Stability Analysis of LPV Time-Delayed Systems

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1 Abstract

This work provides one of the first attempts to derive computationally tractable criteria for analyzing the stability of Linear Parameter Varying (LPV) *time-delayed* systems. We present both delay-independent and delay-dependent stability conditions, which are derived using appropriately selected Lyapunov-Krasovskii functionals. According to the system parameter dependence, these functionals can be selected to obtain increasingly non-conservative results. Gridding techniques may be used to cast these tests as Linear Matrix Inequalities (LMI's). In cases when the system matrices depend affinely or quadratically on the parameter, gridding may be avoided. These LMI's can be solved efficiently using available software. A numerical example of a time-delayed system motivated by a metal removal process is used to demonstrate the theoretical results.

2 LPV Time-Delay Systems

Several linear time-delayed systems depend on parameters whose values are time-varying but not known a priori. Assuming that the parameters enter the system dynamics *without delay*, an LPV time-delayed system has the form

$$\dot{x}(t) = A(\gamma(t))\, x(t) + A_d(\gamma(t))\, x(t-\tau) \tag{1}$$

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In (1) τ is a constant unknown delay with $\tau \in [0, \bar{\tau}]$ and γ is a parameter vector that is assumed to belong to a known polytope Γ. Often, it is also known that the rate of γ belongs to a given polytope, Γ_r. Typically, one is interested in deriving conditions that will guarantee stability for system (1) for all $(\gamma, \dot{\gamma}) \in \Gamma \times \Gamma_r$, and all $\tau \in [0, \bar{\tau}]$. In cases where there are no restrictions in the variation rate of γ we have $\Gamma_r = \mathbb{R}$. In addition, if the stability conditions hold for all $\tau \in [0, \bar{\tau}]$ with $\bar{\tau} < \infty$ then the stability is referred to as *delay-dependent* stability. If the conditions hold for all $\tau \in [0, \infty)$. then the stability is referred to as *delay-independent* stability, since stability is ensured for any amount of delay. In this work we derive both delay-independent and delay-dependent stability conditions. We also restrict our discussion to the scalar parameter LPV case. Our results can be generalized to the case when the parameter γ is a vector, but the derivations become more cumbersome.

The theory of LPV (non-delayed) systems has witnessed an explosion over the recent years. Stability analysis and synthesis results have been reported, for example, in (Becker *et al.*, 1993; Apkarian *et al.*, 1994; Wu *et al.*, 1995; Becker, 1996; Gu, 1997; Lim and How, 1998; Gu, 1999; Wang and Balakrishnan, 1999). The theory of LPV, *time-delayed* systems is less developed, however. Some initial results have been reported in (Wu and Grigoriadis, 2001). In that reference, the authors analyzed a time-delayed LPV system when the state matrices and the time delay are functions of some time-varying parameters that can be measured in real-time. Their analysis uses a Lyapunov-Krasovskii functional in which the kernel of the integral term is parameter independent. In this paper we remove this restriction. On the other hand, Wu and Grigoriadis (2001) also present state-feedback controllers for time-delayed LPV systems that guarantee desired $\mathcal{L}_2$-gain performance. Despite the limited existing results for the analysis and control synthesis for LPV time-delayed systems, there are several cases where time-delayed systems which depend on parameters arise naturally in applications. In milling, for example, the dynamics of the cutting process involve delayed states as well as time-varying parameters (Tlusty, 1985). This particular application provided the motivation of this work.

The first part of this paper presents new delay-independent stability criteria, whereas the second part derives delay-dependent criteria for LPV time-delayed systems. In both cases, these criteria are obtained by application of well-known Lyapunov-Krasovskii stability results (Hale and Lunel, 1993). Since our results are only sufficient, the stability tests may be conservative. To reduce conservatism for the delay-independent stability case we introduce parameter-dependent Lyapunov-Krasovskii functionals. All the stability tests are given in terms of Linear Matrix Inequalities (LMI's). Typically, the resulting LMI's are infinite-dimensional. Thus, gridding and/or relaxation techniques are used to project these LMI's to finite dimensions. Efficient algorithms can then be used to solve these LMI's.

Apart from the recent article of Wu and Grigoriadis (2001), the developments of the present work are the only known results for LPV time-delayed systems. The results in this paper follow closely the corresponding results developed for LTI time-delayed systems (Li and de Souza, 1995; Park, 1999; Zhang *et al.*, 1999). Additional results for LTI time-delayed systems have been developed by Gu (1997,1999)using a discretization scheme. Although these results are necessary and sufficient for the LTI case, they are only sufficient for the uncertain LTI and LPV cases. In addition, they are computationally expensive, and they are not directly extendable to synthesis. Although in this paper we only address the stability of LPV time-delayed systems free of disturbances, it should be mentioned that the results can be easily extended to the analysis of time-delayed LPV systems satisfying an $\mathcal{H}_\infty$ bound.

3 Positive definite functionals

Linear Parameter Varying (LPV) systems can be considered as a special class of Linear Time-Varying (LTV) systems. The main difference with LTV systems is that in LPV systems the time-dependence of the system matrices A and A_d is not known a priori but is given only implicitly by a parameter $\gamma(t)$ which is assumed to be a priori unknown. Stability proofs for LTV (time-delayed) systems hinge on the following well-known facts that we repeat here for completeness. These results can be found, for example, in (Kolmanovskii and Myshkis, 1992; Verriest, 1994).

Let us denote by $\mathcal{C}_\tau$ the set of continuous functions defined over the interval $[-\tau, 0]$ and let $V : \mathbb{R}_+ \times \mathcal{C}_\tau \to \mathbb{R}_+$ be a continuous functional such that $V(t, 0) = 0$. Let also Ω denote the class of scalar, nondecreasing continuous functions α such that $\alpha(r) > 0$ for $r > 0$ and $\alpha(0) = 0$. The functional $V(t, \psi)$ is called *positive definite (negative definite)* if there exist a function $\alpha \in \Omega$ such that $V(t, \psi) \geq \alpha(|\psi(0)|)$ (respectively, $V(t, \psi) \leq -\alpha(|\psi(0)|)$) for all $t \in \mathbb{R}$ and $\psi \in \mathcal{C}_\tau$. It is said to have an *infinitesimal upper bound* if $|V(t, \psi)| \leq \alpha(\sup_t |\psi(t)|)$. The following fact provides the main tool used to show (global) asymptotic stability in this paper.

Theorem 3.1 ((Verriest, 1994; Niculescu *et al.*, 1997)) *Given some $\tau > 0$, assume there exists a positive definite, continuous functional $V : \mathbb{R}_+ \times \mathcal{C}_\tau \to \mathbb{R}_+$, with infinitesimal upper bound whose derivative $\dot{V}$ is a negative definite functional. Then the trivial solution of the LTV, time-delayed system $\dot{x}(t) = A(t)x(t) + A_d(t)x(t - \tau)$ is (globally) uniformly asymptotically stable.*

The following lemma is useful for recognizing positive definite functionals, as the ones used in this paper. In the following, $x_t \in \mathcal{C}_\tau$ denotes the function with domain $[-\tau, 0]$ that coincides with x in the interval $[t-\tau, t]$ i.e., $x_t : [-\tau, 0] \to \mathbb{R}^n$ such that $x_t(\theta) = x(t+\theta)$ for $\theta \in [-\tau, 0]$. In the sequel, $|x|$ will denote the euclidean norm of a vector $x \in \mathbb{R}^n$. Moreover, given a matrix $A(\gamma) \in \mathbb{R}^{n\times m}$, depending continuously on a parameter γ that belongs to a compact set Γ, we denote

$$\|A\|_{\infty,\Gamma} = \left[\max_{\gamma\in\Gamma} \bar{\sigma}^2(A(\gamma))\right]^{1/2} \quad (2)$$

where $\bar{\sigma}(A)$ is the maximum singular value of A. $\|A\|_{\infty,\Gamma}$ is always well-defined since Γ is compact and the singular value is a continuous function of the elements of a matrix (Horn and Johnson, 1985).

Lemma 3.1 *Consider the functional $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \to \mathbb{R}_+$ given by*

$$\begin{aligned} V(t,x_t) &= x^T(t)Px(t) + \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha)]^T P_1 [A(\gamma(\alpha))x(\alpha)] \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &+ \int_{-\tau}^{0}\int_{t+\beta}^{t} [A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2 [A_d(\gamma(\alpha))x(\alpha-\tau)] \ \mathrm{d}\alpha \ \mathrm{d}\beta + \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha) \ \mathrm{d}\alpha \\ &+ \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)] \mathrm{d}\alpha \ \mathrm{d}\beta \end{aligned} \quad (3)$$

where P, P_1, P_2, Q, Y are constant positive-definite matrices, $A(\gamma)$ and $A_d(\gamma)$ are matrices that depend continuously on the parameter $\gamma \in \Gamma$, with Γ compact. Then for every $\tau > 0$ this V is a positive definite functional and it has an infinitesimal upper bound.

Proof. To show that V is a positive definite functional, let $x_t(\theta) = x(t+\theta)$ and notice that

$$V(t,x_t) > x_t^T(0)Px_t(0) \geq \lambda_{\min}(P)|x_t(0)|^2$$

where $\lambda_{\min}(P)$ denotes the minimum eigenvalue of P. In order to show that V has an infinitesimal upper bound, first notice that

$$V_1 = x_t^T(0)Px_t(0) \leq \lambda_{\max}(P)|x_t(0)|^2$$

Also

$$\begin{aligned} V_2 &= \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha)]^T P_1 [A(\gamma(\alpha))x(\alpha)] \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &\leq \int_{-\tau}^{0}\int_{t+\beta}^{t} \lambda_{\max}(P_1)|A(\gamma)x(\alpha)|^2 \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &\leq \int_{-\tau}^{0}\int_{t+\beta}^{t} \lambda_{\max}(P_1)\|A\|_{\infty,\Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &= \lambda_{\max}(P_1)\|A\|_{\infty,\Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \int_{-\tau}^{0}\int_{t+\beta}^{t} \ \mathrm{d}\alpha \ \mathrm{d}\beta \end{aligned}$$

Let now $\lambda_2 = \lambda_{\max}(P_1)\|A\|_{\infty,\Gamma}^2(\frac{1}{2}\tau^2)$. Then

$$V_2 \leq \lambda_2 \max_{\theta\in[-\tau,0]} |x(t+\theta)|^2 = \lambda_2 \max_{\theta\in[-\tau,0]} |x_t(\theta)|^2$$

Moreover,

$$\begin{aligned} V_3 &= \int_{-\tau}^{0}\int_{t+\beta}^{t} [A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2 [A_d(\gamma(\alpha))x(\alpha-\tau)] \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &\leq \int_{-\tau}^{0}\int_{t+\beta}^{t} \lambda_{\max}(P_2)|A_d(\gamma)x(\alpha-\tau)|^2 \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &\leq \int_{-\tau}^{0}\int_{t+\beta}^{t} \lambda_{\max}(P_2)\|A_d\|_{\infty,\Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \ \mathrm{d}\alpha \ \mathrm{d}\beta \\ &= \lambda_{\max}(P_2)\|A_d\|_{\infty,\Gamma}^2 \max_{\theta} |x(t+\theta)|^2 \int_{-\tau}^{0}\int_{t+\beta}^{t} \ \mathrm{d}\alpha \ \mathrm{d}\beta \end{aligned}$$

Let now $\lambda_3 = \lambda_{\max}(P_2)\|A_d\|^2_{\infty,\Gamma}(\frac{1}{2}\tau^2)$. Then

$$V_3 \le \lambda_3 \max_{\theta\in[-2\tau,0]} |x(t+\theta)|^2$$

Similarly,

$$\begin{aligned} V_4 &= \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha)\ \mathrm{d}\alpha \\ &\le \lambda_{\max}(Q)\int_{t-\tau}^{t} \max_{\theta\in[-\tau,0]} |x(t+\theta)|^2\ \mathrm{d}\alpha \\ &= \tau\lambda_{\max}(Q)\max_{\theta\in[-\tau,0]} |x(t+\theta)|^2 = \tau\,\lambda_{\max}(Q)\max_{\theta\in[-\tau,0]} |x_t(\theta)|^2 \end{aligned}$$

Finally,

$$\begin{aligned} V_5 &= \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\mathrm{d}\alpha\ \mathrm{d}\beta \\ &\le \lambda_{\max}(Y)\int_{-\tau}^{0}\int_{t+\beta}^{t} |A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)|^2\mathrm{d}\alpha\ \mathrm{d}\beta \\ &\le \lambda_{\max}(Y)\int_{-\tau}^{0}\int_{t+\beta}^{t} [2|A(\gamma(\alpha))x(\alpha)|^2 + 2|A_d(\gamma(\alpha))x(\alpha-\tau)|^2]\mathrm{d}\alpha\ \mathrm{d}\beta \\ &\le 2\lambda_{\max}(Y)\int_{-\tau}^{0}\int_{t+\beta}^{t} [\|A\|^2_{\infty,\Gamma}\max_{\theta}|x(t+\theta)|^2 + \|A_d\|^2_{\infty,\Gamma}\max_{\theta}|x(t+\theta-\tau)|^2]\mathrm{d}\alpha\ \mathrm{d}\beta \\ &\le \tau^2\lambda_{\max}(Y)[\|A\|^2_{\infty,\Gamma}\max_{\theta}|x(t+\theta)|^2 + \|A_d\|^2_{\infty,\Gamma}\max_{\theta}|x(t+\theta-\tau)|^2] \end{aligned}$$

From the previous inequalities for V_i $(i = 1,\ldots,5)$ it follows that there exist constants c_0, c_1, c_2, c_3 such that

$$\begin{aligned} V(t,x_t) &\le c_0|x_t(0)|^2 + c_1\max_{\theta\in[-\tau,0]}|x_t(\theta)|^2 + c_2\max_{\theta\in[-2\tau,0]}|x_t(\theta)|^2 \\ &\le c_3\max_{\theta\in[-2\tau,0]}|x_t(\theta)|^2 \end{aligned}$$

Therefore, V is a positive definite functional with an infinitesimal upper bound. ■

The following corollary follows immediately.

Corollary 3.1 *Consider the functional* $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \to \mathbb{R}_+$

$$\begin{aligned} V(t,x_t) &= x^T(t)Px(t) + \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha)]^T P_1 [A(\gamma(\alpha))x(\alpha)]\ \mathrm{d}\alpha\ \mathrm{d}\beta \\ &+ \int_{-\tau}^{0}\int_{t+\beta}^{t} [A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2 [A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha\ \mathrm{d}\beta \end{aligned} \tag{4}$$

where P, P_1 *and* P_2 *are constant, positive-definite matrices and* $A(\gamma)$ *and* $A_d(\gamma)$ *are matrices that depend continuously on the parameter* $\gamma \in \Gamma$, *with* Γ *compact. Then for every* $\tau > 0$, V *is positive definite with an infinitesimal upper bound.*

The following lemma also holds.

Lemma 3.2 *Let* Γ *be a compact interval of the real line. Consider the continuous functional* $V : \mathbb{R}_+ \times \mathcal{C}_\tau \to \mathbb{R}_+$ *defined by*

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{-\tau}^{0} x^T(t+\theta)Q(\gamma(t,\theta))x(t+\theta)\ \mathrm{d}\theta$$

where $\gamma(t) \in \Gamma$ *for all* $t \geq 0$ *and* $P(\gamma) > 0$ *and* $Q(\gamma) > 0$ *for all* $\gamma \in \Gamma$. *Then* V *is a positive definite functional with an infinitesimal upper bound.*

Proof. The proof is similar to the one for Lemma 3.1 and will not be repeated here. ∎

4 Delay-Independent Stability

The results in this section deal with systems where the delay $\tau \in [0, \infty)$. Since stability is ensured for every positive delay τ, the stability conditions are delay-independent. Several stability tests are presented.

In the following, the dependence on t has been suppressed for notational simplicity. From now on, it will be tacitly assumed that all parameter-varying matrices depend continuously on the parameter.

Theorem 4.1 *Consider the LPV time-delayed system (1) and* $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$. *Consider a constant matrix* P *and a matrix-valued function* $Q : \Gamma \to \mathbb{R}^{n\times n}$ *such that*

$$P > 0, \quad Q(\gamma) > 0, \quad \forall\ \gamma \in \Gamma \tag{5}$$

and

$$M_1(\gamma_1, \gamma_2) = \begin{bmatrix} PA(\gamma_1) + A^T(\gamma_1)P + Q(\gamma_1) & PA_d(\gamma_1) \\ A_d^T(\gamma_1)P & -Q(\gamma_2) \end{bmatrix} < 0 \tag{6}$$

for all $\gamma_i \in \Gamma,\ i = 1, 2$. *Then the system (1) is asymptotically stable for all* $\gamma \in \Gamma$ *and* $\tau \in [0, \infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional $V : \mathbb{R}_+ \times \mathcal{C}_\tau \to \mathbb{R}_+$

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\ \mathrm{d}\theta$$

where P and $Q(\gamma)$ as in (5). From (5) and Lemma 3.2, it follows that V is positive definite with an infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned} \dot{V}(t, x_t) \;&=\; 2x^T(t)PA(\gamma(t))x(t) + 2x^T(t)PA_d(\gamma(t))x(t-\tau) \\ &\quad + x^T(t)Q(\gamma(t))x(t) - x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau) \end{aligned}$$

or

$$\dot{V}(t, x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_1(\gamma_1, \gamma_2) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{7}$$

where $\gamma_1 = \gamma(t)$ and $\gamma_2 = \gamma(t-\tau)$. Inequality (6) implies that the matrix $M_1(\gamma_1, \gamma_2)$ is negative definite for all $\gamma_1, \gamma_2 \in \Gamma$. Since Γ is compact, then $-\dot{V}(t, x_t) > -\min_{\gamma_1,\gamma_2} \lambda_{\max}[M_1(\gamma_1, \gamma_2)] \, (|x(t)|^2 + |x(t-\tau)|^2) > c\,|x(t)|^2$ where $c = -\min_{\gamma_1,\gamma_2} \lambda_{\max}[M_1(\gamma_1, \gamma_2)] > 0$ and the system (1) is asymptotically stable. ∎

Conditions (5)-(6) represent an infinite-dimensional set of LMI's. Gridding (see Section 4.2) can be used to project on a finite set of LMI's. In case the system matrices have a polynomial dependence on the parameter γ, the following result may be useful.

Theorem 4.2 *Consider the LPV time-delayed system (1) and assume that*

$$A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2 \quad \textit{and} \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}$$

where $\gamma \in \Gamma$, with Γ any compact sub-interval of $\mathbb{R}$. If there exist constant, positive-definite matrices P and Q such that

$$M_2 = \begin{bmatrix} A_0^T P + P A_0 + Q & P A_{d0} & P A_1 \\ A_{d0}^T P & -Q & A_{d1}^T P \\ A_1^T P & P A_{d1} & A_2^T P + P A_2 \end{bmatrix} < 0$$

then system (1) is asymptotically stable for any value of the parameter $\gamma \in \Gamma$ and any $\tau \in [0, \infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional

$$V(t, x) \quad = \quad x^T(t) P x(t) + \int_{-\tau}^{0} x^T(t+\theta) Q x(t+\theta) \, \mathrm{d}\theta$$

From Lemma 3.2, V is positive definite and has an infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned} \dot{V} \quad &= \quad 2x^T(t) P (A_0 + \gamma A_1 + \gamma^2 A_2) x(t) + x^T(t) Q x(t) \\ &\quad + 2x^T(t) P (A_{d0} + \gamma A_{d1}) x(t-\tau) - x^T(t-\tau) Q x(t-\tau) \end{aligned}$$

The last equation can be written as

$$\dot{V}(t) \quad = \quad \begin{bmatrix} x(t) \\ x(t-\tau) \\ \gamma x(t) \end{bmatrix}^T M_2 \begin{bmatrix} x(t) \\ x(t-\tau) \\ \gamma x(t) \end{bmatrix} \tag{8}$$

Since Γ is compact, the previous inequality holds uniformly for all $\gamma \in \Gamma$. Hence $\dot{V}$ is negative definite and from Theorem 3.1 the system (1) is asymptotically stable (Kolmanovskii and Myshkis, 1992). ∎

Remark 1 In Theorem 4.2 the set Γ can be arbitrarily large. Hence, the conditions of the theorem guarantee that system (1) is stable for any (bounded) values of the parameter $\gamma \in \mathbb{R}$. It requires, however, that $A_2^T P + PA_2 < 0$, and $A_0^T P + PA_0 + Q < 0$, i.e., the matrices A_0 and A_2 must be Hurwitz. This condition induces unnecessary conservatism. Assuming that the parameter γ is known to belong to a *known* compact interval, Theorem 4.1 can be used to relax the conditions for delay-independent stability for (1).

4.1 Stability under Bounded Parameter Variation

Theorems 4.1 and 4.2 did not consider the time variation of the parameter γ. In that respect, Theorems 4.1 and 4.2 can be potentially conservative, since they ensure – in principle – stability for arbitrarily fast variations of γ. In particular, in Theorem 4.1 $\gamma(t)$ and $\gamma(t-\tau)$ are treated as independent variables. Nonetheless, this does not induce any extra conservatism even for the case when $\dot\gamma$ is bounded by a (relatively) small upper bound. Unless $\dot\gamma = 0$ then $\gamma(t)$ and $\gamma(t-\tau)$ must be treated as independent since the delay may be arbitrarily large. Hence, for truly delay-independent results, the bound of $\dot\gamma$ does not impose any constraints between $\gamma(t)$ and $\gamma(t-\tau)$. If, on the other hand, the delay is known to belong to a bounded interval, then the treatment of $\gamma(t)$ and $\gamma(t-\tau)$ as independent may cause extra conservatism, especially for small bounds on the parameter variation rate. Similarly, if γ varies very fast then it is expected that (for delay-independent stability) $\gamma(t)$ and $\gamma(t-\tau)$ can still be treated independently, even for small values of the delay. See also the discussion at the end of Section 4.

Next, stability tests are derived that take explicitly into account the knowledge of the bound of the rate of variation of the parameter.

Theorem 4.3 *Consider the LPV time-delayed system (1) with* $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$ *and* $\dot\gamma \in \Gamma_r = [\underline{\dot\gamma}, \overline{\dot\gamma}]$. *Consider matrix valued functions* $P : \Gamma \to \mathbb{R}^{n\times n}$ *and* $Q : \Gamma \to \mathbb{R}^{n\times n}$ *such that*

$$P(\gamma) > 0, \quad Q(\gamma) > 0, \quad \forall\, \gamma \in \Gamma \tag{9}$$

and

$$M_3(\gamma_1, \gamma_2, \nu) = \begin{bmatrix} P(\gamma_1)A(\gamma_1) + (\ )^T + Q(\gamma_1) + \frac{\partial P}{\partial \gamma}\nu & P(\gamma_1)A_d(\gamma_1) \\ \\ A_d^T(\gamma_1)P(\gamma_1) & -Q(\gamma_2) \end{bmatrix} < 0 \tag{10}$$

for all $\gamma_1, \gamma_2 \in \Gamma$ *and* $\nu \in \Gamma_r$. *Then the system (1) is asymptotically stable for all* $(\gamma, \dot\gamma) \in \Gamma \times \Gamma_r$ *and* $\tau \in [0, \infty)$.

Proof. Consider the following Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t)P(\gamma(t))x(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\ \mathrm{d}\theta \tag{11}$$

From (9), V is positive definite with an infinitesimal upper bound. The derivative of V along the trajectories of (1) is

$$\begin{aligned}\dot{V}(t,x_t) &= 2x^T(t)P(\gamma(t))A(\gamma(t))x(t) + x^T(t)\frac{\partial P}{\partial \gamma}\dot{\gamma}x(t)\\ &+ 2x^T(t)P(\gamma(t))A_d(\gamma(t))x(t-\tau)\\ &+ x^T(t)Q(\gamma(t))x(t) - x^T(t-\tau)Q(\gamma(t-\tau))x(t-\tau)\end{aligned}$$

or

$$\dot{V}(t,x_t) = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_3(\gamma(t),\gamma(t-\tau),\dot{\gamma}(t)) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}$$

Inequality (10) implies that M_3 is negative definite for all $\gamma \in \Gamma$ and $\dot{\gamma} \in \Gamma_r$. Since Γ and Γ_r are compact, $-\dot{V}(t,x_t) > -c(|x(t)|^2 + |x(t_\tau)|^2) > c\,|x(t)|^2$ where $c = -\min_{\gamma_1,\gamma_2,\nu} \lambda_{\max}[M_3(\gamma_1,\gamma_2,\nu)] > 0$ and thus, the system (1) is asymptotically stable. ■

4.2 Gridding the Parameter Space

Equations (9)-(10) or (5)-(6) represent an infinite dimensional system of LMI's. A common way to reduce these conditions to a finite set of LMI's is to use gridding of the parameter space. According to this approach, one selects a set of basis functions $f_i(\gamma),\ (i = 1,2,...n_1)$ and $g_i(\gamma),\ (i = 1,2,...n_2)$ and expands P and Q in terms of these basis functions as

$$P(\gamma) = \sum_{i=1}^{n_1} P_i f_i(\gamma) \quad \text{and} \quad Q(\gamma) = \sum_{j=1}^{n_2} Q_j g_j(\gamma) \tag{12}$$

One then seeks matrices $P_i,\ (i = 1,2,\ldots,n_1)$ and $Q_j,\ (j = 1,2,\ldots,n_2)$ such that $\sum_{i=1}^{n_1} P_i f_i(\gamma) > 0$ and $\sum_{j=1}^{n_2} Q_j g_j(\gamma) > 0$ for all $\gamma \in \Gamma$ and

$$\begin{bmatrix} \begin{pmatrix} \left[\sum_{i=1}^{n_1} P_i f_i(\gamma_1)\right] A(\gamma_1) + \\ A^T(\gamma_1)\left[\sum_{i=1}^{n_1} P_i f_i(\gamma_1)\right] + \\ \sum_{i=1}^{n_1} P_i \frac{\partial f_i}{\partial \gamma}\nu + \sum_{j=1}^{n_2} Q_j g_j(\gamma_1) \end{pmatrix} & \left[\sum_{i=1}^{n_1} P_i f_i(\gamma_1)\right] A_d(\gamma_1) \\ \\ A_d^T(\gamma_1)\left[\sum_{i=1}^{n_1} P_i f_i(\gamma_1)\right] & -\sum_{j=1}^{n_2} Q_j g_j(\gamma_2) \end{bmatrix} < 0 \tag{13}$$

for all $\gamma_i \in \Gamma,\ i = 1,2$. The solution of (12)-(13), for instance, is searched over a finite number (grid) of the parameter values. After a solution is found, it is typically validated by testing it on a finer grid. Gridding leads to computationally expensive stability tests. It can be used when the dimension of the parameter vector γ is low. If the number of parameters is large, gridding can be computationally prohibitive, since the number of LMI's to be solved increases exponentially with the number of parameters. In order to get computationally tractable tests, we next assume that the system matrices $A(\gamma)$ and $A_d(\gamma)$ in (1) have a specific polynomial dependence on the parameter γ. Our results will also hold for more complex (non-polynomial) parameter dependencies as long as a polynomial approximation of the parameter dependence holds within the parameter range of interest.

4.3 A Relaxation Approach

The results of Theorem 4.1 and 4.3 require gridding of the parameter spaces Γ and $\Gamma \times \Gamma_r$, respectively. This can be cumbersome since for fine gridding, many matrix inequalities have to be solved simultaneously. In certain cases, the parameter dependence in the matrices A and A_d is relatively simple (low order polynomial) and gridding may be avoided using multi-convexity arguments and relaxation methods at the expense of increasing conservatism (Gahinet *et al.*, 1996; Tuan and Apkarian, 1999). Next, several special cases are explored when gridding can be avoided. In order to prove the main results of this section, the following two lemmas will be used in the sequel.

Lemma 4.1 *Consider the following parameter dependent matrix* $F(\gamma) = \gamma^2 F_2 + \gamma F_1 + F_0$ *where* $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. *If* $F_2 \geq 0$, *then* $F(\gamma)$ *is a convex, matrix-valued function, that is,*

$$\lambda F(\gamma_1) + (1-\lambda)F(\gamma_2) \geq F(\lambda\gamma_1 + (1-\lambda)\gamma_2), \qquad \forall \gamma_1, \gamma_2 \in [\underline{\gamma}, \overline{\gamma}] \tag{14}$$

for any scalar $0 \leq \lambda \leq 1$. *If* $F_2 > 0$ *then* $F(\gamma)$ *is a strictly convex, matrix-valued function, i.e., (14) is satisfied with strict inequality for all* $0 < \lambda < 1$. *Moreover, if* $F_2 \geq 0$ *and* $F(\gamma^{\#}) < 0$ *for* $\gamma^{\#} \in \{\underline{\gamma}, \overline{\gamma}\}$, *then* $F(\gamma) < 0$ *for all* $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

Proof. The proof is straightforward and thus, omitted. ∎

Lemma 4.2 *Consider the following parameter dependent matrix*

$$F(\gamma_1, \gamma_2, \gamma_3) = \gamma_1^2 F_2 + \gamma_1 F_1 + \gamma_2^2 F_3 + \gamma_2 F_4 + F_0(\gamma_3), \quad \textit{where} \quad F_0(\gamma_3) \quad = \quad F_{01} + \gamma_3 F_{02}$$

where $\gamma_i \in [\underline{\gamma}_i, \overline{\gamma}_i] = \Gamma_i$ *for* $i = 1, 2, 3$. *Let* $\Gamma_i^{\#} = \{\underline{\gamma}_i, \overline{\gamma}_i\}$ *denote the vertices of* Γ_i *for* $i = 1, 2, 3$. *If* $F_3 \geq 0, F_2 \geq 0$ *and* $F(\gamma_1^{\#}, \gamma_2^{\#}, \gamma_3^{\#}) < 0$ *for* $(\gamma_1^{\#}, \gamma_2^{\#}, \gamma_3^{\#}) \in \Gamma_1^{\#} \times \Gamma_2^{\#} \times \Gamma_3^{\#}$ *then* $F(\gamma_1, \gamma_2, \gamma_3) < 0$ *for all* $(\gamma_1, \gamma_2, \gamma_3) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3$.

Proof. See (Gahinet *et al.*, 1996). ∎

In the following it is assumed that $\Gamma = [-1, 1]$. In case $\Gamma \neq [-1, 1]$, one can choose $\tilde{\gamma} = [2\gamma - (\overline{\gamma} + \underline{\gamma})]/(\overline{\gamma} - \underline{\gamma})$, such that $\tilde{\gamma} \in [-1, 1]$. This simplification can always be made without loss of generality and results in more compact formulas.

Theorem 4.4 *Consider the system (1) where*

$$A(\gamma) = A_0 + \gamma A_1 + \gamma^2 A_2 \quad \textit{and} \quad A_d(\gamma) = A_{d0} + \gamma A_{d1} + \gamma^2 A_{d2} \tag{15}$$

where $\gamma \in [-1, 1]$, *and* $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. *Assume that there exist negative semi-definite matrices* Q_4, Q_2, P_2, *positive-definite matrices* Q_0, P_0 *and symmetric matrices* Q_1, Q_3, P_1 *such that*

$$Q_0 \pm Q_1 + 2Q_2 > 0, \quad -Q_2 \pm Q_3 + Q_4 \geq 0, \tag{16}$$

$$P_0 \pm P_1 + P_2 > 0 \tag{17}$$

and

$$N_2 + \gamma_1{}^{\#} N_1 + N_3 + \gamma_2^{\#} N_4 + N_0(\nu) < 0 \tag{18}$$

where $\gamma_i{}^{\#} \in \{-1, 1\}$ *and* $\nu \in \{\underline{\dot{\gamma}}, \overline{\dot{\gamma}}\}$, *and where*

$$N_2 = \alpha_1\, \Theta_1 + \alpha_2\, \Theta_2 + \Theta_3 \geq 0 \tag{19}$$

$$N_1 = \frac{1-\alpha_1}{2}\, \Theta_1 + \frac{3-3\alpha_2}{4}\, \Theta_2 + \Theta_4 \tag{20}$$

$$N_0(\dot{\gamma}) = \frac{3\alpha_1 - 3}{16}\, \Theta_1 + \frac{\alpha_2 - 1}{4}\, \Theta_2 + \Theta_5(\dot{\gamma}) \tag{21}$$

$$N_3 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_4 - (1-\beta)Q_3 - Q_2 \end{bmatrix} \geq 0 \tag{22}$$

$$N_4 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{3}{4}\beta Q_3 - Q_1 \end{bmatrix} \tag{23}$$

where

$$\Theta_1 = \begin{bmatrix} A_2^T P_2 + P_2 A_2 + Q_4 & P_2 A_{d2} \\ A_{d2}^T P_2 & 0 \end{bmatrix}$$

$$\Theta_2 = \begin{bmatrix} \begin{pmatrix} A_1^T P_2 + A_2^T P_1 \\ +P_1 A_2 + P_2 A_1 + Q_3 \end{pmatrix} & P_1 A_{d2} + P_2 A_{d1} \\ A_{d2}^T P_1 + A_{d1}^T P_2 & 0 \end{bmatrix}$$

$$\Theta_3 = \begin{bmatrix} \begin{pmatrix} A_2^T P_0 + A_0^T P_2 \\ +A_1^T P_1 + (\ )^T + Q_2 \end{pmatrix} & \begin{pmatrix} P_2 A_{d0} + P_1 A_{d1} \end{pmatrix} \\ A_{d1}^T P_1 + A_{d0}^T P_2 & A_{d2}^T P_0 A_{d2} \end{bmatrix}$$

$$\Theta_4 = \begin{bmatrix} \begin{pmatrix} A_1^T P_0 + P_0 A_1 \\ +A_0^T P_1 + P_1 A_0 + Q_1 \end{pmatrix} & P_0 A_{d1} + P_1 A_{d0} \\ A_{d1}^T P_0 + A_{d0}^T P_1 & 0 \end{bmatrix}$$

$$\Theta_5(\dot{\gamma}) = \begin{bmatrix} \begin{pmatrix} A_0^T P_0 + P_0 A_0 \\ -2\nu_m P_2 + \dot{\gamma} P_1 + Q_0 \end{pmatrix} & P_0 A_{d0} \\ A_{d0}^T P_0 & -Q_0 + \frac{1}{4}\beta Q_3 \end{bmatrix}$$

where $\nu_m = \max\{|\underline{\dot{\gamma}}|, |\overline{\dot{\gamma}}|\}$, *and where the pair* (α_1, α_2) *takes any of the four possible combinations* $(\alpha_1, \alpha_2) \in \{(0,0), (0,1), (1,0), (1,1)\}$ *and* $\beta \in \{0, 1\}$. *Then the system (1) is asymptotically stable for all* $\gamma \in [-1, 1]$, *all* $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$ *and all* $\tau \in [0, \infty)$.

Proof. Consider the Lyapunov-Krasovskii functional

$$V(t, x_t) = x^T(t) P(\gamma(t)) x(t) + \int_{t-\tau}^{t} x^T(\theta) Q(\gamma(\theta)) x(\theta)\ \mathrm{d}\theta$$

where

$$P(\gamma) = P_0 + \gamma P_1 + \gamma^2 P_2 \quad \text{and} \quad Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2 + \gamma^3 Q_3 + \gamma^4 Q_4 \tag{24}$$

Equation (17) implies that $P(\gamma^{\#}) > 0$ for $\gamma^{\#} \in \{-1, 1\}$. Since $P_2 \leq 0$, $-P(\gamma)$ is convex. From Lemma 4.1 it follows that $P(\gamma) > 0$ uniformly, for all $\gamma \in [-1, 1]$. Now write $Q(\gamma)$ as follows

$$Q(\gamma) = [2\gamma^2 Q_2 + \gamma Q_1 + Q_0] + \gamma^2[\gamma^2 Q_4 + \gamma Q_3 - Q_2]$$

Since $Q_2 \leq 0$ and $Q_4 \leq 0$, then (16) along with Lemma 4.1 imply that $Q(\gamma) > 0$ uniformly for all $\gamma \in [-1, 1]$. It follows that V is positive definite with an infinitesimal upper bound. The derivative of V along the system (1) is

$$\begin{aligned}
\dot{V}(t) &= 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_0 + \gamma_1 A_1 + \gamma_1^2 A_2)x(t) \\
&+ x^T(t)[\dot{\gamma}_1 P_1 + 2\gamma_1\dot{\gamma}_1 P_2]x(t) \\
&+ 2x^T(t)(P_0 + \gamma_1 P_1 + \gamma_1^2 P_2)(A_{d0} + \gamma_1 A_{d1} + \gamma_1^2 A_{d2})x(t-\tau) \\
&+ x^T(t)[Q_0 + \gamma_1 Q_1 + \gamma_1^2 Q_2 + \gamma_1^3 Q_3 + \gamma_1^4 Q_4]x(t) \\
&- x^T(t-\tau)[Q_0 + \gamma_2 Q_1 + \gamma_2^2 Q_2 + \gamma_2^3 Q_3 + \gamma_2^4 Q_4]x(t-\tau)
\end{aligned}$$

where $\gamma_1 = \gamma(t), \gamma_2 = \gamma(t-\tau)$. Since $P_2 \leq 0$, it follows that $2\gamma_1\dot{\gamma}_1 x^T(t)P_2x(t) \leq -2\nu_m x^T(t)P_2x(t)$ where $\nu_m = \max\{|\underline{\dot{\gamma}}|, |\bar{\dot{\gamma}}|\}$. One can then rewrite the equation for $\dot{V}$ as

$$\begin{aligned}
\dot{V}(t) \leq & \gamma_1^4\Big\{2x^T(t)[P_2A_2 + 0.5Q_4]x(t) + 2x^T(t)P_2A_{d2}x(t-\tau)\Big\} \\
& - \gamma_2^4 x^T(t-\tau)Q_4x(t-\tau) \\
& + \gamma_1^3\Big\{2x^T(t)[P_1A_2 + P_2A_1 + 0.5Q_3]x(t) + 2x^T(t)[P_1A_{d2} + P_2A_{d1}]x(t-\tau)\Big\} \\
& - \gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) \\
& + \gamma_1^2\Big\{2x^T(t)[P_0A_2 + P_1A_1 + P_2A_0 + 0.5Q_2]x(t) + 2x^T(t)[P_0A_{d2} + P_1A_{d1} + P_2A_{d0}]x(t-\tau)\Big\} \\
& \qquad - \gamma_2^2 x^T(t-\tau)Q_2x(t-\tau) \\
& + \gamma_1\Big\{2x^T(t)[P_0A_1 + P_1A_0 + 0.5Q_1]x(t) + 2x^T(t)[P_0A_{d1} + P_1A_{d0}]x(t-\tau)\Big\} \\
& - \gamma_2 x^T(t-\tau)Q_1x(t-\tau) \\
& + 2x^T(t)[P_0A_0 - \nu_m P_2 + 0.5Q_0 + 0.5\dot{\gamma}P_1]x(t) + 2x^T(t)P_0A_{d0}x(t-\tau) \\
& - x^T(t-\tau)Q_0x(t-\tau)
\end{aligned} \tag{25}$$

Notice now that since $P_0 > 0$ it follows that the inequality

$$2x^T(t)P_0A_{d2}x(t-\tau) \leq x^T(t)P_0x(t) + x^T(t-\tau)A_{d2}^T P_0 A_{d2}x(t-\tau)$$

holds. Also, it can be immediately verified that for all $\gamma \in [-1, 1]$[1] the following inequalities hold

$$\gamma^2 \geq \gamma^4 \geq \frac{1}{2}\gamma - \frac{3}{16}, \qquad \gamma^2 \geq \gamma^3 \geq \frac{3}{4}\gamma - \frac{1}{4} \tag{26}$$

[1] See, for example, (Tuan and Apkarian, 1999).

Then $\gamma^4 y \le \gamma^2 y$ if $y \ge 0$ and $\gamma^4 y \le (\frac{1}{2}\gamma - \frac{3}{16})y$ if $y < 0$. Therefore,

$$\gamma^4 y \le \max\left\{\gamma^2 y, (\frac{1}{2}\gamma - \frac{3}{16})y\right\} \tag{27}$$

and

$$\gamma^3 y \le \max\left\{\gamma^2 y, (\frac{3}{4}\gamma - \frac{1}{4})y\right\} \tag{28}$$

These inequalities imply that

$$\begin{aligned} &\gamma_1^4\Big\{2x^T(t)[P_2A_2 + 0.5Q_4]x(t) + 2x^T(t)P_2A_{d2}x(t-\tau)\Big\} \\ &\le \max\left\{(\frac{\gamma_1}{2} - \frac{3}{16})\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}, \right. \\ &\left. \gamma_1^2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_1 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}\right\} \end{aligned} \tag{29}$$

and

$$\begin{aligned} &\gamma_1^3\Big\{2x^T(t)[P_1A_2 + P_2A_1 + 0.5Q_3]x(t) + 2x^T(t)[P_1A_{d2} + P_2A_{d1}]x(t-\tau)\Big\} \\ &\le \max\left\{\gamma_1^2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}, (\frac{3\gamma_1}{4} - \frac{1}{4})\begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T \Theta_2 \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}\right\} \end{aligned} \tag{30}$$

Since $Q_4 \le 0$ and $\gamma_2 \in [-1, 1]$, it follows that

$$-\gamma_2^4 x^T(t-\tau)Q_4x(t-\tau) \le -\gamma_2^2 x^T(t-\tau)Q_4x(t-\tau) \tag{31}$$

Moreover, if $x^T(t-\tau)Q_3x(t-\tau) > 0$ then

$$-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) < (-\frac{3}{4}\gamma_2 + \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau) \tag{32}$$

whereas if $x^T(t-\tau)Q_3x(t-\tau) < 0$ then

$$-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) < -\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau) \tag{33}$$

Therefore in either case,

$$-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) \le \max\{(-\frac{3}{4}\gamma_2 + \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau), -\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau)\} \tag{34}$$

or that

$$-\gamma_2^3 x^T(t-\tau)Q_3x(t-\tau) \le -(1-\beta)\gamma_2^2 x^T(t-\tau)Q_3x(t-\tau) - \beta(\frac{3}{4}\gamma_2 - \frac{1}{4})x^T(t-\tau)Q_3x(t-\tau)$$

where $\beta \in \{0, 1\}$. Collecting all previous results and substituting in (25), one obtains that

$$\dot{V}(x) \quad \le \quad \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix}^T M_4(\gamma_1, \gamma_2, \dot{\gamma}) \begin{bmatrix} x(t) \\ x(t-\tau)\end{bmatrix} \tag{35}$$

where

$$M_4(\gamma_1, \gamma_2, \dot{\gamma}) = \gamma_1^2 N_2 + \gamma_1 N_1 + \gamma_2^2 N_3 + \gamma_2 N_4 + N_0(\dot{\gamma}) \tag{36}$$

and where N_0, N_1, N_2, N_3, N_4 as in (19)-(21). The inequalities $N_3 \geq 0$ and $N_2 \geq 0$ along with inequality (18) and using Lemma 4.2 imply that $M_4(\gamma_1, \gamma_2, \dot{\gamma}) < 0$ for all $(\gamma_1, \gamma_2) \in [-1, 1] \times [-1, 1]$ and $\dot{\gamma} \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. The asymptotic stability of (1) then follows immediately from (35). ■

Remark 2 If $A_2 = A_{d2} = 0$ and $\dot{\gamma} \in (-\infty, +\infty)$, it can be shown that condition (18) of Theorem 4.4 reduces to condition (6) of Theorem 4.1 with $Q(\gamma) = Q_0 + \gamma Q_1$.

In case the system matrices $A(\gamma)$ and $A_d(\gamma)$ depend only affinely on γ we have the following result.

Theorem 4.5 *Consider the LPV time-delayed system (1) with*

$$A(\gamma) = A_0 + \gamma A_1, \qquad A_d(\gamma) = A_{d0} + \gamma A_{d1} \tag{37}$$

where $(\gamma, \dot{\gamma}) \in \mathcal{G} = [\underline{\gamma}, \overline{\gamma}] \times [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$. Assume that there exist a negative semi-definite matrix $Q_2 \leq 0$, a positive semi-definite matrix $P_1 \geq 0$, and symmetric matrices P_0, Q_0, Q_1 which satisfy the following LMI's

$$Q(\gamma^{\#}) = Q_0 + \gamma^{\#} Q_1 + {\gamma^{\#}}^2 Q_2 > 0 \tag{38a}$$

$$P(\gamma^{\#}) = P_0 + \gamma^{\#} P_1 > 0 \tag{38b}$$

for all $\gamma^{\#} \in \{\underline{\gamma}, \overline{\gamma}\}$ and

$${\gamma_1^{\#}}^2 L_2 + {\gamma_1}^{\#} L_1 + {\gamma_2^{\#}}^2 L_3 + \gamma_2^{\#} L_4 + L_0(\nu) < 0 \tag{39}$$

for all $\nu \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$ and $\gamma_i \in [\underline{\gamma}, \overline{\gamma}]$, $i = 1, 2$, and where

$$L_2 = \begin{bmatrix} P_1 A_1 + A_1^T P_1 + Q_2 + P_1 & 0 \\ 0 & A_{d1}^T P_1 A_{d1} \end{bmatrix} \geq 0 \tag{40}$$

$$L_1 = \begin{bmatrix} P_1 A_0 + P_0 A_1 + (\ )^T + Q_1 & P_1 A_{d0} + P_0 A_{d1} \\ A_{d0}^T P_1 + A_{d1}^T P_0 & 0 \end{bmatrix} \tag{41}$$

$$L_3 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_2 \end{bmatrix} \geq 0 \tag{42}$$

$$L_4 = \begin{bmatrix} 0 & 0 \\ 0 & -Q_1 \end{bmatrix} \tag{43}$$

$$L_0(\dot{\gamma}) = \begin{bmatrix} P_0 A_0 + A_0^T P_0 + Q_0 + \dot{\gamma} P_1 & P_0 A_{d0} \\ A_{d0}^T P_0 & -Q_0 \end{bmatrix} \tag{44}$$

Then the system (1) is delay-independent stable for all $(\gamma, \dot{\gamma}) \in \mathcal{G}$.

Proof. Consider the parameter-dependent Lyapunov-Krasovskii functional

$$V(t,x_t) = x^T(t)P(\gamma(t))x(t) + \int_{t-\tau}^{t} x^T(\theta)Q(\gamma(\theta))x(\theta)\ \mathrm{d}\theta$$

where $P(\gamma) = P_0 + \gamma P_1$ and $Q(\gamma) = Q_0 + \gamma Q_1 + \gamma^2 Q_2$. Since $Q_2 \leq 0$, and from (38a) and Lemma 4.1 it follows that $Q(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. From (38b), it also follows that $P(\gamma) > 0$ for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. Therefore, $V(t, x_t)$ is a positive definite functional with an infinitesimal upper bound. Calculation of the derivative of V yields

$$\begin{aligned}
\dot{V}(t) \;=\; & \gamma_1 \left\{2x^T(t)[P_1A_0 + P_0A_1 + 0.5Q_1]x(t) + 2x^T(t)[P_1A_{d0} + P_0A_{d1}]x(t-\tau)\right\} \\
& +\gamma_1^2 \left\{2x^T(t)[P_1A_1 + 0.5Q_2]x(t) + 2x^T(t)P_1A_{d1}x(t-\tau)\right\} && (45) \\
& -\gamma_2\, x^T(t-\tau)Q_1x(t-\tau) - \gamma_2^2\, x^T(t-\tau)Q_2x(t-\tau) \\
& +2x^T(t)[P_0A_0 + 0.5Q_0 + 0.5\dot{\gamma}P_1]x(t) \\
& +2x^T(t)P_0A_{d0}x^T(t-\tau) - x^T(t-\tau)Q_0x(t-\tau) && (46)
\end{aligned}$$

where $\gamma_1 = \gamma(t)$ and $\gamma_2 = \gamma(t-\tau)$. Since $P_1 > 0$ it follows that

$$2x^T(t)P_1A_{d1}x(t-\tau) \leq x^T(t)P_1x(t) + x^T(t-\tau)A_{d1}^TP_1A_{d1}x(t-\tau) \tag{47}$$

Substituting (47) in (45) and collecting terms yields

$$\dot{V} \leq \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T M_5(\gamma_1, \gamma_2, \dot{\gamma}) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \tag{48}$$

where

$$M_5(\gamma_1, \gamma_2, \dot{\gamma}) = \gamma_1^2 L_2 + \gamma_1 L_1 + \gamma_2^2 L_3 + \gamma_2 L_4 + L_0(\dot{\gamma}) \tag{49}$$

where L_0, L_1, L_2, L_3, L_4 as in (40)-(44). Since $L_2 \geq 0$ and $L_3 \geq 0$ then (39) implies, using Lemma 4.1, that $M_5(\gamma_1, \gamma_2, \nu) < 0$ for all $\nu \in [\underline{\dot{\gamma}}, \overline{\dot{\gamma}}]$, $\gamma_i \in [\underline{\gamma}, \overline{\gamma}]$, $i = 1, 2$. Thus, the derivative of V is negative definite and the system (1) is asymptotically stable for all $(\gamma, \dot{\gamma}) \in \mathcal{G}$ and $\tau \in [0, \infty)$. ∎

Remark 3 It can be shown that the condition (39) of Theorem 4.5 reduces to condition (6) of Theorem 4.1 when $P_1 = Q_2 = 0$.

It should be noted that in the previous results no additional conservatism is introduced by treating $\gamma(t)$ and $\gamma(t-\tau)$ as independent, even if the bound on $\dot{\gamma}$ is arbitrarily small. However, for $\dot{\gamma} = 0$, one has that $\gamma(t) = \gamma(t-\tau)$ for all $t \geq 0$ and in this special case $\gamma(t)$ and $\gamma(t-\tau)$ are related (they are, in fact, equal). One possible method to account for the dependence of $\gamma(t)$ and $\gamma(t-\tau)$ is to eliminate $\gamma(t-\tau)$ using the fact that $\gamma(t) = \gamma(t-\tau) + \dot{\gamma}(\xi)\tau$, for some $\xi \in [t-\tau, t]$ and then take into account any known bounds for $\dot{\gamma}$. The resulting stability tests are then *delay-dependent*. We do not investigate this approach further since it follows from the previous results in a straightforward manner. Generally speaking, for small variation rates, a delay-dependent stability test should be used. Next, we present several delay-dependent stability results for LPV systems, albeit without consideration to parameter variation rates. The latter problem is left for future investigation.

5 Delay-Dependent Stability

We now derive stability conditions for the system (1) that take explicitly into account the delay bound $\bar{\tau}$. Before we give the main results, we re-write system (1) in the following equivalent forms.

$$\begin{aligned} \dot{x}(t) &= [A(\gamma(t)) + A_d(\gamma(t))]x(t) \\ &\quad - A_d(\gamma(t)) \int_{-\tau}^{0} [A(\gamma(t+\alpha))x(\alpha+t) + A_d(\gamma(t+\alpha))x(\alpha+t-\tau)]d\alpha \end{aligned} \tag{50}$$

$$\begin{aligned} \dot{x}(t) &= [A(\gamma(t)) + M\ A_d(\gamma(t))]x(t) + (I-M)A_d(\gamma(t))x(t-\tau) \\ &\quad - MA_d(\gamma(t)) \int_{t-\tau}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\,\mathrm{d}\alpha \end{aligned} \tag{51}$$

where M in system (51) is an arbitrary matrix. The previous system transformation is similar to the one presented in (Niculescu, 1999) and (Zhang *et al.*, 2001). Zhang *et al.* (2001) used (51) to obtain stability tests for LTI time-delayed systems using frequency domain techniques. These results provided the exact counterpart of similar stability conditions developed in the time-domain (Verriest *et al.*, 1993; Verriest and Ivanov, 1994; Li and de Souza, 1996; Park *et al.*, 1998). Most importantly, this frequency domain framework later allowed the derivation of new stability criteria with a very low degree of conservatism (Zhang *et al.*, 1999; Zhang *et al.*, 2000*b*; Zhang *et al.*, 2000*a*). To see how systems (1) and (50) are related, notice that

$$x(t) - x(t-\tau) = \int_{-\tau}^{0} \dot{x}\ (t+\alpha)\,\mathrm{d}\alpha \tag{52}$$

Then

$$\begin{aligned} \dot{x}(t) &= A(\gamma(t))\ x(t) + A_d(\gamma(t))\ x(t-\tau) + A_d(\gamma(t))\ x(t) - A_d(\gamma(t))\ x(t) \\ &= [A(\gamma(t)) + A_d(\gamma(t))]\ x(t) - A_d(\gamma(t))[x(t) - x(t-\tau)] \\ &= [A(\gamma(t)) + A_d(\gamma(t))]\ x(t) - A_d(\gamma(t)) \int_{-\tau}^{0} \dot{x}\ (t+\alpha)d\alpha \\ &= [A(\gamma(t)) + A_d(\gamma(t))]\ x(t) - A_d(\gamma(t)) \int_{-\tau}^{0} [A(\gamma(\alpha+t))x(\alpha+t) + A_d(\gamma(\alpha+t))x(\alpha+t-\tau)]\mathrm{d}\alpha \end{aligned}$$

Similarly, for system (51), one has

$$\begin{aligned} \dot{x}(t) &= A(\gamma(t))\ x(t) + A_d(\gamma(t))\ x(t-\tau) + MA_d(\gamma(t))[x(t) - x(t-\tau) - \int_{t-\tau}^{t} \dot{x}\ (\alpha)\mathrm{d}\alpha] \\ &= A(\gamma(t))\ x(t) + A_d(\gamma(t))\ x(t-\tau) + MA_d(\gamma(t))x(t) - MA_d(\gamma(t))x(t-\tau) \\ &\quad - MA_d(\gamma(t)) \int_{t-\tau}^{t} \dot{x}\ (\alpha)d\alpha \\ &= [A(\gamma(t)) + M\ A_d(\gamma(t)))x(t) + (I-M)A_d(\gamma(t)]x(t-\tau) \\ &\quad - MA_d(\gamma(t)) \int_{t-\tau}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\mathrm{d}\alpha \end{aligned}$$

It follows that the trajectories of (50) or (51) are also trajectories of (1). Hence if systems (50) or (51) are stable, the original system (1) is also stable. It should be pointed out, however, that system (1) and systems (50) or (51) are not equivalent. Systems (50) and (51) include additional dynamics arising due to the eigenvalues of the matrix A_d (Gu and Niculescu, 1999). Hence any stability test based on either of these two systems is conservative.

5.1 Delay-dependent stability conditions in LMI form

Our first delay-dependent result is given by the following theorem.

Theorem 5.1 *Let $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$. Then the system (50) is asymptotically stable for any constant delay τ, with $0 \le \tau \le \bar{\tau}$, if there exist positive-definite matrices P, Q_1, Q_2 such that the following matrix inequality is satisfied*

$$\begin{bmatrix} A(\gamma)^T P + PA(\gamma) - Q_2 & -Q_2 & Q_2 + PA_d(\gamma) & \bar{\tau} A^T(\gamma) Q_1 & 0 \\ -Q_2 & -(Q_1 + Q_2) & Q_2 & 0 & 0 \\ A_d^T(\gamma) P + Q_2 & Q_2 & -Q_2 & 0 & \bar{\tau} A_d^T(\gamma) Q_2 \\ \bar{\tau} Q_1 A(\gamma) & 0 & 0 & -Q_1 & 0 \\ 0 & 0 & \bar{\tau} Q_2 A_d(\gamma) & 0 & -Q_2 \end{bmatrix} < 0 \quad (53)$$

for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

Proof. Consider the following Lyapunov-Krasovskii functional $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \to \mathbb{R}_+$

$$\begin{aligned} V(t, x_t) &= x^T(t) P x(t) + \int_{-\tau}^{0} \int_{t+\beta}^{t} [A(\gamma(\alpha)) x(\alpha)]^T P_1 [A(\gamma(\alpha)) x(\alpha)] \, \mathrm{d}\alpha \, \mathrm{d}\beta \\ &+ \int_{-\tau}^{0} \int_{t+\beta}^{t} [A_d(\gamma(\alpha)) x(\alpha - \tau)]^T P_2 [A_d(\gamma(\alpha)) x(\alpha - \tau)] \, \mathrm{d}\alpha \, \mathrm{d}\beta \end{aligned} \quad (54)$$

Where P, P_1 and P_2 are constant, positive-definite matrices. According to Corollary 3.1, V is positive definite with an infinitesimal upper bound. The derivative of V along the trajectories of the system in Eq. (50) is

$$\begin{aligned} \dot{V}(t) &= x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))] x(t) \\ &-2x^T(t) P A_d(\gamma(t)) \int_{t-\tau}^{t} [A(\gamma(\alpha)) x(\alpha) + A_d(\gamma(\alpha)) x(\alpha - \tau)] \, \mathrm{d}\alpha \\ &+ \int_{-\tau}^{0} [A(\gamma(t)) x(t)]^T P_1 [A(\gamma(t)) x(t)] \, \mathrm{d}\beta - \int_{-\tau}^{0} [A(\gamma(t+\beta)) x(t+\beta)]^T P_1 [A(\gamma(t+\beta)) x(t+\beta)] \, \mathrm{d}\beta \\ &+ \int_{-\tau}^{0} [A_d(\gamma(t)) x(t-\tau)]^T P_2 [A_d(\gamma(t)) x(t-\tau)] \, \mathrm{d}\beta \\ &- \int_{-\tau}^{0} [A_d(\gamma(t+\beta)) x(t+\beta-\tau)]^T P_2 [A_d(\gamma(t+\beta)) x(t+\beta-\tau)] \, \mathrm{d}\beta \end{aligned} \quad (55)$$

or

$$\begin{aligned}
\dot{V}(t) &= x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))]x(t) \\
&\quad -2x^T(t)PA_d(\gamma(t))\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ \mathrm{d}\alpha - 2x^T(t)PA_d(\gamma(t))\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ \mathrm{d}\alpha \\
&\quad +\tau\,[A(\gamma(t))x(t)]^T P_1[A(\gamma(t))x(t)]\ - \int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha)]^T P_1[A(\gamma(\alpha))x(\alpha)]\ \mathrm{d}\alpha \\
&\quad +\tau\,[A_d(\gamma(t))x(t-\tau)]^T P_2[A_d(\gamma(t))x(t-\tau)] \\
&\quad -\int_{t-\tau}^{t}[A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2[A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha. \qquad (56)
\end{aligned}$$

By the Cauchy-Schwartz inequality one can show that for any positive definite matrix P and any $v(\alpha) \in \mathbb{R}^n$,

$$\tau\int_{t-\tau}^{t} v(\alpha)^T P v(\alpha)\mathrm{d}\alpha \geq [\int_{t-\tau}^{t} v(\alpha)\mathrm{d}\alpha]^T P[\int_{t-\tau}^{t} v(\alpha)\mathrm{d}\alpha] \qquad (57)$$

Using this result, and because $0 \leq \tau \leq \bar{\tau}$, we have

$$\begin{aligned}
&-\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha)]^T P_1[A(\gamma(\alpha))x(\alpha)]\ \mathrm{d}\alpha \leq -(\frac{1}{\bar{\tau}})[\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ \mathrm{d}\alpha]^T\ P_1\ [\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ \mathrm{d}\alpha] \\
&-\int_{t-\tau}^{t}[A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2[A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha. \\
&\qquad \leq -(\frac{1}{\bar{\tau}})[\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ \mathrm{d}\alpha]^T\ P_2\ [\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ \mathrm{d}\alpha] \qquad (58)
\end{aligned}$$

Define now the new variables

$$y(t) = -\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ \mathrm{d}\alpha \quad \text{and} \quad z(t) = -\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ \mathrm{d}\alpha \qquad (59)$$

Using (1) one obtains

$$z(t) = -x(t) + x(t-\tau) - y(t) \qquad (60)$$

and because $0 \leq \tau \leq \bar{\tau}$, one obtains the inequality

$$\begin{aligned}
\dot{V} &\leq x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))]x(t) \\
&\quad +2x^T(t)PA_d(\gamma(t))y(t) + 2x^T(t)PA_d(\gamma(t))z(t) \\
&\quad +\bar{\tau}x^T(t)A^T(\gamma(t))P_1A(\gamma(t))x(t) + \bar{\tau}x^T(t-\tau)A_d^T(\gamma(t))P_2A_d(\gamma(t))x(t-\tau) \\
&\quad -(\frac{1}{\bar{\tau}})[\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ d\alpha]^T\ P_1\ [\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha)\ d\alpha] \\
&\quad -(\frac{1}{\bar{\tau}})[\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ d\alpha]^T\ P_2\ [\int_{t-\tau}^{t} A_d(\gamma(\alpha))x(\alpha-\tau)\ d\alpha] \qquad (61)
\end{aligned}$$

where we made use of Eq. (57). The previous inequality can be rewritten in the form,

$$\begin{aligned}
\dot{V} &\leq x^T(t)[(A(\gamma(t)) + A_d(\gamma(t)))^T P + P(A(\gamma(t)) + A_d(\gamma(t)))]x(t) + 2x^T(t)PA_d(\gamma(t))y(t) \\
&\quad +2x^T(t)PA_d(\gamma(t))z(t) + \bar{\tau}x^T(t)A^T(\gamma(t))P_1A(\gamma(t))x(t) \\
&\quad +\bar{\tau}x^T(t-\tau)A_d^T(\gamma(t))P_2A_d(\gamma(t))x(t-\tau)\ - y^T(t)(P_1/\bar{\tau})y(t) - z^T(t)(P_2/\bar{\tau})z(t) \qquad (62)
\end{aligned}$$

Substituting (60) into (62), one obtains

$$\dot{V} \leq \begin{bmatrix} x(t) \\ y(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \begin{pmatrix} A(\gamma(t))^T P + PA(\gamma(t)) \\ +\bar{\tau} A^T(\gamma(t)) P_1 A(\gamma(t)) \\ -P_2/\bar{\tau} \end{pmatrix} & -P_2/\bar{\tau} & P_2/\bar{\tau} + PA_d(\gamma(t)) \\ -P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} \\ P_2/\bar{\tau} + A_d^T(\gamma(t))P & P_2/\bar{\tau} & \begin{pmatrix} \bar{\tau} A_d^T(\gamma(t)) P_2 A_d(\gamma(t)) \\ -P_2/\bar{\tau} \end{pmatrix} \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ x(t-\tau) \end{bmatrix} \quad (63)$$

Hence, if the following inequality is satisfied, the time-delayed system (1) will be asymptotically stable for any $0 \leq \tau \leq \bar{\tau}$

$$\begin{bmatrix} \begin{pmatrix} A(\gamma)^T P + PA(\gamma) \\ +\bar{\tau} A^T(\gamma) P_1 A(\gamma) - P_2/\bar{\tau} \end{pmatrix} & -P_2/\bar{\tau} & P_2/\bar{\tau} + PA_d(\gamma) \\ -P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} \\ P_2/\bar{\tau} + A_d^T(\gamma)P & P_2/\bar{\tau} & \bar{\tau} A_d^T(\gamma) P_2 A_d(\gamma) - P_2/\bar{\tau} \end{bmatrix} < 0, \quad \forall \gamma \in [\underline{\gamma}, \overline{\gamma}] \quad (64)$$

Inequality (64) can be rewritten as follows

$$\begin{bmatrix} A(\gamma)^T P + PA(\gamma) - P_2/\bar{\tau} & -P_2/\bar{\tau} & P_2/\bar{\tau} + PA_d(\gamma) & A^T(\gamma) P_1 & 0 \\ -P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} & 0 & 0 \\ A_d^T(\gamma)P + P_2/\bar{\tau} & P_2/\bar{\tau} & -P_2/\bar{\tau} & 0 & A_d^T(\gamma) P_2 \\ P_1 A(\gamma) & 0 & 0 & -P_1/\bar{\tau} & 0 \\ 0 & 0 & P_2 A_d(\gamma) & 0 & -P_2/\bar{\tau} \end{bmatrix} < 0 \quad (65)$$

Let $Q_1 = P_1/\bar{\tau}$ and $Q_2 = P_2/\bar{\tau}$. Then, inequality (65) can be rewritten as (53), which has to be satisfied for all $\gamma \in [\underline{\gamma}, \overline{\gamma}]$. Since the parameter γ lies in a compact interval, $\dot{V}$ is uniformly negative definite with respect to γ and the system (1) is asymptotically stable. ■

Gridding of the interval $[\underline{\gamma}, \overline{\gamma}]$ is required to reduce the infinite system of LMI's in (53) to a finite set of LMI's. Alternatively, for an LPV time-delayed system, for which the system matrices are affine functions of γ, the inequalities (53) need only to be checked at the boundary points of the interval $\Gamma = [\underline{\gamma}, \overline{\gamma}]$. This statement is formalized in the following corollary.

Corollary 5.1 *Consider the system (50) with*

$$A(\gamma) = A_0 + \gamma A_1, \qquad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \qquad \gamma \in [\underline{\gamma}, \overline{\gamma}] \quad (66)$$

Suppose that there exist positive-definite matrices P, Q_1, Q_2 such that the following matrix inequality is satisfied

$$\begin{bmatrix} A(\gamma^\#)^T P + PA(\gamma^\#) - Q_2 & -Q_2 & Q_2 + PA_d(\gamma^\#) & \bar{\tau} A^T(\gamma^\#) Q_1 & 0 \\ -Q_2 & -(Q_1 + Q_2) & Q_2 & 0 & 0 \\ A_d^T(\gamma^\#)P + Q_2 & Q_2 & -Q_2 & 0 & \bar{\tau} A_d^T(\gamma^\#) Q_2 \\ \bar{\tau} Q_1 A(\gamma^\#) & 0 & 0 & -Q_1 & 0 \\ 0 & 0 & \bar{\tau} Q_2 A_d(\gamma^\#) & 0 & -Q_2 \end{bmatrix} < 0 \quad (67)$$

for all $\gamma^{\#} \in \{\underline{\gamma}, \overline{\gamma}\}$. *Then (50) is asymptotically stable for any constant* τ*, such that* $0 \leq \tau \leq \bar{\tau}$*, and all* $\gamma \in [\underline{\gamma}, \overline{\gamma}]$.

Another delay-dependent stability test is given below. The approach is based on a generalization of the stability test in (Park, 1999) to LPV systems.

Theorem 5.2 *Consider the system (51) where* $0 \leq \tau \leq \bar{\tau}$*, and* $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$*. Suppose that there exist positive-definite matrices* P, Z *and* Q *such that for any constant matrix* $\bar{M}, \bar{W}$*, the following LMI condition is satisfied*

$$\begin{bmatrix} \begin{pmatrix} A(\gamma)^T P + A_d(\gamma)^T \bar{M}^T \\ +PA(\gamma) + \bar{M}A_d(\gamma) \\ +Q + A(\gamma)^T Z A(\gamma) \end{pmatrix} & \bar{\tau}\bar{M}A_d(\gamma) + \bar{W} & \begin{pmatrix} PA_d(\gamma) - \bar{M}A_d(\gamma) \\ +A(\gamma)^T Z A_d(\gamma) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Z + \bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T \end{pmatrix} & -\bar{W}^T & \bar{\tau}\bar{W}^T \\ * & * & -Q + A_d(\gamma)^T Z A_d(\gamma) & 0 \\ * & * & * & -Z \end{bmatrix} < 0 \qquad (68)$$

for all $\gamma \in \Gamma$*. Then (51) is asymptotically stable for all* $\tau \in [0, \bar{\tau}]$.

Before we proceed with the proof of this theorem, we present a lemma that will be used later on.

Lemma 5.1 ((Park *et al.*, 1998)) *For any* $X > 0$ *and any* W*, and* $\Sigma = (W^T X + I)X^{-1}(XW + I)$ *the following inequality holds*

$$-2\int_{\Omega} b^T(\alpha)a(\alpha)\mathrm{d}\alpha \leq \int_{\Omega} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix}^T \begin{bmatrix} X & XW \\ W^T X & \Sigma \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} \mathrm{d}\alpha \qquad (69)$$

Proof. [of Theorem 5.2] Consider the Lyapunov-Krasovskii functional $V : \mathbb{R}_+ \times \mathcal{C}_{2\tau} \to \mathbb{R}_+$ given by

$$V(t, x_t) = x^T(t)Px(t) + \int_{t-\tau}^{t} x^T(\alpha)Qx(\alpha)\,\mathrm{d}\alpha$$
$$+\int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\mathrm{d}\alpha\,\mathrm{d}\beta \quad (70)$$

Where P, Q, Y are positive-definite matrices. According to Corollary 3.1, V is positive definite and has an infinitesimal upper bound. Taking the derivative of V along the trajectories of the system

(51) which is equivalent to system (1), one obtains,

$$\begin{aligned}\dot{V}(t) &= x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^T P + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t)\\ &+ x^T(t)[P(I-M)A_d(\gamma(t))]x(t-\tau) + x^T(t-\tau)[P(I-M)A_d(\gamma(t))]^T x(t)\\ &- 2x^T(t)PMA_d(\gamma(t))\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha\\ &+ x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau)\\ &+ \int_{-\tau}^{0}[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)]^T Y[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)]\ \mathrm{d}\beta\\ &- \int_{-\tau}^{0}[A(\gamma(t+\beta))x(t+\beta) + A_d(\gamma(t+\beta))x(t+\beta-\tau)]^T Y[A(\gamma(t+\beta))x(t+\beta) + A_d(\gamma(t+\beta))x(t+\beta-\tau)]\ \mathrm{d}\beta\end{aligned}$$

The last integral can be written as

$$\begin{aligned}&\int_{-\tau}^{0}[A(\gamma(t+\beta))x(t+\beta) + A_d(\gamma(t+\beta))x(t+\beta-\tau)]^T Y[A(\gamma(t+\beta))x(t+\beta) + A_d(\gamma(t+\beta))x(t+\beta-\tau)]\ \mathrm{d}\beta\\ &\qquad = \int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha \qquad (71)\end{aligned}$$

Hence,

$$\begin{aligned}\dot{V}(t) &\leq x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^T P + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t)\\ &+ x^T(t)[P(I-M)A_d(\gamma(t))]x(t-\tau) + x^T(t-\tau)[P(I-M)A_d(\gamma(t))]^T x(t)\\ &- 2x^T(t)PMA_d(\gamma(t))\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha\\ &+ x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau)\\ &+ \bar{\tau}[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)]^T Y[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)]\\ &- \int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha \qquad (72)\end{aligned}$$

Let now $a(\alpha) = [A(\gamma)x(\alpha) + A_d(\gamma)x(\alpha-\tau)]$ and $b(\alpha) = A_d^T(\gamma(t))M^T Px(t)$ and use Lemma 5.1 to obtain

$$\begin{aligned}&-2x^T(t)PMA_d(\gamma(t))\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha\\ \leq\ &\tau x^T(t)PMA_d(\gamma(t))[W^T Y + I]Y^{-1}[YW + I]A_d^T(\gamma(t))M^T Px(t)\\ +\ &2x^T(t)PMA_d(\gamma(t))W^T Y\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha\\ +\ &\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha \qquad (73)\end{aligned}$$

for any matrix W. Using the fact

$$\int_{t-\tau}^{t}[A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]\ \mathrm{d}\alpha = \int_{t-\tau}^{t}\dot{x}(\alpha)\ \mathrm{d}\alpha = x(t) - x(t-\tau) \qquad (74)$$

and substituting (73) and (74) into (72), we have

$$\begin{aligned}
\dot{V}(t) \leq\ & x^T(t)[(A(\gamma(t)) + MA_d(\gamma(t)))^T P + P(A(\gamma(t)) + MA_d(\gamma(t)))]x(t) \\
& + x^T(t)[P(I-M)A_d(\gamma(t))]x(t-\tau) + x^T(t-\tau)[P(I-M)A_d(\gamma(t))]^T x(t) \\
& \bar{\tau} x^T(t) PMA_d(\gamma(t))[W^T Y + I]Y^{-1}[YW + I]A_d^T(\gamma(t))M^T Px(t) \\
& + 2x^T(t)PMA_d(\gamma(t))W^T Y[x(t) - x(t-\tau)] \\
& + x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau) \\
& + \bar{\tau}[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)]^T Y[A(\gamma(t))x(t) + A_d(\gamma(t))x(t-\tau)] \qquad (75)
\end{aligned}$$

In the above inequality, W is an arbitrary matrix. Noticing that

$$\begin{aligned}
& \bar{\tau} x^T(t) PMA_d(\gamma)[W^T Y + I]Y^{-1}[YW + I]A_d^T(\gamma)M^T Px(t) \\
=\ & \bar{\tau} x^T(t) PMA_d(\gamma)[W^T YW + W^T + W - Y^{-1}]A_d^T(\gamma)M^T Px(t) \\
& + 2\bar{\tau} x^T(t)PMA_d(\gamma)Y^{-1}A_d^T(\gamma)M^T Px(t) \\
=\ & \bar{\tau} x^T(t) PMA_d(\gamma)Y^{-1}[YW^T YY^{-1}YWY + YW^T Y + YWY - Y]Y^{-1}A_d^T(\gamma)M^T Px(t) \\
& + 2\bar{\tau} x^T(t)PMA_d(\gamma)Y^{-1}A_d^T(\gamma)M^T Px(t) \\
=\ & \bar{\tau} y^T(t)[\bar{W}^T Y^{-1}\bar{W} + \bar{W}^T + \bar{W} - Y]y(t) + 2\bar{\tau} x^T(t)PMA_d(\gamma)y(t) \qquad (76)
\end{aligned}$$

where $y(t)$ and $\bar{W}$ are defined by

$$y(t) = Y^{-1}A_d^T(\gamma)M^T Px(t), \qquad \bar{W} = YWY$$

because W is an arbitrary matrix, $\bar{W}$ is also an arbitrary matrix, and

$$2x^T(t)PMA_d(\gamma)W^T Y[x(t) - x(t-\tau)] = 2y^T(t)\bar{W}^T[x(t) - x(t-\tau)] \qquad (77)$$

and substituting (76) and (77) into (75), one obtains

$$\begin{aligned}
\dot{V}(t) \leq\ & x^T(t)[(A(\gamma) + MA_d(\gamma))^T P + P(A(\gamma) + MA_d(\gamma)) + Q]x(t) \\
+\ & 2x^T(t)[P(I-M)A_d(\gamma)]x(t-\tau) - x^T(t-\tau)Qx(t-\tau) \\
+\ & \bar{\tau} y^T(t)[\bar{W}^T Y^{-1}\bar{W} + \bar{W}^T + \bar{W} - Y]y(t) + 2\bar{\tau} x^T(t)PMA_d(\gamma)y(t) \\
+\ & 2y^T(t)\bar{W}^T[x(t) - x(t-\tau)] \\
+\ & \bar{\tau}\,[A(\gamma)x(t) + A_d(\gamma)x(t-\tau)]^T Y[A(\gamma)x(t) + A_d(\gamma)x(t-\tau)] \qquad (78)
\end{aligned}$$

Let now $\bar{X}^T = [x(t)\ \ y(t)\ \ x(t-\tau)]^T$. Inequality (78) can be rewritten as follows:

$$\dot{V}(t) \leq \bar{X}^T \begin{bmatrix} \begin{pmatrix} (A(\gamma) + MA_d(\gamma))^T P \\ +P(A(\gamma) + MA_d(\gamma)) \\ +Q + A(\gamma)^T \bar{\tau} YA(\gamma) \end{pmatrix} & \bar{\tau} PMA_d(\gamma) + \bar{W} & \begin{pmatrix} P(I-M)A_d(\gamma) \\ +A(\gamma)^T \bar{\tau} YA_d(\gamma) \end{pmatrix} \\ * & \begin{pmatrix} -Y\bar{\tau} + \bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T + \bar{\tau}\bar{W}^T Y^{-1}\bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & -Q + A_d(\gamma)^T \bar{\tau} YA_d(\gamma) \end{bmatrix} \bar{X} \qquad (79)$$

Using Schur complements, the matrix in the right hand side of (79) is negative if

$$\begin{bmatrix} \begin{pmatrix} (A(\gamma)+MA_d(\gamma))^T P \\ +P(A(\gamma)+MA_d(\gamma)) \\ +Q+A(\gamma)^T\bar{\tau}YA(\gamma) \end{pmatrix} & \bar{\tau}PMA_d(\gamma)+\bar{W} & \begin{pmatrix} P(I-M)A_d(\gamma) \\ +A(\gamma)^T\bar{\tau}YA_d(\gamma) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Y\bar{\tau}+\bar{\tau}\bar{W} \\ +\tau\bar{W}^T \end{pmatrix} & -\bar{W}^T & \tau\bar{W}^T \\ * & * & -Q+A_d(\gamma)^T\bar{\tau}YA_d(\gamma) & 0 \\ * & * & * & -\bar{\tau}Y \end{bmatrix} < 0 \qquad (80)$$

Let $Z = \bar{\tau}Y$ and $\bar{M} = PM$. Then (80) is equivalent to (68). Because M is an arbitrary matrix and P is positive definite, $\bar{M}$ is also an arbitrary matrix. If inequality (68) is satisfied for all $\gamma \in \Gamma$, the derivative of V is uniformly negative definite and the system (1) is asymptotically stable. This completes the proof of the theorem. ■

Remark 4 It can be shown that – when restricted to LTI systems – (68) is slightly more general than the condition in (Park, 1999) assuming that A_d is invertible. This is due to the extra variable $\bar{W}$. In fact, it can be shown (Zhang *et al.*, 2001) that (68) reduces to the condition in (Park, 1999) when $\bar{W} = 0$. In case A_d is not invertible, condition (68) (with $\bar{W} = 0$) is implied by the condition in (Zhang *et al.*, 2001).

Inequalities (68) is an infinite-dimensional set of LMI's in the unknowns $P, \bar{M}, \bar{W}, Z$ and Q. These LMI's can be checked by gridding the parameter space. As before, gridding can be avoided in case the system matrices are affine functions of the parameter γ.

Corollary 5.2 *Consider the system (51) with*

$$A(\gamma) = A_0 + \gamma A_1, \qquad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \qquad \gamma \in [\underline{\gamma}, \overline{\gamma}] \qquad (81)$$

Suppose there exist positive-definite matrices P, Q and Z such that for any constant matrix $\bar{M}, \bar{W}$, the following matrix inequality is satisfied

$$\begin{bmatrix} \begin{pmatrix} A(\gamma^\#)^T P + A_d(\gamma^\#)^T \bar{M}^T \\ +PA(\gamma^\#)+\bar{M}A_d(\gamma^\#) \\ +Q+A(\gamma^\#)^T ZA(\gamma^\#) \end{pmatrix} & \bar{\tau}\bar{M}A_d(\gamma^\#)+\bar{W} & \begin{pmatrix} PA_d(\gamma^\#)-\bar{M}A_d(\gamma^\#) \\ +A(\gamma^\#)^T ZA_d(\gamma^\#) \end{pmatrix} & 0 \\ * & \begin{pmatrix} -Z+\bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T \end{pmatrix} & -\bar{W}^T & \bar{\tau}\bar{W}^T \\ * & * & -Q+A_d(\gamma^\#)^T ZA_d(\gamma^\#) & 0 \\ * & * & * & -Z \end{bmatrix} < 0 \quad (82)$$

where $\gamma^\# \in \{\underline{\gamma}, \overline{\gamma}\}$. Then (1) is asymptotically stable for any constant delay τ, such that $0 \leq \tau \leq \bar{\tau}$.

The third delay-dependent stability result for (1) is given next.

Theorem 5.3 *Consider the system (1) with $\gamma \in \Gamma = [\underline{\gamma}, \overline{\gamma}]$. If there exist positive-definite matrices Q_1, Q_2, Q, Z and P, for any constant matrix $R, \bar{W}$, such that the following condition is satisfied*

$$
\begin{bmatrix}
H_{11}(\gamma) & -Q_2 & \bar{\tau} R A_d(\gamma) + \bar{W} & H_{14}(\gamma) & \bar{\tau} A^T(\gamma) Q_1 & 0 & 0 \\
* & -(Q_1 + Q_2) & 0 & Q_2 & 0 & 0 & 0 \\
* & * & H_{33}(\gamma) & -\bar{W}^T & 0 & 0 & \bar{\tau}\bar{W} \\
* & * & * & H_{44}(\gamma) & 0 & \bar{\tau} A^T(\gamma) Q_2 & 0 \\
* & * & * & * & -Q_1 & 0 & 0 \\
* & * & * & * & * & -Q_2 & 0 \\
 & * & * & * & * & * & -Z
\end{bmatrix} < 0, \quad \forall \gamma \in \Gamma \tag{83}
$$

where,

$$
\begin{aligned}
H_{11}(\gamma) &= A^T(\gamma)P + A_d^T(\gamma)R^T + PA(\gamma) + RA_d(\gamma) + Q - Q_2 + A^T(\gamma) Z A(\gamma) && (84)\\
H_{14}(\gamma) &= PA_d(\gamma) + Q_2 - RA_d(\gamma) + A^T(\gamma) Z A_d(\gamma) && (85)\\
H_{33}(\gamma) &= -Z + \bar{\tau}\bar{W} + \bar{\tau}\bar{W}^T && (86)\\
H_{44}(\gamma) &= -Q - Q_2 + A_d^T(\gamma) Z A_d(\gamma) && (87)
\end{aligned}
$$

then (1) is asymptotically stable for any constant delay $\tau \in [0, \bar{\tau}]$.

Proof. Consider the following positive-definite functional,

$$
\begin{aligned}
V(t, x_t) &= 2x^T(t)Px(t) + \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha)]^T P_1 [A(\gamma(\alpha))x(\alpha)] \, \mathrm{d}\alpha \, \mathrm{d}\beta \\
&+ \int_{-\tau}^{0}\int_{t+\beta}^{t} [A_d(\gamma(\alpha))x(\alpha-\tau)]^T P_2 [A_d(\gamma(\alpha))x(\alpha-\tau)] \, \mathrm{d}\alpha \, \mathrm{d}\beta + \int_{t-\tau}^{t} x^T(\alpha) Q x(\alpha) \, \mathrm{d}\alpha \\
&+ \int_{-\tau}^{0}\int_{t+\beta}^{t} [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)]^T Y [A(\gamma(\alpha))x(\alpha) + A_d(\gamma(\alpha))x(\alpha-\tau)] \mathrm{d}\alpha \, \mathrm{d}\beta \quad (88)
\end{aligned}
$$

where P, P_1, P_2, Q, Y are constant positive-definite matrices. This functional $V(t, x_t)$ is a positive definite functional and has an infinitesimal upper bound according to the Lemma 3.1. Taking now the derivative of V along the trajectories of the system (50) or (51) and defining

$$
y(t) = -\int_{t-\tau}^{t} A(\gamma(\alpha))x(\alpha) \, \mathrm{d}\alpha, \qquad g(t) = Y^{-1} A_d^T(\gamma(t)) M^T P x(t) \tag{89}
$$

one obtains,

$$
\dot{V}(t) \leq \begin{bmatrix} x(t) \\ y(t) \\ x(t-\tau) \end{bmatrix}^T
\begin{bmatrix}
\begin{pmatrix} A(\gamma)^T P + PA(\gamma) \\ + \bar{\tau} A^T(\gamma) P_1 A(\gamma) \\ -P_2/\bar{\tau} \end{pmatrix} & -P_2/\bar{\tau} & P_2/\bar{\tau} + PA_d(\gamma) \\
-P_2/\bar{\tau} & -P_1/\bar{\tau} - P_2/\bar{\tau} & P_2/\bar{\tau} \\
P_2/\bar{\tau} + A_d^T(\gamma)P & P_2/\bar{\tau} & \begin{pmatrix} \bar{\tau} A_d^T(\gamma) P_2 A_d(\gamma) \\ -P_2/\bar{\tau} \end{pmatrix}
\end{bmatrix}
\begin{bmatrix} x(t) \\ y(t) \\ x(t-\tau) \end{bmatrix}
$$

$$+\begin{bmatrix} x(t) \\ g(t) \\ x(t-\tau) \end{bmatrix}^T \begin{bmatrix} \begin{pmatrix} (A(\gamma)+MA_d(\gamma))^T P \\ +P(A(\gamma)+MA_d(\gamma)) \\ +Q+A(\gamma)^T\bar{\tau}YA(\gamma) \end{pmatrix} & \bar{\tau}PMA_d(\gamma)+\bar{W} & \begin{pmatrix} P(I-M)A_d(\gamma) \\ +A(\gamma)^T\bar{\tau}YA_d(\gamma) \end{pmatrix} \\ * & \begin{pmatrix} -Y\bar{\tau}+\bar{\tau}\bar{W} \\ +\bar{\tau}\bar{W}^T+\bar{\tau}\bar{W}^TY^{-1}\bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & -Q+A_d(\gamma)^T\bar{\tau}YA_d(\gamma) \end{bmatrix} \begin{bmatrix} x(t) \\ g(t) \\ x(t-\tau) \end{bmatrix}$$

which can be rewritten as

$$\dot{V}(x(t)) \quad \leq \quad \begin{bmatrix} x(t) \\ y(t) \\ g(t) \\ x(t-\tau) \end{bmatrix}^T [\Sigma] \begin{bmatrix} x(t) \\ y(t) \\ g(t) \\ x(t-\tau) \end{bmatrix} \tag{90}$$

where,

$$[\Sigma] = \begin{bmatrix} \begin{pmatrix} (A(\gamma)+MA_d(\gamma))^T P \\ +P(A(\gamma)+MA_d(\gamma)) \\ +Q+\bar{\tau}A^T(\gamma)YA(\gamma) \\ A^T(\gamma)P+PA(\gamma) \\ +\bar{\tau}A^T(\gamma)P_1A(\gamma)-P_2/\bar{\tau} \end{pmatrix} & -P_2/\bar{\tau} & \bar{\tau}PMA_d(\gamma)+\bar{W} & \begin{pmatrix} PA_d(\gamma)+P_2/\bar{\tau} \\ +P(I-M)A_d(\gamma) \\ \bar{\tau}A^T(\gamma)YA_d(\gamma) \end{pmatrix} \\ * & -(P_1+P_2)/\bar{\tau} & 0 & P_2/\bar{\tau} \\ * & * & \begin{pmatrix} -Y\bar{\tau}+\bar{\tau}\bar{W}+\bar{\tau}\bar{W}^T \\ +\bar{\tau}\bar{W}^TY^{-1}\bar{W} \end{pmatrix} & -\bar{W}^T \\ * & * & * & \begin{pmatrix} -Q-P_2/\bar{\tau} \\ +\bar{\tau}A_d^T(\gamma)P_2A_d(\gamma) \\ +\bar{\tau}A_d^T(\gamma)YA_d(\gamma) \end{pmatrix} \end{bmatrix} \tag{91}$$

The stability of system (1) can be guaranteed by satisfying $[\Sigma] < 0$. This inequality is not an LMI. However, using $Q_1 = P_1/\bar{\tau}$,$Q_2 = P_2/\bar{\tau}$, $R = PM$, $Z = \bar{\tau}Y$ and using the Schur complement theorem, the requirement for $[\Sigma]$ to be negative definite is equivalent to (83) which is an LMI. ■

The solution of the infinite-dimensional set of matrix inequalities in (83) can be checked by gridding the parameter space. As with Theorems 5.1-5.2 gridding can be avoided if the state matrices are an affine function of the parameter.

Corollary 5.3 *Consider system (51) where*

$$A(\gamma) = A_0 + \gamma A_1, \quad A_d(\gamma) = A_{d0} + \gamma A_{d1}, \quad \gamma \in [\underline{\gamma}, \overline{\gamma}] \tag{92}$$

Suppose there exist constant positive-definite matrices $Q_1, Q_2, Q, Z, P,$ and any constant matrix

$R, \bar{W}$, such that the following condition is satisfied

$$\begin{bmatrix} H_{11}(\gamma^{\#}) & -Q_2 & \bar{\tau}RA_d(\gamma^{\#}) + \bar{W} & H_{14}(\gamma^{\#}) & \bar{\tau}A^T(\gamma^{\#})Q_1 & 0 & 0 \\ * & -(Q_1+Q_2) & 0 & Q_2 & 0 & 0 & 0 \\ * & * & H_{33}(\gamma^{\#}) & -\bar{W}^T & 0 & 0 & \bar{\tau}\bar{W} \\ * & * & * & H_{44}(\gamma^{\#}) & 0 & \bar{\tau}A^T(\gamma^{\#})Q_2 & 0 \\ * & * & * & * & -Q_1 & 0 & 0 \\ * & * & * & * & * & -Q_2 & 0 \\ & * & * & * & * & * & -Z \end{bmatrix} < 0 \quad (93)$$

for all $\gamma^{\#} \in \{\underline{\gamma}, \overline{\gamma}\}$, with $H_{11}(\gamma)$, $H_{14}(\gamma)$ $H_{33}(\gamma)$ and $H_{44}(\gamma)$ as in (84)-(87). Then (51) is asymptotically stable for all constant delays $\tau \in [0, \bar{\tau}]$.

Proof. It suffices to show that (93) implies (83). Let $F(\gamma)$ denote the matrix on the left hand side of (83). Since $F(\gamma) = F_0 + \gamma F_1 + \gamma^2 F_2$, where

$$F_2 = \begin{bmatrix} A_1^T Z A_1 & 0 & 0 & A_1^T Z A_{d1} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & A_{d1}^T Z A_{d1} & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix} \quad (94)$$

and $F_2 \geq 0$, $F(\gamma)$ is convex matrix function of γ. Using Lemma 4.1, inequality (93) implies (83). ■

6 Numerical Example

In this section we consider a numerical example motivated by control of chatter during milling. The dynamics depend both on the cutter speed, as well as on the machine tool and piece contact geometry. The force acting on the tool is a function not only of the current displacement of the tool, but also the surface characteristics, hence the displacement at the previous tool pass. This induces a delay into the system. The force depends also on the angular position of the blade, which plays the role of a time-varying parameter. Figure 1 depicts the geometry of the cutting process. As shown in the figure, for this example, the cutter has two blades that are used to remove the material of the workpiece. The blades are assumed to rotate at a constant speed ω. The equations of this system can be derived directly from figure 1 as follows

$$m_1\ddot{x}_1 + k_1(x_1 - x_2) = f \quad (95a)$$

$$m_2\ddot{x}_2 + c\dot{x}_2 + k_1(x_2 - x_1) + k_2 x_2 = 0 \quad (95b)$$

$$f = k\sin(\phi + \beta)h(t) \quad (95c)$$

$$h(t) = h_{ave} + \sin(\phi)[x_1(t-\tau) - x_1(t)] \quad (95d)$$

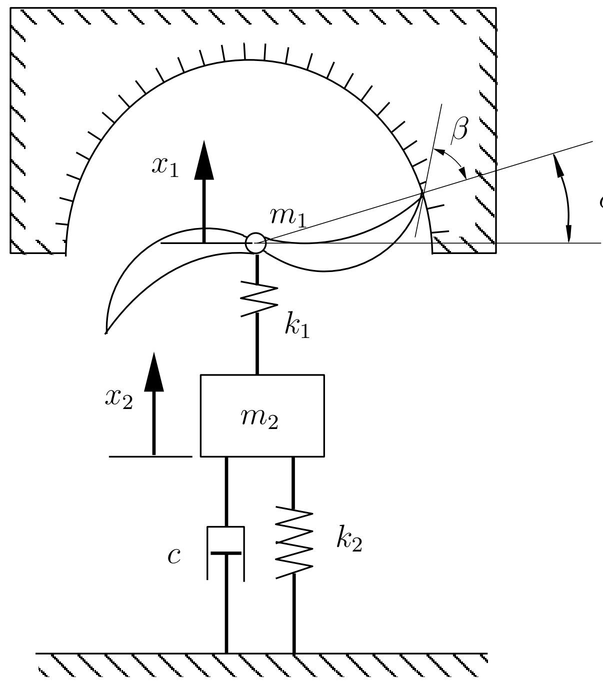

Figure 1: A simplified geometry of a milling process.

where k_1 and k_2 are the stiffnesses of the two springs, c is the damping coefficient, m_1 is the mass of the cutter, m_2 is the mass of the "spindle". The displacements of the blade and tool are x_1 and x_2 respectively. The angle β depends on the particular material and tool used, and is constant. The angle ϕ denotes the angular position of the blade and k is the cutting stiffness. h_{ave} is the average chip thickness (here assumed, without loss of generality, that $h_{ave} = 0$) and $\tau = \pi/\omega$ is the delay between successive passes of the blades. The previous equations can be written as

$$\ddot{x}_1 = \frac{1}{m_1}[-k_1 x_1 + k_1 x_2 - k\sin(\phi+\beta)\sin(\phi)x_1 + k\sin(\phi+\beta)\sin(\phi)x_1(t-\tau)] \quad (96a)$$

$$\ddot{x}_2 = \frac{1}{m_2}[k_1 x_1 - k_1 x_2 - k_2 x_2 - c\dot{x}_2] \quad (96b)$$

or in state-space form,

$$\dot{X}(t) = A(\phi)X(t) + A_d(\phi)X(t-\tau) \quad (97)$$

where $X = [x_1,\ x_2,\ \dot{x}_1,\ \dot{x}_2]^T$ and

$$A(\phi) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1 + k\sin(\phi)\sin(\phi+\beta)}{m_1} & \frac{k_1}{m_1} & 0 & 0 \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & 0 & -\frac{c}{m_2} \end{bmatrix}, \quad A_d(\phi) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{k\sin(\phi)\sin(\phi+\beta)}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (98)$$

Assume that the problem data are as follows: $m_1 = 1,\ m_2 = 2,\ k_1 = 10,\ k_2 = 20,\ c = 0.5,\ \beta = 70\,\text{deg}$. Since

$$\sin(\phi)\sin(\phi+\beta) = \frac{1}{2}[\cos(\beta) - \cos(2\phi+\beta)] = 0.1710 - 0.5\cos(2\phi+\beta) \quad (99)$$

the system matrices take the form

$$A(\gamma) = A_0 + \gamma\, A_1, \qquad A_d(\gamma) = A_{d0} + \gamma\, A_{d1} \tag{100}$$

where,

$$\begin{aligned} A_0 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10+0.1710\,k) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5\,k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ A_{d0} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1710\,k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5\,k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{101}$$

and $\gamma = \cos(2\phi + \beta) \in [-1, 1]$. We have formulated the process as an LPV time-delayed system which is also parametrically dependent on the cutting stiffness k (constant in this example). Our objective is to find, for a given angular velocity ω of the blades, the value of the cutting stiffness k, such that this LPV system will be stable for all $\gamma(t) \in [-1, 1]$ and delay $\tau = \pi/\omega$. In addition, we wish to find, for a given cutting stiffness k and speed ω, the maximal allowed time delay $\bar{\tau}$. If $\pi/\omega < \bar{\tau}$, the system will be stable. Several tests were performed, using the results of Corollaries 5.1-5.3. The results of the analysis are shown in figure 2. These results indicate that the stability conditions of Corollary 5.2 and Corollary 5.3 are better than the one of Corollary 5.1. This is because the system in (50), which is used in Corollary 5.1, is a special case of the system in (51), which is used in Corollaries 5.2 and 5.3. Moreover, the Lyapunov functionals used in Corollaries 5.1 and 5.2 are both special cases of the functional in Corollary 5.3. It is therefore expected that Corollary 5.3 should give a less conservative stability condition than Corollaries 5.1 and 5.2. This is verified by the calculation results shown in figure 2. From the same figure, Corollaries 5.2 and 5.3 seem to provide an almost "delay-independent" stability condition if the parameter $k < 0.275$. For comparison, we also applied the results of (Gu, 1997) to this example. The results of (Gu, 1997) assume that the delay interval is known and thus, are not readily applicable to the calculation of the maximum delay interval. A bisection method was used to calculate $\bar{\tau}$ for each value of k in this case. On the contrary, the maximum delay from Corollaries 5.1-5.3 can be calculated directly, since these conditions can be cast as generalized eigenvalue problems (Boyd *et al.*, 1994). Using three discrete elements, the result from (Gu, 1997) is shown as the solid line in figure 2. The prediction for the stability region in this case is less conservative than the one predicted by Corollaries 5.1-5.3. However, the computer CPU time was an order of magnitude larger than the one required for the stability tests of Corollaries 5.1-5.3.

We also applied the delay-independent stability conditions of Theorems 4.1-4.5. Note that for the milling example investigated here, $A_2 = A_{d2} = 0$ and $\dot{\gamma} \in (-\infty, \infty)$. The system will be delay-independent stable if $k \le K_m$. Table 6 summarizes the results of the tests that ensure delay-independent stability.

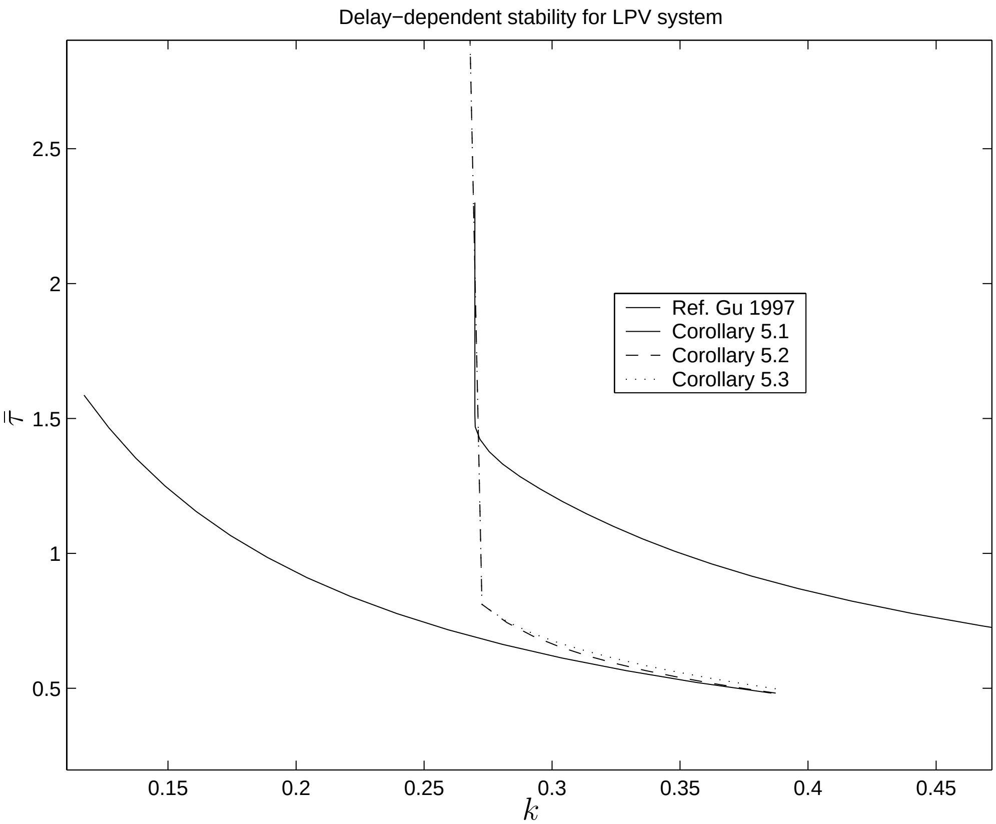

Figure 2: Maximal allowable time delay predicted by Corollaries 5.1-5.3.

Finally, we emphasize that all results shown in figure 2 are conservative. The maximal value of the delay that ensures stability is not known for this example.

7 Conclusions

We have developed stability tests for Linear Parameter Varying (LPV) systems subject to delays. Both delay-independent and delay-dependent stability conditions are derived. Delay-independent criteria ensure that stability is maintained for all parameter values and every value of the time-delay. The delay-dependent tests incorporate explicitly the maximum allowable delay before loss of stability. Using gridding of the parameter space both delay-dependent and delay-independent stability tests can be cast as convex optimization problems involving LMI's, which can be solved efficiently using current computer software. A numerical example motivated by the problem of machine milling is used to compare the developed stability analysis tests. In the delay-independent case, bounds on the parameter variation can also be incorporated using parameter-dependent Lyapunov-Krasovskii functionals. The ensuing stability tests assume that $\gamma(t)$, $\gamma(t-\tau)$ and $\dot{\gamma}(t)$ are independent. The independence of $\gamma(t)$ and $\gamma(t-\tau)$ is a reasonable assumption for delay-independent stability ($\bar{\tau} \to \infty$) and for no variation bounds on γ. As the bound on $\dot{\gamma}$ becomes increasingly smaller the

Table 1: Results for delay-independent stability. The system will be delay-independent stable if $k \leq K_m$.

Theorem	K_m	**Notes**
Th. 4.1	0.2671	$Q(\gamma) = Q_0$
Th. 4.1	0.2695	$Q(\gamma) = Q_0 + \gamma Q_1$
Th. 4.1	0.3043	$Q(\gamma)$, gridding
Th. 4.2	N/A	$A_2 = 0$
Th. 4.3	same as Th. 4.1	$P = P_0$, constant
Th. 4.4	0.2695	Remark 2
Th. 4.5	0.2695	Remark 3

conservativeness of the results increases, as $\gamma(t)$ and $\gamma(t-\tau)$ cannot be treated as independent. Delay-dependent stability tests may be more appropriate in this case. Our delay-dependent results do not incorporate any parameter variation bounds. Nonetheless, it is expected that for several problems, delay-dependent results may not be very conservative even if they do not incorporate explicitly any bounds on the parameter variation. This is because any such bounds may be implicit in the maximum allowable delay. That is, for some LPV time-delayed systems, the rate of the parameter variation and the amount of delay may be related. For the milling problem, for example, the rate of the parameter γ depends on the amount of the maximal delay as well as the parameter itself. In particular, since $\dot{\phi} = \omega$, one immediately obtains that

$$\dot{\gamma} = -2\sin(2\phi + \beta)\dot{\phi} = -2\sin(2\phi + \beta)(\pi/\tau) \tag{102}$$

Notice that γ, $\dot{\gamma}$ and τ are related via

$$\gamma^2 + \left(\frac{\tau\dot{\gamma}}{2\pi}\right)^2 = 1$$

Notice that as $\tau \to 0$, then $\dot{\gamma} \to \infty$ and vice versa. This implies that in order to capture the effect of small delays in stability (i.e., truly delay-dependent stability conditions) we need to allow arbitrarily fast variations of the parameter γ. In general, however, parameter-dependent functionals must be introduced in Theorems 5.1-5.3 to reduce the conservatism even further. Finally, it should be pointed out that the *analysis* of LPV time-delayed systems can be addressed from the standard point of view of uncertain time-delay systems. This is the approach followed, for instance, by Gu (1997). However, such approaches lead to conservative results during *synthesis*. For LPV systems, the parameter γ – although a priori unknown – can be measured on-line and thus used to adjust the controller gains. It is during controller synthesis for LPV time-delayed systems that the results of this paper have a distinct advantage over existing results in the literature. Such synthesis results are left for future investigation.

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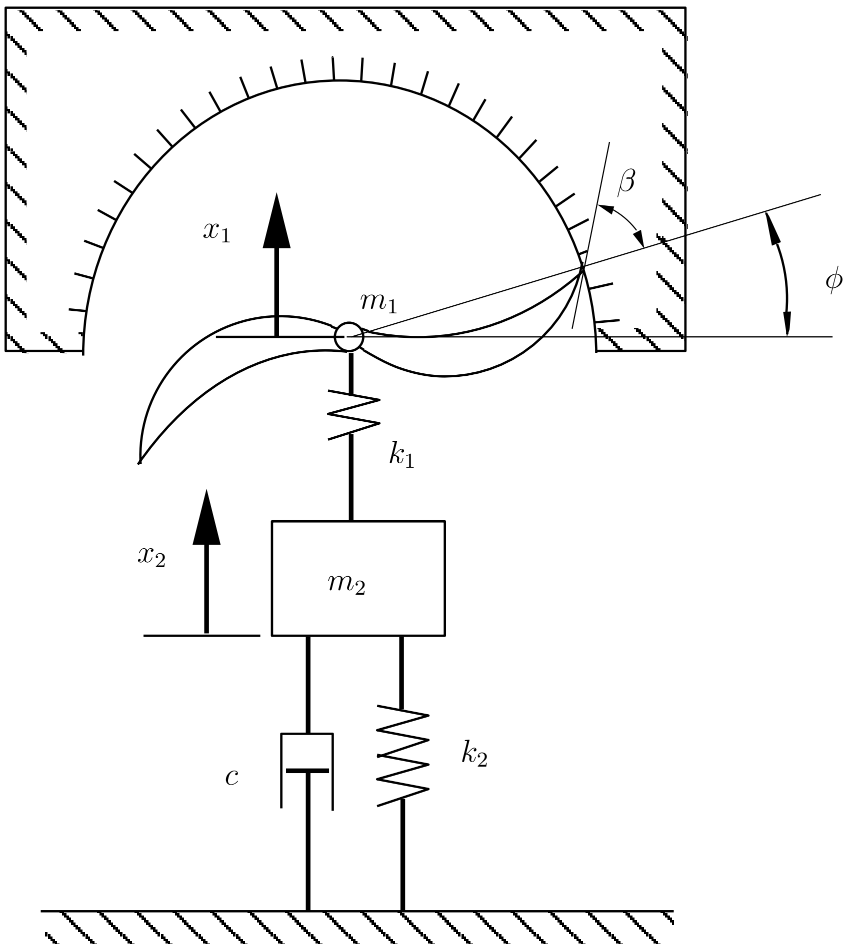
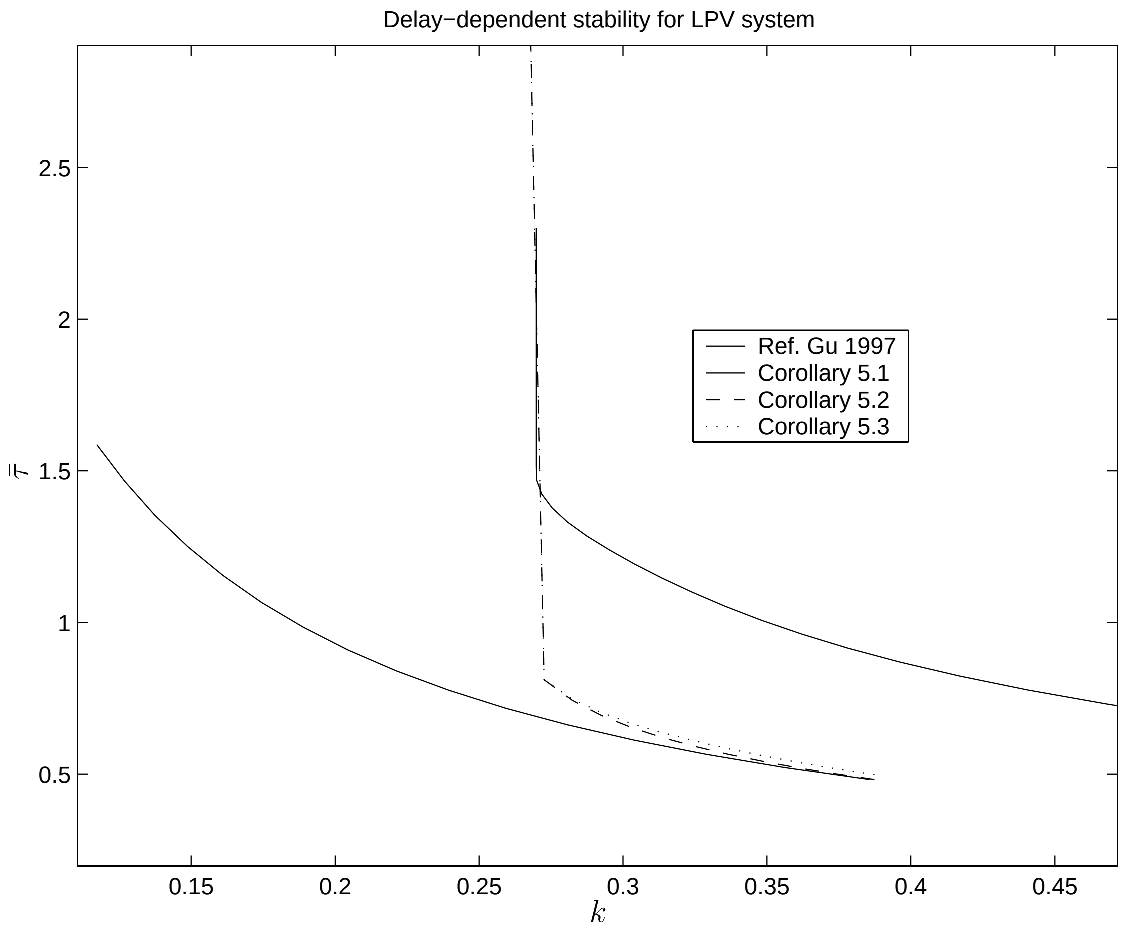
Theorem	K_m	**Notes**
Th. 4.1	0.2671	$Q(\gamma) = Q_0$
Th. 4.1	0.2695	$Q(\gamma) = Q_0 + \gamma Q_1$
Th. 4.1	0.3043	$Q(\gamma)$, gridding
Th. 4.2	N/A	$A_2 = 0$
Th. 4.3	same as Th. 4.1	$P = P_0$, constant
Th. 4.4	0.2695	Remark 2
Th. 4.5	0.2695	Remark 3


x_1
m_1
β
ϕ
k_1
x_2
m_2
c
k_2



Delay−dependent stability for LPV system
2.5
2
1.5
1
0.5
$\bar{\tau}$
0.15
0.2
0.25
0.3
0.35
0.4
0.45
k
Ref. Gu 1997
Corollary 5.1
Corollary 5.2
Corollary 5.3

Figure Captions

Figure 1 A simplified geometry of a milling process.

Figure 2 Maximal allowable time delay predicted by Corollaries 5.1-5.3.