenvalue problem for the pair of matrices $(\bar{A}_0, \bar{A}_g)$ or $(\tilde{A}_0, \tilde{A}_g)$, respectively. The solution of the generalized eigenvalue problem for a pair of matrices (A, B) of dimension $n\times n$ using the QZ algorithm, for example (including the initial reduction of the matrix A in Hessenberg form), is of order $O(n^3)$ (Moller and Stewart 1973), which is of the same order as before. The benefits from using the generalized eigenvalue problem formulation stems from the fact that its solution is more numerically stable than computing the inverse of a matrix, especially when the matrix dimensions are large.

5. Generalized stability condition for multi-parameter dependent LTI systems

In this section, the robust stability condition for the following multi-parameter dependent LTI system will be studied

$$\dot{x} = \left(A_0 + \sum_{i=1}^{k} \rho_i A_{g,i}\right)x. \tag{29}$$

Rern *et al.* (1994) give a stability condition for a system of the form (29), however that condition is only sufficient. Saydy *et al.* (1988) used a semi-guardian map (A map ν from the set of $n\times n$ real Hurwitz matrices $\mathcal{A}$ onto $\mathbb{R}$ is a semi-guardian map if it is continuous, not identically zero and $A \in \partial\mathcal{A} \Rightarrow \nu(A) = 0$) to investigate robust stability for the following two-parameter quadratically-dependent matrix over the domain $(r_1, r_2) \in [0,1]\times[0,1]$

$$A(\rho_1, \rho_2) = \sum_{i_1, i_2=0}^{i_1+i_2=m} \rho_1^{i_1}\rho_2^{i_2} A_{i_1, i_2}. \tag{30}$$

The stability test in (Saydy *et al.* 1988) requires the parameter domain to be known a priori. Consequently, the test checks whether the matrix is Hurwitz for all values of the parameters in a given domain. In this section, we extend the results of § 4.2 to synthesize the entire stability region for systems of the form (29).

Lemma 3: *Given the vector* $(\rho_1, \rho_2, \ldots, \rho_k)^T \in \mathbb{R}^k$, $k \geq 2$, *there exists a real number* r *and* $k-1$ *scalars* $\theta_i \in [0, \pi)$, $i = 2, \ldots, k$ *such that*

$$(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta), \quad (31)$$

where $\theta := (\theta_2, \ldots, \theta_k)^T \in [0, \pi)^{k-1}$ *and*

$$v(\theta) := (\cos\theta_2,\ \sin\theta_2\cos\theta_3,\ \sin\theta_2\sin\theta_3\cos\theta_4, \ldots, \sin\theta_2\sin\theta_3\ldots\sin\theta_{k-1}\cos\theta_k, \sin\theta_2\sin\theta_3\cdots\sin\theta_k)^T \in [-1, +1]^k, \quad (32)$$

Proof: The proof follows by induction.

1. Let $k = 2$. Define

$$r = r_2 = \operatorname{sgn}(\rho_2)\sqrt{\rho_1^2 + \rho_2^2} \quad \text{and} \quad \theta_2 = \cos^{-1}\left(\frac{\rho_1}{r_2}\right) \in [0, \pi). \quad (33)$$

Note that $\sin\theta_2 = \rho_2/r_2 \geq 0$. Let $v(\theta) = (\cos\theta_2, \sin\theta_2)^T$. It follows that $rv(\theta) = (\rho_1, \rho_2)$ as required.

2. Suppose for $k > 2$, we have that

$$(\rho_2, \rho_3, \ldots, \rho_{k+1}) = r_k\big(\cos\theta_2^k, \sin\theta_2^k\cos\theta_3^k, \ldots, \sin\theta_2^k\sin\theta_3^k\ldots\sin\theta_{k-1}^k\cos\theta_k^k, \sin\theta_2^k\sin\theta_3^k\ldots\sin\theta_k^k\big) = r_k v(\theta^k)^T \quad (34)$$

where $\theta^k := (\theta_2^k, \theta_3^k, \ldots, \theta_k^k) \in [0, \pi)^{k-1}$.

3. For $k+1$, we have that

$$(\rho_1, \rho_2, \ldots, \rho_k, \rho_{k+1}) = (\rho_1, (\rho_2, \ldots, \rho_k, \rho_{k+1})) = \big(\rho_1, r_k v(\theta^k)^T\big). \quad (35)$$

Let $r_{k+1} = \operatorname{sgn}(r_k)\sqrt{\rho_1^2 + r_k^2}$ and $\theta_2^{k+1} = \cos^{-1}(\rho_1/r_{k+1}) \in [0, \pi)$. Note that $\sin\theta_2^{k+1} = r_k/r_{k+1} \geq 0$. It follows that

$$\begin{aligned}(\rho_1, \rho_2, \ldots, \rho_k, \rho_{k+1}) &= r_{k+1}\big(\cos\theta_2^{k+1}, \sin\theta_2^{k+1}v(\theta^k)^T\big) \\ &= r_{k+1}\big(\cos\theta_2^{k+1}, \sin\theta_2^{k+1}\cos\theta_2^k, \ldots, \sin\theta_2^{k+1}\sin\theta_2^k, \ldots, \sin\theta_{k-1}^k\cos\theta_k^k, \sin\theta_2^{k+1}\sin\theta_2^k, \ldots, \sin\theta_k^k\big).\end{aligned}$$

Letting now $\theta_{i+1}^{k+1} = \theta_i^k$, $i = 2, 3, \ldots, k$, the proof is complete. □

We now use the stability condition of Theorem 5, to obtain the following stability condition for the LTIPD system in (29).

Theorem 9: *Given* $A_0, A_{g,i} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$ *with* $\det(A_0 \oplus A_0) \neq 0$, *define* $\bar{A}_0 := A_0 \oplus A_0$ *and let* $(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta)$ *as in Lemma 3. Let* $p = \mathcal{B}^{\#}(\bar{A}_0^{-1}\bar{A}_g(\theta)) - 2$ *where* $\bar{A}_g(\theta) := A_g(\theta) \oplus A_g(\theta)$, $A_g(\theta) := \sum_{i=1}^k A_{g,i}v_i(\theta)$ *and* $v_i(\theta)$ *are the components of vector* $v(\theta)$. *Then define the following open set:*

$$\Omega_\epsilon(\theta) = \bigcup_{i\in\mathcal{I}(\theta)} (r_i, r_{i+1}), \quad (36)$$

where r_i, r_{i+1} *are consecutive members of* $\mathcal{B}(\bar{A}_0^{-1}\bar{A}_g(\theta))$ *and the index set* $\mathcal{I}(\theta)$ *is given by*

$$\mathcal{I}(\theta) = \big\{i \in \mathcal{I}_p^0\colon A_0 + r_i'A_g(\theta) \text{ is Hurwitz for some } r_i' \in (r_i, r_{i+1}),\ r_i, r_{i+1} \text{ consecutive members of } \mathcal{B}(\bar{A}_0^{-1}\bar{A}_g(\theta))\big\}.$$

Furthermore, let

$$\Omega_\epsilon' := \bigcup_{\theta\in[0,\pi)^{k-1}} \big\{y(\theta) \in \mathbb{R}^k\colon y(\theta) = rv(\theta),\ r \in \Omega_\epsilon(\theta)\big\}. \quad (37)$$

Then $A_0 + \sum_{i=1}^k \rho_i A_{g,i}$ *is Hurwitz if and only if* $(\rho_1, \ldots, \rho_k)^T \in \Omega_\epsilon'$.

Proof: Applying Lemma 3, the vector $(\rho_1, \ldots, \rho_k)^T \in \mathbb{R}^k$ can be expressed as $(\rho_1, \ldots, \rho_k)^T = r(v_1(\theta), \ldots, v_k(\theta))^T$. The system matrix in equation (29) can then be rewritten as

$$A_0 + \sum_i^k \rho_i A_{g,i} = A_0 + r\sum_{i=1}^k A_{g,i}v_i(\theta) = A_0 + rA_g(\theta). \quad (38)$$

When the angle vector $\theta \in [0, \pi)^{k-1}$ is given, the system matrix in (38) is a single-parameter matrix which dependents on $r \in \mathbb{R}$. Applying Theorem 5, the complete stability domain for r in the direction θ can be calculated as in (36). The set defined by (37) is the union of the exact stability domains for the parameter r for each $\theta \in [0, \pi)^{k-1}$. Therefore Ω_ϵ' is the exact stability domain for $(\rho_1, \rho_2, \ldots, \rho_k)^T \in \mathbb{R}^k$. □

In a similar manner, the entire stability domain using the guardian map induced by the bialternate sum for multi-parameter dependent systems is given below.

Theorem 10: *Given* $A_0, A_{g,i} \in \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$ *with* $\det(A_0 \oplus A_0) \neq 0$, *define* $\tilde{A}_0 := 2A_0 \star I$ *and let* $(\rho_1, \rho_2, \ldots, \rho_k)^T = rv(\theta)$ *as in Lemma 3. Let* $p = \big(\mathcal{B}(A_0^{-1}A_g(\theta))\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta))\big)^{\#} - 2$ *where* $\tilde{A}_g(\theta) := 2A_g(\theta) \star I_n$, $A_g(\theta) := \sum_{i=1}^k A_{g,i}v_i(\theta)$ *and* $v_i(\theta)$ *are the components of vector* $v(\theta)$. *Then define the following open set:*

$$\Omega_\epsilon(\theta) = \bigcup_{i\in\mathcal{I}(\theta)} (r_i, r_{i+1}), \quad (39)$$

where r_i, r_{i+1} *are consecutive members of* $\mathcal{B}(A_0^{-1}A_g(\theta))\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta))$ *and the index set* $\mathcal{I}(\theta)$ *is given by*

$$\begin{aligned}\mathcal{I}(\theta) = \big\{ i \in \mathcal{I}_p^0\text{: } & A_0 + r_i' A_g(\theta) \text{ is Hurwitz for} \\ & \text{some } r_i' \in (r_i, r_{i+1}), \\ & r_i, r_{i+1} \text{ consecutive members of} \\ & \mathcal{B}(A_0^{-1}A_g(\theta))\bar{\cup}\mathcal{B}(\tilde{A}_0^{-1}\tilde{A}_g(\theta))\big\}.\end{aligned}$$

Furthermore, let

$$\Omega_\epsilon' := \bigcup_{\theta\in[0,\pi)^{k-1}} \big\{ y(\theta) \in \mathbb{R}^k\text{: } y(\theta) = rv(\theta),\ r \in \Omega_\epsilon(\theta)\big\}. \quad (40)$$

Then $A_0 + \sum_{i=1}^k \rho_i A_{g,i}$ *is Hurwitz if and only if* $(\rho_1, \ldots, \rho_k)^T \in \Omega_\epsilon'$.

Proof: The proof is similar to the one of Theorem 7 and thus, it is omitted. □

Theorems 9 and 10 give the complete stability domain for multi parameter-dependent matrices. Moreover, these two results do not require that the matrix A_0 is Hurwitz. The drawback of the approach is that for $k > 1$ the calculation of Ω_ϵ' requires gridding of the space $[0, \pi)^{k-1}$, implying exponential dependence on the problem data. However, this is probably the best we can expect, as it is well known that the problem of determining the exact domain of stability of a multi-parameter LTI system is NP-hard (Blondel and Tsitsiklis 1995, Peaucelle and Arzelier 2001). Given the inherent computational difficulty of the problem (in the multi-variable case) in § 7 we will summarize some observations concerning the computational complexity of the proposed algorithms.

6. Numerical examples

6.1 *Single-parameter case*

In the following examples, the stability domain for the matrix $A(\rho) = A_0 + \rho A_g$, with $A_0, A_g \in \mathbb{R}^{n\times n}$, $\rho \in \Omega$, will be calculated by means of Theorems 1 and 2.

Example 1: Consider the system matrix $A(\rho) = A_0 + \rho A_g$ with

$$A_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of $A(\rho)$ are $\{-1, -1\}$ for all $\rho \in \mathbb{R}$. Hence, the largest stability domain for this example is $(-\infty, +\infty)$.

From Theorem 1, we calculate

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} 0 & -0.5 & -0.5 & 0 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\text{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0, 0, 0, 0\}$. The largest continuous interval of ρ which includes zero and guarantees stability for the matrix $A(\rho)$ is $(-\infty, +\infty)$. This agrees with the eigenvalue analysis. Using Theorem 2, we have $\tilde{A}_0 = -2$, $\tilde{A}_g = 0$, $\tilde{A}_0^{-1}\tilde{A}_g = 0$, and

$$\begin{aligned}&\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) = \mathcal{N}(0) = (-\infty, +\infty) \\ &\mathcal{N}(A_0^{-1}A_g) = \mathcal{N}\left(\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}\right) = (-\infty, +\infty).\end{aligned}$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty, +\infty)$, which coincides with the result from Theorem 1 and the direct eigenvalue analysis.

Example 2: Consider the system matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Since the eigenvalues of matrix $A(\rho)$ are $\{-2 \pm \rho i\}$, the largest stability interval of ρ is $(-\infty, +\infty)$. Using Theorem 1, one obtains

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} 0 & -0.25 & -0.25 & 0 \\ 0.25 & 0 & 0 & -0.25 \\ 0.25 & 0 & 0 & -0.25 \\ 0 & 0.25 & 0.25 & 0 \end{bmatrix},$$

and $\text{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0, 0, -0.5i, 0.5i\}$. The largest continuous interval which includes 0 and guarantees stability for $A(\rho)$ is $(-\infty, +\infty)$. Applying Theorem 1, one obtains that $\tilde{A}_0 = -4$, $\tilde{A}_g = 0$, $\tilde{A}_0^{-1}\tilde{A}_g = 0$ and

$$\begin{aligned}&\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) = \mathcal{N}(0) = (-\infty, +\infty) \\ &\mathcal{N}(A_0^{-1}A_g) = \mathcal{N}\left(\begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) \\ &\qquad = \mathcal{N}\left(\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right) = (-\infty, +\infty).\end{aligned}$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty, +\infty)$, which coincides with the result from Theorem 1 and the direct eigenvalue analysis.

Example 3: Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Direct eigenvalue analysis of $A(\rho)$ shows that $\lambda_1 = -2 - \rho$ and $\lambda_2 = -1 - \rho$. Hence $A(\rho)$ is Hurwitz when $\rho > -1$. Using Theorem 1, one obtains

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\mathrm{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{0.5, 0.6667, 0.6667, 1\}$. The largest continuous interval of ρ which includes zero and guarantees stability for $A(\rho)$ is $(-1, \infty)$ according to Theorem 1, which agrees with the eigenvalue analysis. Using Theorem 2, one obtains $\tilde{A}_0 = -3$, $\tilde{A}_g = -2$, $\tilde{A}_0^{-1}\tilde{A}_g = 2/3$ and

$$\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) = \mathcal{N}(2/3) = (-3/2, +\infty)$$
$$\mathcal{N}\left(A_0^{-1}A_g\right) = \mathcal{N}\left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\right) = (-1, +\infty).$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-1, +\infty) \cap -\frac{3}{2}, +\infty) = (-1, +\infty)$, which coincides with the result by Theorem 1 and the eigenvalue analysis.

Example 4: Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Direct eigenvalue analysis of $A(\rho)$ gives $\lambda_1 = -2 + \rho$, $\lambda_2 = -1 - \rho$, hence $A(\rho)$ is Hurwitz when $-1 < \rho < 2$. Using Theorem 1,

$$\bar{A}_0^{-1}\bar{A}_g = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $\mathrm{mspec}(\bar{A}_0^{-1}\bar{A}_g) = \{-0.5, 0, 0, 1\}$. The largest continuous interval of ρ which includes 0 that guarantees stability for $A(\rho)$ is $(-1, 2)$, which agrees with the eigenvalue analysis. Using Theorem 2, one obtains $\tilde{A}_0 = -3$, $\tilde{A}_g = 0$, $\tilde{A}_0^{-1}\tilde{A}_g = 0$ and

$$\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) = \mathcal{N}(0) = (-\infty, +\infty)$$
$$\mathcal{N}\left(A_0^{-1}A_g\right) = \mathcal{N}\left(\begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix}\right) = (-1, +2).$$

The stability domain is $\mathcal{N}(\tilde{A}_0^{-1}\tilde{A}_g) \cap \mathcal{N}(A_0^{-1}A_g) = (-\infty, +\infty) \cap (-1, +2) = (-1, +2)$, which coincides with the result of Theorem 1 and the direct eigenvalue analysis.

In order to compare Theorems 1 and 2 and Theorems 5 and 7 we consider the following three examples.

Example 5: Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix} 62.563 & -121.34 & -217.75 & -111.86 & 309.77 \\ -64.806 & 123.09 & 214.78 & 115.44 & -319.39 \\ -7.6195 & 19.044 & 25.231 & 21.651 & -52.037 \\ 4.3314 & 1.9045 & -9.3643 & -3.8729 & 1.8837 \\ -44.276 & 91.392 & 150.51 & 85.741 & -235.05 \end{bmatrix},$$

$$A_g = \begin{bmatrix} -5.9399 & -21.242 & 23.809 & 11.251 & -6.9852 \\ -8.8534 & -35.439 & 24.579 & 22.030 & 0.98018 \\ -10.049 & -21.452 & 20.026 & 13.640 & -4.3113 \\ 0.77706 & -24.138 & 15.174 & 9.3705 & 1.5890 \\ 2.2073 & -14.157 & 13.148 & 3.8678 & -8.9941 \end{bmatrix}.$$

Notice that for this example, both A_0 and A_g are Hurwitz, but $A_0 + A_g$ is not Hurwitz. It is clear that in this case the maximal stability interval Ω_ϵ is composed of at least two disjoint open intervals. Theorems 1 and 2 give the maximal continuous stability domain as $(-0.02306, 0.11802)$, which includes the origin. Theorems 5 and 8, on the other hand, give the whole stability domain, which is equal to $(-0.02306, 0.11802) \cup (4.30818, +\infty)$.

Example 6: Consider the matrix $A(\rho) = A_0 + \rho A_g$, where

$$A_0 = \begin{bmatrix} -10.64 & 3.395 & 8.841 & 4.558 & -10.25 \\ -11.28 & -0.1536 & 14.67 & 9.852 & -13.53 \\ 0.7320 & 3.811 & -0.6074 & 2.408 & -10.44 \\ -12.14 & 4.938 & 9.649 & 1.152 & -6.297 \\ -11.66 & 6.451 & 11.70 & 9.453 & -17.28 \end{bmatrix},$$

$$A_g = \begin{bmatrix} -110.9 & -247.0 & 162.4 & -57.61 & 194.2 \\ 241.82 & 731.3 & -446.6 & 87.68 & -511.8 \\ 366.8 & 987.5 & -617.4 & 181.9 & -777.1 \\ 385.3 & 1118.5 & -666.7 & 137.4 & -809.4 \\ 100.8 & 237.1 & -142.4 & 57.89 & -234.3 \end{bmatrix}$$

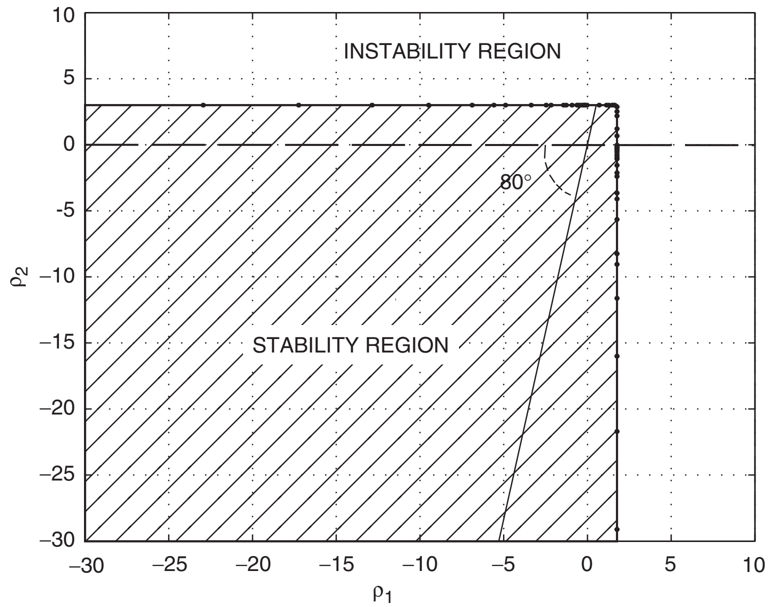

Figure 1. Robust stability domain for Example 7.

Theorems 1 and 2 give the maximal stability interval around the origin, which is $(-0.04632, 0.00241)$. Theorems 5 and 8 give the exact stability domain, which is $(-0.04632, 0.00241) \cup (4.2279, +\infty)$.

6.2 *Multi-parameter case*

Example 7: This example is from Rern *et al.* (1994). Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A(\rho) = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} + \rho_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \rho_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \qquad (41)$$

The exact robust stability region for this problem is $(-\infty, 1.75) \times (-\infty, 3)$ (see Rern *et al.* 1994). Theorem 9 or Theorem 10, give this exact stability domain as seen in figure 1. The figure shows how the exact two-dimensional stability region was constructed. This was done by considering the maximal one-dimensional stability region along a particular direction, which for the case, say, $\theta = 80^\circ$ is

$$(\rho_1\ \rho_2)^T \in \{\rho = r\,v(\theta) \colon r \in (-\infty, 3.0463),\ v(\theta) = (\cos 80^\circ\ \sin 80^\circ)^T\},$$

and repeating this over all directions $\theta \in [0, \pi)$.

Example 8: Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.15087 & 0.86001 & 0.49655 \\ 0.69790 & 0.85366 & 0.89977 \\ 0.37837 & 0.59356 & 0.82163 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.64491 & 0.34197 & 0.53408 \\ 0.81797 & 0.28973 & 0.72711 \\ 0.66023 & 0.34119 & 0.30929 \end{bmatrix}.$$

Both Theorem 9 and Theorem 10 give the same stability domain for the matrix $A(\rho)$, which is shown in figure 2.

Example 9: Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.98017 & -0.0033774 & -0.35993 \\ 0.57772 & -0.57207 & 0.92020 \\ -0.12268 & 0.28698 & 0.45326 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.26414 & -0.18023 & -0.86232 \\ 0.73370 & 1.3001 & 1.0177 \\ -0.69616 & 0.55000 & 0.38635 \end{bmatrix}.$$

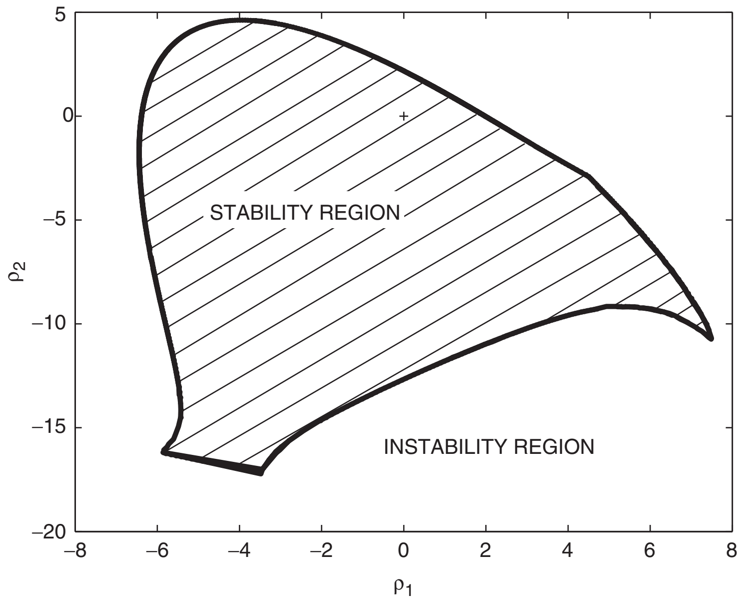

Figure 2. Robust stability domain for Example 8.

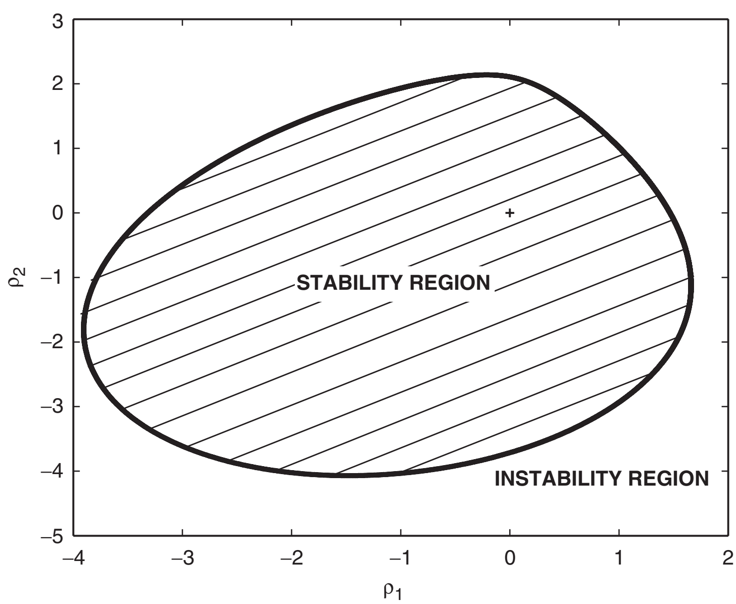

Figure 3. Robust stability domain for Example 9.

It can be easily verified that both Theorems 9 and 10 give the same stability domain for the matrix $A(\rho)$, which is depicted in figure 3.

Example 10: Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.916 & -0.8119 & -0.2168 \\ -0.6863 & -0.1001 & -0.4944 \\ -0.1673 & 0.7383 & -0.2912 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.215 & 1.664 & -2.209 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.199 & 0.6493 & -0.2214 \end{bmatrix}. \quad (42)$$

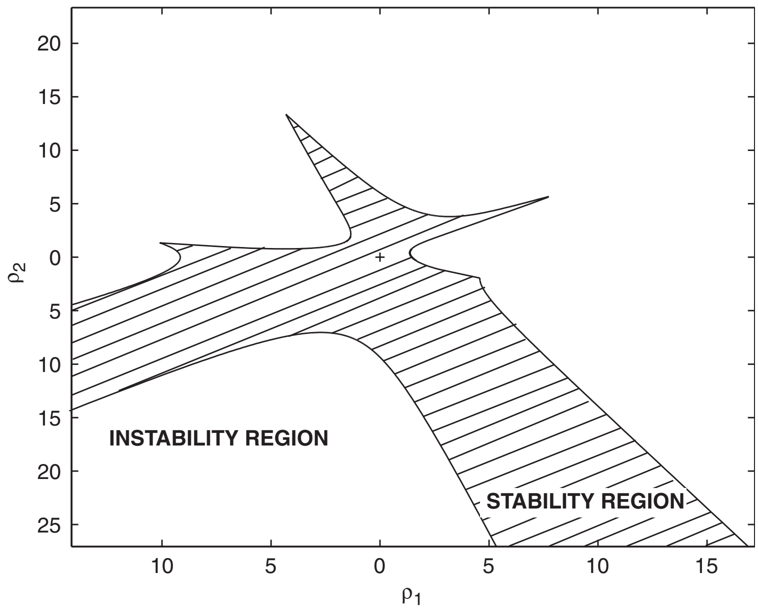

Figure 4. Robust stability domain for Example 10.

Both Theorem 9 and Theorem 10 give the same stability domain for the matrix $A(\rho)$, shown in figure 4.

Example 11: Consider the matrix $A(\rho) = A_0 + \rho_1 A_1 + \rho_2 A_2$, where

$$A_0 = \begin{pmatrix} 62.56 & -121.3 & -217.7 & -111.9 & 309.7e+002 \\ -64.81 & 123.1 & 214.78 & 115.4 & -319.4 \\ -7.619 & 19.04 & 25.23 & 21.651 & -52.04 \\ 4.331 & 1.904 & -9.364 & -3.873 & 1.884 \\ -44.28 & 91.39 & 150.5 & 85.74 & -235.0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} -0.1340 & 0.1139 & 0.2959 & 0.03392 & 0.2288 \\ 0.1747 & -0.2621 & 0.1509 & 0.2436 & 0.2165 \\ 0.1528 & 0.2313 & -0.06069 & 0.2725 & 0.1955 \\ 0.02228 & 0.09418 & 0.2484 & -0.2981 & 0.2262 \\ 0.05797 & 0.1914 & 0.2753 & 0.03664 & -0.1461 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -5.940 & -21.24 & 23.81 & 11.25 & -6.985 \\ -8.853 & -35.44 & 24.58 & 22.03 & 0.9802 \\ -10.05 & -21.45 & 20.03 & 13.64 & -4.311 \\ 0.7771 & -24.14 & 15.17 & 9.370 & 1.589 \\ 2.207 & -14.16 & 13.15 & 3.868 & -8.994 \end{pmatrix}.$$

Again, both Theorems 9 and 10 give the same stability domain for the matrix $A(\rho)$, which is shown figure 5. In this case, the two-dimensional parameter stability space is composed of two disconnected sets. The area close the origin is zoomed in and is depicted in figure 5(b).

7. Computational complexity

The proposed method suffers from the "curse of dimensionality" (the term typically reflects the fact that the problem scales exponentially – as opposed to polynomially – with the problem data) since it requires gridding of the parameter space (in the multi-parameter case). Although some improvements are possible (see Remark 6) the unfavorable growth of the problem complexity is nonetheless unavoidable since the problem is known to be NP-hard (Blondel and Tsitsiklis 1995, Peaucelle and Arzelier 2001). In this section, via several numerical examples, we provide some insights of the complexity of the methods in terms of the dimension of the matrices involved, n, and the dimension of the parameter vector, m.

Specifically, several examples were conducted for several values of n and m. For $m > 1$ gridding of the parameter space is performed using 50 points in each dimension. The computation times (cpu-seconds) were recorded and are shown in table 1. All computations were performed on a 2.0GHz/525RAM Wintel PC running MATLAB version 6.5.0.18091 3a Release 13.

From table 1 it is evident that the complexity is dominated by the exponential growth owing to the gridding of the parameter space.

8. Conclusions

In this paper we address the problem of stability for Linear Time Invariant Parameter Dependent (LTIPD) systems. We extend previous results in the

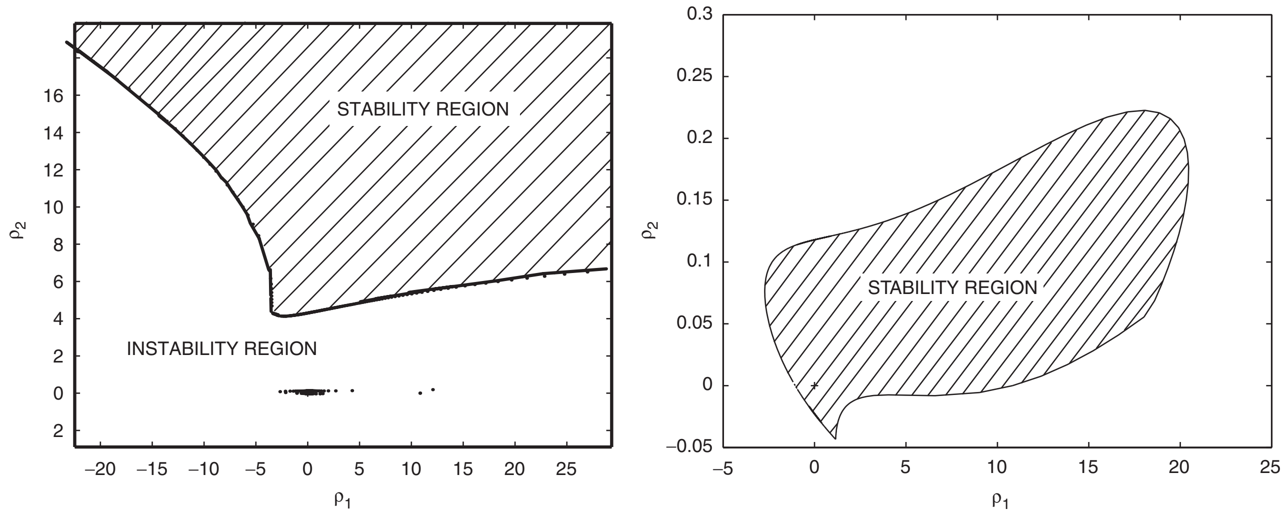

Figure 5. Robust stability domain for the Example 11.

Table 1. Comparison of CPU times (sec) in terms of the problem data.

	Theorem 8			Theorem 9		
	$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
$n=2$	0.000344	0.05620	1.918143	0.000180	0.017155	0.719733
$n=3$	0.000718	0.069767	3.069083	0.000338	0.024211	1.113533
$n=4$	0.001464	0.103167	4.194083	0.000566	0.038282	1.929333
$n=5$	0.002919	0.158367	7.120857	0.001020	0.056049	2.9842
$n=6$	0.005803	0.277567	12.34900	0.001740	0.090268	4.8782
$n=7$	0.010699	0.488567	24.5470	0.002724	0.140683	7.593
$n=8$	0.019291	0.879500	43.329	0.004437	0.219927	12.391
$n=9$	0.031115	1.359417	64.876	0.007053	0.347323	18.828
$n=10$	0.052756	2.229417	113.767	0.010610	0.518161	29.344
$n=15$	0.438118	20.563	1030.521	0.063326	3.193182	181.799
$n=20$	2.642591	135.44	–	0.294556	14.735	737.4122
$n=30$	28.28175	1420.12	–	3.59000	192.299	–
$n=50$	1626.8020	–	–	106.798	–	–

literature and derive conditions that can be used to compute the exact stability region in the parameter space. Our methodology makes use of the guardian maps induced by the Kronecker and the bialternate sum of a matrix with itself. Although both these maps can be used (yielding identical results), the latter has the benefit of a reduced number of computations. Both single-parameter and multi-parameter LTIPD systems are treated. The computational complexity of both approaches in terms of the problem data is briefly commented upon. In the multi-parameter case, our approach requires gridding which may limit the applicability of the results to low-parameter dimensions. Several examples are presented to demonstrate the applicability of the derived results.

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