SUBOPTIMAL CONTROL OF RIGID BODY MOTION WITH A QUADRATIC COST

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Abstract. This paper considers the problem of controlling the rotational motion of a rigid body using three independent control torques. Given a quadratic cost we seek stabilizing state feedback controllers which guarantee that all motions starting within a specified bounded set satisfy a specified bound on a quadratic performance index or cost. For a special class of cost functions, we present explicit expressions for the optimal stabilizing controllers. For the general case, we present sufficient conditions which guarantee the existence of linear, suboptimal, stabilizing controllers.

Keywords. Attitude control; quadratic cost; linear matrix inequalities.

1 INTRODUCTION

In this paper we consider the problem of controlling the rotational motion of a rigid body using three independent control torques. The minimal requirement on the controller is to stabilize the body about a specified orientation. In addition, we require the controller to guarantee that a quadratic performance index or cost be bounded for all initial states lying in a given set. Ideally, we would like to minimize the cost; since this is, in general, a difficult task we are contented with minimizing an upper bound for the cost. By minimizing this upper bound, it is hoped that one can achieve acceptable performance.

The problem addressed in this paper is of importance in aerospace engineering since it corresponds to the control of the orientation of a spacecraft. Previous research on this problem has been mostly directed towards the time or fuel-optimal control problem; see, for example, (Athans, *et al.*, 1963; Branets *et al.*, 1984; Dixon *et al.*, 1970) and the recent survey paper by Scrivener and Thomson (1994).

As far as the optimal regulation of the angular velocity or the angular momentum vector is concerned, the earliest results seem to be the ones reported in (Debs and Athans, 1969; Kumar, 1965; and Windeknecht, 1963). More recent results on the same problem were reported by Dabbous and Ahmet (1982) and Bourdache-Siguerdidjane (1991). In the present work we are interested however with the more complicated problem of optimal control of the complete attitude equations, i.e. dynamics and kinematics.

The equations describing the rotational motion of a rigid body are nonlinear. Thus, in general, to obtain optimal feedback controllers for nonlinear systems one has to solve the associated Hamilton-Jacobi equation (HJE). This is a partial differential equation and except for very special cases, it is difficult to obtain solutions. As a result of the difficulty in obtaining optimal controllers, in this paper we look for suboptimal stabilizing feedback controllers. More specifically, we consider the following problem: Given a bounded set $\mathcal{C}$ in the state space, find a feedback controller which results in an asymptotically stable closed loop system (with $\mathcal{C}$ contained in the region of attraction), and such that a quadratic cost satisfies a specified bound for all initial conditions in $\mathcal{C}$. We call this problem the Quadratic Regulation Problem (QRP).

The paper is organized as follows. In section 2 we present the equations of motion of a rotating rigid body and we state the QRP. The description of the dynamics is standard, whereas for the kinematics we choose the Cayley-Rodrigues parameters.

In section 3 we show that there exist linear controllers which render the system under consideration *globally asymptotically stable* and give bounded quadratic cost. In section 4 we consider the QRP for the *kinematics only* with the angular velocity as the control input. It turns out that for this case we can compute globally optimal controllers for a special class of quadratic cost functions. The resulting optimal controllers are *linear*.

The main results are in section 5. For a general quadratic performance index, we present sufficient conditions which, if satisfied, guarantee the existence of a linear stabilizing controller that minimizes an upper bound on a quadratic cost for all initial conditions in a given set. That is, we give sufficient conditions for the solvability of the QRP with linear controllers. The conditions comprise certain matrix inequalities. The paper concludes with a (finite dimensional) nonlinear programming problem that can be used to test our sufficient conditions. We show that the resulting nonlinear programming problem can be solved by solving a sequence of Linear Matrix Inequality (LMI) problems.

The notation used is standard. The transpose of a matrix A is A'. If A is real and symmetric, we use $A < 0$ to denote that A has (strictly) negative eigenvalues; $A > 0$ is equivalent to $-A < 0$. The identity matrix of size n is denoted by I_n. The euclidean norm is $\|x\| = \sqrt{x'x}$. The function $V : \mathbb{R}^n \to \mathbb{R}$ is positive definite if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(x) = 0$ only if $x = 0$. The gradient of $V(x)$ is denoted by $V_x(x)$ (row vector). Finally, the notation $\mathbf{Co}\{\mathcal{S}\}$ stands for the convex hull of the set $\mathcal{S}$.

2 PROBLEM FORMULATION

We consider the rotational motion of a rigid body subject to three independent scalar control torques; these torques are applied about axes which are fixed in the body and aligned with the body principal axes. The rotational motion of a rigid body can be described by a system of six first order differential equations. Three of these equations govern the angular velocity (dynamic equations) while the other three describe the evolution of the body orientation (kinematic equations).

Choosing a body-fixed coordinate system aligned with the torque axes, the dynamic equations can be written in the form

$$\dot{\omega} = F(\omega)\omega + J^{-1}u, \quad \omega(0) = \omega_0, \tag{1}$$

where $\omega = [\omega_1\ \omega_2\ \omega_3]'$ is the angular velocity vector. The matrix $F(\omega)$ is given by

$$F(\omega) = \begin{bmatrix} 0 & -J_3\omega_3/J_1 & J_2\omega_2/J_1 \\ J_3\omega_3/J_2 & 0 & -J_1\omega_1/J_2 \\ -J_2\omega_2/J_3 & J_1\omega_1/J_3 & 0 \end{bmatrix} \tag{2}$$

where J_1, J_2, and J_3 are the principal moments of inertia of the rigid body at the mass center. The matrix J is the diagonal matrix

$$J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}.$$

To describe the orientation of the rigid body in space, kinematic equations are necessary. The orientation of a rigid body can be described with a *rotation matrix* which transforms the basis unit vectors between the body-fixed and the inertial coordinate systems. Every orientation of the body corresponds to an element of the group of orthogonal 3×3 matrices with determinant $+1$, called the rotation group, which may be viewed as the configuration space for all rotations of a rigid body. The different ways of parameterizing the rotation group give rise to alternative parameterizations of the kinematics, leading to different sets of kinematic equations (Kane *et al.*, 1983).

One possible choice of kinematic parameters are the so-called Cayley-Rodrigues parameters ρ_1, ρ_2, and ρ_3 (Kane *et al.*, 1983). These parameters lead to a minimal three-dimensional representation of the rotation group. The corresponding kinematic equations are

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0 \tag{3}$$

where $\rho = [\rho_1\ \rho_2\ \rho_3]'$ is the kinematic vector,

$$G(\rho) := \frac{1}{2}(I_3 + S(\rho) + \rho\rho'), \tag{4}$$

and $S(\rho)$ is the skew-symmetric matrix defined by

$$S(\rho) := \begin{bmatrix} 0 & -\rho_3 & \rho_2 \\ \rho_3 & 0 & -\rho_1 \\ -\rho_2 & \rho_1 & 0 \end{bmatrix}. \tag{5}$$

A useful property of the Cayley-Rodrigues representation is that, for any $\rho \in \mathbb{R}^3$, we have

$$\rho' G(\rho) = \frac{1}{2}(1 + \|\rho\|^2)\rho'. \tag{6}$$

This property will be used when computing Lyapunov derivatives associated with the nonlinear system given by (1) and (3).

When the Cayley-Rodrigues parameters are zero, $\rho_1 = \rho_2 = \rho_3 = 0$, the rotation matrix is the identity matrix and the body and inertial coordinate systems coincide. *This will be the equilibrium (rest) or desired orientation in this paper.*

We associate with the system given by (1) and (3) a *performance output*

$$z = C \begin{bmatrix} \rho \\ \omega \end{bmatrix} + Du, \tag{7}$$

where C and D are given real matrices. For each initial state $[\rho_0'\ \omega_0']' \in \mathbb{R}^6$, and control input u, the performance index or cost associated with this output is given by

$$\mathcal{J}(\rho_0, \omega_0, u) := \int_0^\infty \|z(t)\|^2\, dt, \tag{8}$$

where $\|z\| = \sqrt{z'z}$.

The objective of this paper is to solve the following problem.

Quadratic Regulation Problem (QRP). Consider the nonlinear system

$$\dot{\rho} = G(\rho)\omega \tag{9a}$$

$$\dot{\omega} = F(\omega)\omega + J^{-1}u \tag{9b}$$

where $F(\cdot)$ is defined in (2), $G(\cdot)$ is defined in (4), and the performance index is given by (7) and (8). Given any bounded set $\mathcal{C} \subset \mathbb{R}^6$ containing zero and any positive scalar γ, obtain a memoryless state-feedback controller $u = k(\rho, \omega)$ such that,

(i) the closed loop system is asymptotically stable about zero with $\mathcal{C}$ as a region of attraction;

(ii) for each initial state $[\rho_0'\ \omega_0']' \in \mathcal{C}$ the performance index satisfies the bound

$$\mathcal{J}(\rho_0, \omega_0, u) \leq \gamma. \tag{10}$$

3 GLOBALLY ASYMPTOTICALLY STABILIZING LINEAR CONTROLLERS

In this section we present simple linear controllers which render the nonlinear system (9) *globally* asymptotically stable. The result given here provides the motivation for the later developments.

Lemma 1: *The linear controller*

$$u = -\kappa_1\omega - \kappa_2\rho, \tag{11}$$

where κ_1 and κ_2 are any positive scalars, globally asymptotically stabilizes the system (9). Moreover,

$$\mathcal{J}(\rho_0, \omega_0, u) < \infty.$$

Proof. Define the positive definite function

$$V(\omega, \rho) := \tfrac{1}{2}\omega' J\omega + \kappa_2 \ln(1 + \|\rho\|^2) \tag{12}$$

where $\ln(\cdot)$ denotes the natural logarithm. We show that this is a Lyapunov function for the closed-loop system. Differentiating (12) along the trajectories of the closed-loop system obtained by applying (11) to (9), and using (6), we obtain

$$\dot{V} = -\kappa_1\omega' J\omega \leq 0. \tag{13}$$

Since V is radially unbounded, it now follows that all trajectories are bounded. Note also that $\dot{V} \equiv 0$ implies $\omega \equiv 0$; this gives $\dot{\omega} \equiv 0$ and $\rho \equiv 0$. Hence, from LaSalle's theorem, it follows that the closed-loop system is globally asymptotically stable about zero.

To show that the cost (8) is bounded, we will first show that the control law (11) is (locally) exponential stabilizing. This can be done by showing that the linearization of the closed-loop system has all eigenvalues with negative real parts (Khalil, 1992). The linearization of the closed-loop system about the origin is given by

$$\begin{bmatrix} \dot{\rho} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I_3/2 \\ -J^{-1}\kappa_2 & -J^{-1}\kappa_1 \end{bmatrix} \begin{bmatrix} \rho \\ \omega \end{bmatrix}. \tag{14}$$

A simple calculation shows that, if λ is an eigenvalue of (14), then

$$\det(J\lambda^2 + \kappa_1 I_3 \lambda + \frac{\kappa_2}{2} I_3) = 0.$$

Since $J > 0$, $\kappa_1 > 0$, and $\kappa_2 > 0$, we conclude that λ has strictly negative real part.

In order now to show that the cost (8) is bounded, consider a closed loop trajectory $x(\cdot) = [\rho(\cdot)' \; \omega(\cdot)']'$ starting from an arbitrary initial condition $[\rho_0' \; \omega_0']'$. Using the fact that the closed loop is globally asymptotically stable and locally exponentially stable, it is quite elementary to show that

$$\lim_{T\to\infty} \int_0^T \|x(t)\|^2 \, dt < \infty$$

Since the control law is linear in the state, and the cost is quadratic in the state and the control we conclude that

$$\mathcal{J}(\rho_0, \omega_0, u) < \infty.$$

□

This lemma provides the main motivation for the methodology used in the paper. According to this lemma, the system (9) has the – rather unusual for a nonlinear system – property that it admits linear *globally* asymptotically stabilizing control laws. In addition, the linear control law (11) provides a bounded value for the cost (8). It is natural then to search over the class of linear controllers to find the one yielding the minimum value of the cost.

4 SPECIAL COST FUNCTIONS

In this section we consider the special class of QRP problems with performance output z of the form

$$z = \begin{bmatrix} r_1 \rho \\ r_2 \omega \end{bmatrix} \tag{15}$$

where r_1, r_2 are positive scalars; the corresponding cost is given by

$$\mathcal{J}(\rho_0, \omega_0, u) = \int_0^\infty \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} \, dt. \tag{16}$$

Note that (9) is a system in cascade form; i.e., ρ does not enter the right-hand-side of (9b) and u does not enter (9a). In essence, ω acts as a "control" for the subsystem (9a). Therefore, when u does not enter in the cost function, it is natural to consider first the optimal control problem for the kinematics only with ω treated as a control-like variable. Such problems are simpler than optimal control problems for both (9a) and (9b). Optimal control problems with ω as the control provide lower bounds on the optimal performance that can be achieved when u is the control variable.

Lemma 2: *Consider the nonlinear system*

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0 \tag{17}$$

with ω as the control input. Let r_1 and r_2 denote two positive scalars and define $r = r_1/r_2$. The controller

$$\omega_{\text{opt}}(\rho) = -r\rho \tag{18}$$

has the following properties:

(i) The corresponding closed-loop system is globally exponentially stable about zero.

(ii) For every initial state ρ_0, the controller (18) minimizes the performance index

$$\mathcal{H}(\rho_0, \omega) := \int_0^\infty \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} \, dt \tag{19}$$

over the set of control inputs $\omega(\cdot)$ which result in $\lim_{t\to\infty} \rho(t) = 0$, and the minimum of the performance index is

$$\mathcal{H}_{\text{opt}}(\rho_0) = 2 r_1 r_2 \ln(1 + \|\rho_0\|^2). \tag{20}$$

Proof. To demonstrate global exponential stability of the closed loop system

$$\dot{\rho} = -rG(\rho)\rho \tag{21}$$

introduce the Lyapunov function candidate

$$W(\rho) = \rho'\rho.$$

From (6) it follows that the derivative of W along any solution of the closed loop system satisfies

$$\begin{aligned} \dot{W} &= -r(1 + \|\rho\|^2)\|\rho\|^2 \\ &\leq -rW. \end{aligned}$$

This guarantees global exponential stability about zero with rate of convergence $r/2$.

To demonstrate the optimality properties of controller (18), consider the positive definite function

$$V(\rho) := 2 r_1 r_2 \ln(1 + \|\rho\|^2). \tag{22}$$

Take now any initial state ρ_0 and any control input $\omega(\cdot)$ which results in $\lim_{t\to\infty} \rho(t) = 0$. The derivative of V along the corresponding solution of system (17) satisfies (this computation makes use of (6))

$$\begin{aligned} \dot{V} &= 4 r_1 r_2 (1 + \|\rho\|^2)^{-1} \rho' G(\rho) \omega \\ &= 2 r_1 r_2 \rho' \omega \\ &= -r_1^2 \|\rho\|^2 - r_2^2 \|\omega\|^2 + \|r_1 \rho + r_2 \omega\|^2. \end{aligned}$$

Considering any time $T \geq 0$ and integrating this last equality over the interval $[0, T]$ yields

$$\begin{aligned} \int_0^T \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} dt &= V(\rho_0) - V(\rho(T)) \\ &+ \int_0^T \|r_1 \rho(t) + r_2 \omega(t)\|^2 \, dt. \end{aligned}$$

Since $\lim_{T\to\infty} \rho(T) = 0$, we have $\lim_{T\to\infty} V(\rho(T)) = 0$ and

$$\mathcal{H}(\rho_0, \omega) = V(\rho_0) + \int_0^\infty \|r_1 \rho(t) + r_2 \omega(t)\|^2 \, dt. \tag{23}$$

The optimality properties of controller (18) now follow from (23). □

5 MAIN RESULTS

We now consider a control problem for the nonlinear system (9) with a more general performance index than the one in Lemma 2. In particular, we now include a penalty in the control input u. Unfortunately, when the performance index is arbitrary, we cannot solve the optimal control problem. Instead, we give sufficient conditions for the solvability of the QRP problem introduced in section 2.

In order to state our main result, we need to compute a few preliminary quantities. First, note that we can write the nonlinear system (9) in the form

$$\begin{aligned} \dot{x} &= A(x)x + Bu \quad x(0) = x_0 && \text{(24a)} \\ z &= Cx + Du, && \text{(24b)} \end{aligned}$$

where $x := [\rho' \, \omega']'$ and

$$A(x) := \begin{bmatrix} 0 & G(\rho) \\ 0 & F(\omega) \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix}. \tag{25}$$

Using (25) it can be shown that $A(x)$ can also be written as

$$A(x) = A_0 + \sum_{i=1}^{6} x_i A_i + B_0 x x' C_0, \tag{26}$$

where $A_0, A_1, \ldots, A_6, B_0, C_0$ are real matrices in $\mathbb{R}^{6\times 6}$, determined by $G(\cdot)$ and $F(\cdot)$. These matrices are fairly easy to compute and they are given in the appendix. Equation (26) shows that the matrix $A(x)$ is the sum of two parts; the first part is affine in the state x and the second is quadratic in the state x.

Let $\mathcal{B}_\infty(d)$ denote the hypercube of radius d in $\mathbb{R}^6$; i.e,

$$\mathcal{B}_\infty(d) = \left\{ x \in \mathbb{R}^6 \;\middle|\; |x_i| \le d, i = 1, 2, \ldots, 6 \right\}.$$

Compute real matrices $A_1^\#, \ldots, A_p^\#$ such that

$$\left\{ A_0 + \sum_{i=1}^{6} x_i A_i \;\middle|\; x \in \mathcal{B}_\infty(d) \right\} = \mathbf{Co}\{A_1^\#, \ldots, A_p^\#\}. \tag{27}$$

The matrices $A_1^\#, \ldots, A_p^\#$ exist because the set in the left hand side of (27) is a polytope; these matrices are given in the appendix.

The next result yields a solution to the suboptimal quadratic regulation problem for the nonlinear system (24). The basic idea is to give conditions that guarantee the existence of Lyapunov functions of the form

$$V(x) = \lambda \ln(1 + \|\rho\|^2) + x' P x,$$

(the positive definite matrix $P \in \mathbb{R}^{6\times 6}$ and the nonnegative scalar λ are free), that can be used to prove stability and compute an upper bound for the quadratic cost. (Recall from the previous sections that Lyapunov functions which include a logarithmic term in the kinematic parameters give rise to linear controllers and, in addition, are optimal for certain special cost functions.)

Theorem 1: *Consider the nonlinear system (24) together with the cost function*

$$\mathcal{J}(x_0, u) = \int_0^\infty \|Cx(t) + Du(t)\|^2 \, dt. \tag{28}$$

Suppose that $D'[C \;\; D] = [0 \;\; I]$. Let d denote a positive constant and let the matrices $A_1^\#, \ldots A_p^\#$, B_0, and C_0 be defined by (27) and (26). Suppose there exists a positive definite symmetric matrix $P \in \mathbb{R}^{6\times 6}$, positive scalars $\sigma_1, \ldots, \sigma_p$, and $\lambda \ge 0$ such that, for each $i = 1, \ldots, p$, we have

$$\begin{aligned} A_i^{\#\prime} P + P A_i^\# &+ 3d^2(\sigma_i P B_0 + \sigma_i^{-1} C_0')(\sigma_i P B_0 + \sigma_i^{-1} C_0')' \\ &- PBB'P + C'C + \lambda \Pi < 0, \end{aligned} \tag{29}$$

where

$$\Pi := \tfrac{1}{2} \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}.$$

Define the positive definite function

$$V(x) := \lambda \ln(1 + \|\rho\|^2) + x' P x, \tag{30}$$

where ρ denotes the first 3 components of $x \in \mathbb{R}^6$, and define the set

$$\Omega(\gamma) := \left\{ x \in \mathbb{R}^6 \mid V(x) \le \gamma \right\}, \tag{31}$$

where γ is a given positive number. If $\Omega(\gamma) \subset \mathcal{B}_\infty(d)$, then the linear state-feedback control law

$$u = -B' P x \tag{32}$$

is such that, given any initial condition $x_0 \in \Omega(\gamma)$, the resulting closed loop trajectory converges to zero, and the closed loop cost satisfies the bound

$$\begin{aligned} \mathcal{J}(x_0, -B'Px) &= \int_0^\infty \|(C - DB'P)x(t)\|^2 \, dt \\ &\le \lambda \ln(1 + \|\rho_0\|^2) + x_0' P x_0 \le \gamma. \end{aligned} \tag{33}$$

The intuition behind this theorem is as follows. If the matrix inequalities in (29) hold, one can show, using the Lyapunov function (30), that the control $u = -B'Px$ asymptotically stabilizes the nonlinear system (24) whenever the initial state belongs to an invariant set of closed loop trajectories contained in $\mathcal{B}_\infty(d)$. The set $\Omega(\gamma)$ is one such invariant set because $x(t) \in \Omega(\gamma)$ implies $x(t) \in \mathcal{B}_\infty(d)$; hence, $\dot{V}(x(t)) \le 0$. Moreover, the same Lyapunov function can be used to show the performance bound in (33) by a simple "completion of squares argument." Although simple, the proof of Theorem 1 is lengthy and (due to space limitations) it is omitted; the interested reader may find a proof of this result in Rotea *et al.* (1995).

6 NUMERICAL SOLUTION OF THE QRP PROBLEM

Let $\mathcal{C}$ be the bounded set of initial states where the QRP should be solved. Then, the best suboptimal controller that can be obtained from Theorem 1 is obtained by solving

$$\begin{aligned} \gamma_{\text{opt}} = \inf_{(\gamma, \lambda, \sigma_1, \ldots, \sigma_p, P)} \; &\gamma \\ \text{subject to } &\lambda \ge 0, \; \sigma_1 > 0, \ldots, \sigma_p > 0, \\ &P = P' > 0, \text{ and (29)} \\ &\mathcal{C} \subset \Omega(\gamma) \subset \mathcal{B}_\infty(d) \end{aligned} \tag{34}$$

where $\Omega(\gamma)$ is defined in (31). Indeed, the state-feedback gain $K_{\text{qrp}} = -B' P_{\text{opt}}$, where P_{opt} denotes a solution to (34), stabilizes the set $\mathcal{C}$ and guarantees that the quadratic performance index is bounded by γ_{opt} for all initial conditions in $\mathcal{C}$.

It turns out that (34) does not exhibit any convexity properties that can be exploited to compute a global solution. To see this, suppose that all optimization variables except P are fixed. Then the matrix inequalities (29) cannot be made convex in P due to the presence of an *indefinite* quadratic term in P; similarly, if we write (29) in terms of P^{-1}, the presence of an *indefinite* quadratic term in P^{-1} shows that (29) is not convex in P^{-1} either. Below, we will give an iterative method for finding local solutions that can be implemented by solving a sequence of Linear Matrix Inequalities (LMIs). The reduction of the problem to one involving LMIs has computational advantages (Boyd *et al.*, 1994; Gahinet and Nemirovskii, 1993).

Notice first that, with $K = -B'P$, we can write each matrix inequality in (29) as

$$\begin{aligned}&(A_i^{\#} + BK)'P + P(A_i^{\#} + BK)\\ &+3d^2(\sigma_i PB_0 + \sigma_i^{-1}C_0')(\sigma_i PB_0 + \sigma_i^{-1}C_0')'\\ &\quad +(C + DK)'(C + DK) + \lambda\Pi < 0.\end{aligned} \tag{35}$$

Fix $\gamma > 0$ and K. It follows that there exist $\lambda \geq 0$, positive numbers $\sigma_1, \ldots, \sigma_p$, and $P = P' > 0$, such that (35) holds and

$$\mathcal{C} \subset \Omega(\gamma) \subset \mathcal{B}_\infty(d) \tag{36}$$

if and only if there exist $\beta_0 \geq 0$, positive numbers $\beta_1, \ldots, \beta_p$, and $X = X' > 0$ such that

$$\begin{aligned}&(A_i^{\#} + BK)'X + X(A_i^{\#} + BK) +\\ &3d^2\beta_i^{-1}(XB_0 + \beta_i C_0')(XB_0 + \beta_i C_0')' +\\ &\gamma^{-1}(C + DK)'(C + DK) + \beta_0\Pi < 0\end{aligned} \tag{37}$$

and

$$\mathcal{C} \subset \Phi \subset \mathcal{B}_\infty(d). \tag{38}$$

where

$$\Phi := \left\{ x \in \mathbb{R}^6 \;\middle|\; \beta_0 \ln(1 + \|\rho\|^2) + x'Xx \leq 1 \right\} \tag{39}$$

(To see this, introduce the change of variables $P = \gamma X$, $\beta_0 = \lambda/\gamma$, and $\beta_i = 1/(\sigma_i^2\gamma)$)

Introducing the change of variables $\alpha = \gamma^{-1}$, and using the Schur complement formula, (37) is equivalent to

$$\begin{bmatrix} (A_i^{\#} + BK)'X + X(A_i^{\#} + BK) + \\ \alpha(C + DK)'(C + DK) + \beta_0\Pi & XB_0 + \beta_i C_0' \\ B_0'X + \beta_i C_0 & -\frac{1}{3d^2}\beta_i I \end{bmatrix} < 0. \tag{40}$$

Finally, from the equivalence between the pair of conditions (35)-(36) and the pair of conditions (40)-(38), we get that the optimization problem (34) is equivalent to

$$\begin{aligned}\gamma_{\rm opt}^{-1} = \ &\sup_{(\alpha,\beta_0,\beta_1,\ldots,\beta_p,X,K)} \alpha\\ &\text{subject to } \beta_0 \geq 0, X = X' > 0, \beta_i > 0\\ &\qquad (40) \text{ holds for } i = 1, \ldots, p,\\ &\qquad \text{the set inclusion (38) holds,}\\ &\qquad \text{and } K = -\tfrac{1}{\alpha}B'X.\end{aligned} \tag{41}$$

We will now show how to compute a local solution to (41) by solving a sequence of LMI problems when the set $\mathcal{C}$ in (38) is the polytope defined by

$$\mathcal{C} = \mathcal{P} := \mathbf{Co}\{v_1, \ldots, v_r\}.$$

for some vectors $v_i \in \mathbb{R}^6$, for $i = 1, ..., r$.

First we show how that if certain LMIs in the variables β_0 and X hold, the set inclusion in (38) holds. Consider first the inclusion $\mathcal{P} \subset \Phi$. Suppose that, given $\beta_0 \geq 0$, $X > 0$ and $h = 1, \ldots, r$, we have

$$\beta_0 \ \|[I_3\ 0]v_h\|^2 + v_h'Xv_h \leq 1. \tag{42}$$

Then, since $\mathcal{P}$ is a polytope and

$$\Phi_- := \left\{ x \;\middle|\; \beta_0 \ \|[I_3\ 0]x\|^2 + x'Xx \leq 1 \right\}$$

is convex, we get $\mathcal{P} \subset \Phi_-$. Since $\ln(1 + \theta^2) \leq \theta^2$ for all $\theta \in \mathbb{R}$, we obtain $\Phi_- \subset \Phi$. Hence, if (42) holds, $\mathcal{P} \subset \Phi$. The important point is that (42) is affine in β_0 and X.

Now, to enforce $\Phi \subset \mathcal{B}_\infty(d)$ we use

$$\begin{bmatrix} d^2 & e_s' \\ e_s & X \end{bmatrix} \geq 0, \tag{43}$$

for $s = 1, \ldots, 6$, where e_s denotes the unit vector in $\mathbb{R}^6$. To show that this condition implies $\Phi \subset \mathcal{B}_\infty(d)$, take $x \in \Phi$. Since $\beta_0 \geq 0$ we get $x'Xx \leq 1$. If Φ_+ denotes the ellipsoid

$$\Phi_+ := \left\{ x \;\middle|\; x'Xx \leq 1 \right\}$$

then clearly, $\Phi \subset \Phi_+$.

If (43) holds and $x \in \Phi_+$, then given any real number y we have

$$d^2y^2 + 2ye_s'x + 1 \geq 0.$$

Taking $y = d^{-1}$ and $y = -d^{-1}$ in this last inequality, we obtain $|e_s'x| \leq d$; since s is arbitrary, it follows that $x \in \mathcal{B}_\infty(d)$, thus $\Phi_+ \subset \mathcal{B}_\infty(d)$.

From these (conservative) characterizations of the set inclusion (38), we get

$$\begin{aligned}\gamma_{\rm opt}^{-1} \ \geq \ &\sup_{(\alpha,\beta_0,\beta_1,\ldots,\beta_p,X,K)} \alpha\\ &\text{subject to } \beta_0 \geq 0, X = X' > 0, \beta_i > 0\\ &\qquad (40) \text{ holds for } i = 1, \ldots, p,\\ &\qquad (42) \text{ holds for } h = 1, \ldots, r,\\ &\qquad (43) \text{ holds for } s = 1, \ldots, 6,\\ &\qquad \text{and } K = -\tfrac{1}{\alpha}B'X.\end{aligned} \tag{44}$$

Notice that, when K is fixed and the last equality constraint in the optimization problem (44) is ignored, (44) is a convex LMI problem in the variables $(\alpha, \beta_0, \beta_1, \ldots, \beta_p, X)$. Notice also that given $(\alpha, \beta_0, \beta_1, \ldots, \beta_p, X, K)$ satisfying all the constraints of (44) but the last one, a new gain can be generated according to the formula

$$K_{\rm new} = -\frac{1}{\alpha}B'X.$$

Hence, local solutions to (44) can be obtained by iteratively computing (α, X) and K. This is summarized in the following algorithm.

The $(\alpha, X) - K$ iteration

1. Choose d with $\mathcal{P} \subset \mathcal{B}_\infty(d)$ and compute the data necessary to write down the LMIs (40), (42), and (43).
2. Compute K_0 the solution of the LQR problem corresponding to the linearized (about $x = 0$) system. (A unique K_0 exists.) Set the iteration index $\ell = 0$ and goto 3.
3. Fix $K = K_\ell$ in (44) and solve it (without the last equality constraint) to obtain (α_ℓ, X_ℓ). (This is a standard LMI problem which may be solved with the package described in Gahinet and Nemirovskii, 1993.)
4. Compute $K_{\ell+1} = -\alpha_\ell^{-1} B' X_\ell$. Stop if $\|K_{\ell+1} - K_\ell\|_m$ is less than a specified tolerance; otherwise, set $\ell = \ell + 1$ and goto 3. (Here $\|\cdot\|_m$ denotes a matrix norm.)

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APPENDIX

Let

$$\Sigma_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Sigma_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\Sigma_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrices in equation (26) are given by

$$A_0 := \tfrac{1}{2} \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}$$

$$A_1 := \tfrac{1}{2} \begin{bmatrix} 0 & \Sigma_1 \\ 0 & 0 \end{bmatrix},$$

$$A_2 := \tfrac{1}{2} \begin{bmatrix} 0 & \Sigma_2 \\ 0 & 0 \end{bmatrix},$$

$$A_3 := \tfrac{1}{2} \begin{bmatrix} 0 & \Sigma_3 \\ 0 & 0 \end{bmatrix},$$

$$A_4 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_1 J_1 \end{bmatrix},$$

$$A_5 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_2 J_2 \end{bmatrix},$$

$$A_6 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_3 J_3 \end{bmatrix}$$

$$B_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & I_3 \\ 0 & 0 \end{bmatrix}$$

The matrices $A_1^\#, \ldots, A_p^\#$ required in equation (27) are given by

$$A_k^\# := A_0 + \sum_{i=1}^{6} \hat{d}_{k,i} A_i$$

where $\hat{d}_{k,i}$ denotes the i*th* component of the k*th* vertex ($1 \le k \le p = 2^6$) of the 6-dimensional cube $\mathcal{B}_\infty(d)$.