Stability of Time-Delay Systems: Equivalence between Lyapunov and Scaled Small-Gain Conditions

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***Abstract*—It is demonstrated that many previously reported Lyapunov-based stability conditions for time-delay systems are equivalent to the robust stability analysis of an uncertain comparison system free of delays via the use of the scaled small-gain lemma with constant scales. The novelty of this note stems from the fact that it unifies several existing stability results under the same framework. In addition, it offers insights on how new, less conservative results can be developed.**

***Index Terms*—Stability, time-delay systems.**

II. Introduction

The analysis of linear time-delay systems (LTDS) has attracted much interest in the literature over the half century, especially in the last decade. Two types of stability conditions, namely delay-independent and delay-dependent, have been studied [17]. As the name implies, delay-independent results guarantee stability for arbitrarily large delays. Delay-dependent results take into account the maximum delay that can be tolerated by the system and, thus, are more useful in applications. One of the first stability analysis results was the polynomial criteria [8]–[10]. An important result was later provided by [3], which gives necessary and sufficient conditions for efficient computation of the delay margin for the linear systems with *commensurate* delays. This result only requires the computation of the eigenvalues and generalized eigenvalues of constant matrices. Unfortunately, it is not straightforward to extend this to many problems of interest, such as the stability of general (noncommensurate) delays systems, $\mathcal{H}_\infty$ performance of LTDS with exogenous disturbances, robust stability of LTDS with dynamical uncertainties, and robust controller synthesis, etc. Recently, much effort has been devoted to developing frequency-domain and time-domain based techniques which may be extendable to such problems. The frequency-domain approaches include integral quadratic constraints [6], singular value tests [25], μ framework-based criteria [4], and other similar techniques. In [20], the traditional μ-framework was extended for time-delay systems to obtain a necessary and sufficient stability condition, which was then relaxed to a convex sufficient condition.

Other recent stability analysis results have been developed in the time-domain, based on Lyapunov's Second Method using either Lyapunov–Krasovskii functionals or Lyapunov–Razumikhin functions [26], [12], [13], [16], [22], [14], [17], [19]. These results are formulated in terms of linear matrix inequalities (LMIs), and, hence, can be solved efficiently [1]. While these results are often extendable to the systems with general multiple delays and/or dynamical uncertainties, they can be rather conservative and the corresponding Lyapunov functionals are complex. A formal procedure for constructing Lyapunov functionals for LTDS was proposed in [11], but a Lyapunov functional, in general, does not provide direct information on how conservative the resultant condition may be in practice.

In this note, we show that several existing Lyapunov-based results, both delay-independent and delay-dependent, are equivalent to the scaled small-gain condition for robust stability of a comparison system that is free of delay. This result provides a new frequency-domain interpretation to some common Lyapunov-based results in the literature. Via a numerical example, we investigate the potential conservatism of the stability conditions, and demonstrate that a major source of conservatism is the embedding of the delay uncertainties in unit disks that the comparison system employs. This source of conservatism is hidden in the Lyapunov-based framework but is quite apparent in the comparison system interpretation. These results also provide insight into how to reduce the conservatism of the stability tests.

After a conference version of this note appeared in [28], we became aware of the results of [15] and [7] which are related to our approach. Unlike the model transformation class in [15], which contains distributed delays, the comparison system employed herein is a *delay-free uncertain system* stated in frequency domain and permits the immediate application of the standard frequency-domain techniques, such as the μ framework. The results in [7] are based on a special case of our comparison system, namely $M = I_n$. Neither [15] nor [7] examined the equivalence of existing Lyapunov-based criteria and the scaled small-gain conditions, which is the contribution of this note.

The notation is conventional. Let $\mathbb{R}^{n\times m}$ $(\mathbb{C}^{n\times m})$ be the set of all real (complex) $n\times m$ matrices, $\mathbb{R}_e := \mathbb{R}\cup\{\infty\}$, I_n be $n\times n$ identity matrix, W^T be the transpose of real matrix W, and $RH_\infty := \{H(s) : H(s)\in\mathcal{H}_\infty, H(s)$ is a real rational transfer matrix$\}$. $P>0$ indicates that P is a symmetric and positive definite matrix, and $\|\cdot\|_\infty$ indicates the $\mathcal{H}_\infty$ norm defined by $\|G\|_\infty := \sup_{\omega\in\mathbb{R}}\overline{\sigma}[G(j\omega)]$ where $\overline{\sigma}(M)$ is the maximum singular value of complex matrix M. The structured singular value of a matrix $M\in\mathbb{C}^{n\times n}$ with respect to a block structure $\underline{\Lambda}$ is defined by $\mu_{\underline{\Lambda}}(M)=0$ if there is no $\Lambda\in\underline{\Lambda}$ such that $I-M\Lambda$ is singular, and

$$\mu_{\underline{\Lambda}}(M) = [\min\{\overline{\sigma}(\Lambda) : \det(I-M\Lambda)=0,\ \Lambda\in\underline{\Lambda}\}]^{-1}$$

otherwise. We also define the set $\underline{\mathbf{\Delta}}_r := \{\mathrm{diag}[\lambda_1 I_{n_1}, \lambda_2 I_{n_2}, \cdots, \lambda_r I_{n_r}]\colon \lambda_i\in\mathbb{C}\}$ and the closed norm-bounded set $B\underline{\mathbf{\Delta}}_r := \{\Delta\in\mathcal{H}_\infty\colon \|\Delta\|_\infty\le 1,\ \Delta(s)\in\underline{\mathbf{\Delta}}_r\}$. Finally, for linear time-invariant system $P(s)$ and its input $x(t)$, we define a signal $P(s)[x](t)$ as

$$P(s)[x](t) := \mathcal{L}^{-1}[P(s)X(s)]$$

where $X(s)$ is the Laplace transform of $x(t)$, and $\mathcal{L}^{-1}[\,\cdot\,]$ is the inverse Laplace operator.

III. Comparison System

For ease of exposition, we will examine the single-delay case. However, the Lyapunov stability conditions examined here may all be straightforwardly extended to the case of systems with multiple (noncommensurate) delays. Consider the linear time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \tag{1}$$

where $A\in\mathbb{R}^{n\times n}$ and $A_d\in\mathbb{R}^{n\times n}$ are constant matrices, and the delay τ is constant, unknown, but bounded by a known bound as $0\le\tau\le\overline{\tau}$. The following assumption is a necessary condition when investigating asymptotic stability of the system (1).

Assumption 1: The system (1) free of delay is asymptotically stable, that is, the matrix $\overline{A} := A + A_d$ is Hurwitz.

Taking Laplace transforms of both sides, the system (1) can be expressed in the s domain as

$$sX(s) = AX(s) + A_d e^{-\tau s}X(s). \tag{2}$$

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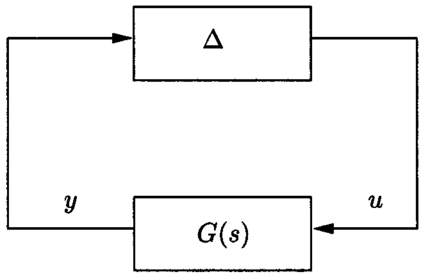

Fig. 1. A system with uncertainty.

The results of this note depend on the notion of robust stability of a feedback interconnection of a finite-dimensional, linear, time-invariant (FDLTI) system and an uncertain system with known uncertainty structure. The following definition clarifies the type of robust stability used herein. More on this definition can be found in [32].

Definition 1: Consider a linear, time-invariant (finite-dimensional) system $G(s)$ interconnected with an uncertain block Δ, as shown in Fig. 1. The uncertain block Δ belongs to a known, uncertainty structure set $\Delta \in \underline{\mathbf{\Delta}}$. Then, the system is said to be *robustly stable* if $G(s)$ is internally stable and the interconnection is well posed and remains internally stable for all $\Delta \in \underline{\mathbf{\Delta}}$.

To proceed with our analysis, we need the following preliminary results.

Lemma 1: Let $M \in \mathbb{R}^{n\times n}$ be any constant matrix. The system (1) is asymptotically stable for all $\tau \in [0, \overline{\tau}]$, if the following comparison system

$$sX = (A + MA_d)X + \Delta_1(I_n - M)A_dX + \Delta_2\overline{\tau}MA_dAX + \Delta_1\Delta_2\overline{\tau}MA_dA_dX \tag{3}$$

where $\mathrm{diag}[\Delta_1, \Delta_2] \in B\underline{\mathbf{\Delta}}_2$, is robustly stable.

Proof: Using (2), we have

$$\begin{aligned} sX(s) &= AX(s) + (I-M)A_de^{-\tau s}X(s) + MA_de^{-\tau s}X(s) \\ &= (A+MA_d)X(s) + (I-M)A_de^{-\tau s}X(s) \\ &\quad + \left(\frac{e^{-\tau s}-1}{\overline{\tau}s}\right)\overline{\tau}MA_dsX(s) \\ &= (A+MA_d)X(s) + (I-M)A_de^{-\tau s}X(s) \\ &\quad + \left(\frac{e^{-\tau s}-1}{\overline{\tau}s}\right)\overline{\tau}MA_d\left[AX(s)+A_de^{-\tau s}X(s)\right] \\ &= (A+MA_d)X(s) + (I-M)A_de^{-\tau s}X(s) \\ &\quad + \left(\frac{e^{-\tau s}-1}{\overline{\tau}s}\right)\overline{\tau}MA_dAX(s) + e^{-\tau s} \\ &\quad \cdot \left(\frac{e^{-\tau s}-1}{\overline{\tau}s}\right)\overline{\tau}MA_dA_dX(s). \end{aligned}$$

In view of the fact that $\|e^{-\tau s}\|_\infty = 1$ and $\|(e^{-\tau s}-1)/(\overline{\tau}s)\|_\infty = \tau/\overline{\tau} \le 1$, it follows from the above equation that (2) is a special case of the uncertain system (3) with $\Delta_1 = e^{-\tau s}I_n$, and $\Delta_2 = (e^{-\tau s} - 1)/(\overline{\tau}s)I_n$. Therefore, the robust stability of (3) guarantees that (1) is asymptotically stable for all $\tau \in [0, \overline{\tau}]$. ■

As shown in the next section, the comparison system (3) can be rewritten as an interconnection of an FDLTI system $G(s)$ with a block Δ, where $\Delta = \mathrm{diag}[\Delta_1, \Delta_2] \in B\underline{\mathbf{\Delta}}_2$. Hence, the analysis of the robust stability of the system (3) may be performed via μ-analysis, since the small-μ theorem applies even to the case where the uncertainty is nonrational [23]. Because the calculation of μ is NP-hard in general [2], its upper bound with D scales is typically used instead. In particular, the interconnection in Fig. 1 is robustly stable if $G(s) \in RH_\infty$ is internally stable and

$$\sup_{\omega\in\mathbb{R}} \inf_{D\in\underline{\mathbf{D}}} \overline{\sigma}\left(DG(j\omega)D^{-1}\right) < 1 \tag{4}$$

where

$$\underline{\mathbf{D}} := \{\mathrm{diag}\,[D_1, D_2]\,|D_i \in \mathbb{C}^{n\times n}, D_i = D_i^* > 0\}.$$

The test (4), although a convex optimization problem, requires a frequency sweep. Alternatively, the analysis of robust stability may be performed without the frequency sweep by solving an LMI. The following lemma states this result. Additional conservatism is introduced in this formulation, however, since it implies satisfaction of (4) with the same *constant real* scaling matrix used for all frequencies.

Lemma 2[21] (Scaled Small-Gain LMI)[1]: Consider the system interconnection shown in Fig. 1 where the plant $G(s)$ is FDLTI and the uncertainty block is such that $\Delta \in B\underline{\mathbf{\Delta}}_r$. Let (A, B, C, D) be a minimal realization of $G(s)$ with

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then, the closed-loop system is robustly stable if there exist matrices $X > 0$ and $Q = \mathrm{diag}[Q_1, Q_2, \cdots, Q_r] > 0$, $Q_i \in \mathbb{R}^{n_i\times n_i}$, $i = 1, 2, \ldots, r$, satisfying the following LMI:

$$\begin{bmatrix} A^TX + XA & XB & C^TQ \\ B^TX & -Q & D^TQ \\ QC & QD & -Q \end{bmatrix} < 0. \tag{5}$$

Definitiom 2: If a system satisfies (5), then we say that this system satisfies the scaled small-gain sufficiency (SSGS) condition for robust stability.

IV. Main Result

Herein, we introduce our main result, namely, the equivalence between several Lyapunov-based results [25], [13], [16], [19] and the scaled small-gain conditions for the comparison system (3).

First, we restate these stability analysis results.

Theorem 1: Consider the system (1) under Assumption 1. Then, we have

1) (*Delay-Independent Stability*) [25]: The system (1) is asymptotically stable for any $\tau \ge 0$, if there exist matrices $X > 0$ and $Q > 0$ satisfying
$$A^TX + XA + XA_dQ^{-1}A_d^TX + Q < 0. \tag{6}$$
2) (*Delay-Dependent Stability*): The system (1) is asymptotically stable for any $0 \le \tau \le \overline{\tau}$, if one of the following conditions hold.
   a) [16] There exist matrix $X > 0$ and constants $\beta_1 > 0$ and $\beta_2 > 0$ satisfying
   $$\begin{bmatrix} \Omega & XA_dA & XA_dA_d \\ A^TA_d^TX & \beta_1^{-1}X & 0 \\ A_d^TA_d^TX & 0 & \beta_2^{-1}X \end{bmatrix} > 0 \tag{7}$$
   where $\Omega = -\overline{\tau}^{-1}[(A+A_d)^TX + X(A+A_d)] - (\beta_1^{-1} + \beta_2^{-1})X$.
   b) [13] There exist matrices $P > 0$, $P_1 > 0$ and $P_2 > 0$ satisfying
   $$\begin{bmatrix} H & \overline{\tau}PA^T & \overline{\tau}PA_d^T \\ \overline{\tau}AP & -\overline{\tau}P_1 & 0 \\ \overline{\tau}A_dP & 0 & -\overline{\tau}P_2 \end{bmatrix} < 0 \tag{8}$$

[1]The small gain theorem applies to the case where the uncertainty blocks contain infinite dimensional dynamic systems [32].

where $H = P(A + A_d)^T + (A + A_d)P + \bar{\tau} A_d (P_1 + P_2) A_d^T$.

c) [19] There exist matrices $X > 0,\ U > 0,\ V > 0$ and W satisfying

$$\begin{bmatrix} \Omega_1 & -WA_d & A^T A_d^T V & \bar{\tau}(W + X) \\ -A_d^T W^T & -U & A_d^T A_d^T V & 0 \\ VA_d A & VA_d A_d & -V & 0 \\ \bar{\tau}\left(W^T + X\right) & 0 & 0 & -V \end{bmatrix} < 0 \qquad (9)$$

where $\Omega_1 = (A + A_d)^T X + X(A + A_d) + WA_d + A_d^T W^T + U$.

The following proposition shows that all of above conditions are equivalent to the SSGS conditions for the special case of the comparison system (3).

Proposition 1: For the comparison system (3), if $M = 0$, the SSGS condition is equivalent to the condition (6),[2] and, if $M = I_n$, the SSGS condition is equivalent to the condition (8) and can also be reduced to the condition (7). Moreover, the delay-dependent condition (9) is equivalent to the SSGS condition for (3) with M as a free-matrix variable.

Proof: First, let $M = 0$, then the comparison system (3) becomes

$$sX(s) = AX(s) + \Delta_1 A_d X(s)$$
$$\Delta_1 \in B\underline{\boldsymbol{\Delta}}_1$$

which can be described as the following closed-loop system:

$$\begin{aligned} \dot{x} &= Ax + A_d u \\ y &= x \\ u &= \Delta_1[y](t). \end{aligned}$$

With

$$G(s) = \left[\begin{array}{c|c} A & A_d \\ \hline I_n & 0 \end{array}\right]$$

the SSGS condition becomes (6).

Next, we let $M = I_n$ and $\Delta_3 = \Delta_1 \Delta_2$. Equation (3) then becomes

$$sX(s) = (A + A_d)X(s) + \Delta_2 \bar{\tau} A_d A X(s) + \Delta_3 \bar{\tau} A_d A_d X(s) \qquad (10)$$

with $\mathrm{diag}[\Delta_2,\ \Delta_3] \in B\underline{\boldsymbol{\Delta}}_2$. The last equation can be rewritten as the closed-loop system

$$\begin{aligned} \dot{x} &= (A + A_d)x + \bar{\tau} A_d u_1 + \bar{\tau} A_d u_2 \\ y_1 &= Ax \\ y_2 &= A_d x \\ u_1 &= \Delta_2[y_1](t) \\ u_2 &= \Delta_3[y_2](t). \end{aligned}$$

Then, by applying Lemma 2 with

$$G(s) = \left[\begin{array}{c|c} A + A_d & [\bar{\tau} A_d \quad \bar{\tau} A_d] \\ \hline \begin{bmatrix} A \\ A_d \end{bmatrix} & 0 \end{array}\right]$$

we see that the system (1) is asymptotically stable for any constant τ, $0 \le \tau \le \bar{\tau}$, if there exist $X > 0$ and $Q = \mathrm{diag}[Q_1,\ Q_2] > 0$ such that

$$\begin{bmatrix} R & \bar{\tau} X A_d & \bar{\tau} X A_d & A^T Q_1 & A_d^T Q_2 \\ \bar{\tau} A_d^T X & -Q_1 & 0 & 0 & 0 \\ \bar{\tau} A_d^T X & 0 & -Q_2 & 0 & 0 \\ Q_1 A & 0 & 0 & -Q_1 & 0 \\ Q_2 A_d & 0 & 0 & 0 & -Q_2 \end{bmatrix} < 0$$

where $R = (A + A_d)^T X + X(A + A_d)$. Multiplying by $\mathrm{diag}[X^{-1},\ I,\ I,\ \bar{\tau} Q_1^{-1},\ \bar{\tau} Q_2^{-1}]$ on both sides and using Schur complements, the above inequality is equivalent to

$$\begin{bmatrix} H & \bar{\tau} X^{-1} A^T & \bar{\tau} X^{-1} A_d^T \\ \bar{\tau} A X^{-1} & -\bar{\tau}^2 Q_1^{-1} & 0 \\ \bar{\tau} A_d X^{-1} & 0 & -\bar{\tau}^2 Q_2^{-1} \end{bmatrix} < 0 \qquad (11)$$

where $H = X^{-1}(A + A_d)^T + (A + A_d)X^{-1} + \bar{\tau}^2 A_d (Q_1^{-1} + Q_2^{-1}) A_d^T$. Defining $P = X^{-1} > 0$, $P_1 = \bar{\tau} Q_1^{-1} > 0$ and $P_2 = \bar{\tau} Q_2^{-1} > 0$, (11) becomes (8).

In addition, if we rewrite (10) in a slightly different form

$$\begin{aligned} \dot{x} &= (A + A_d)x + \bar{\tau} A_d A u_1 + \bar{\tau} A_d A_d u_2 \\ y_1 &= x \\ y_2 &= x \\ u_1 &= \Delta_2[y_1](t) \\ u_2 &= \Delta_3[y_2](t) \end{aligned}$$

then, similarly, we can obtain the following stability condition:

$$\begin{bmatrix} H & \bar{\tau} X A_d A & \bar{\tau} X A_d A_d \\ \bar{\tau} A^T A_d^T X & -Q_1 & 0 \\ \bar{\tau} A_d^T A_d^T X & 0 & -Q_2 \end{bmatrix} < 0$$

$$X > 0, \quad Q_1 > 0, \quad Q_2 > 0 \qquad (12)$$

where $H = (A + A_d)^T X + X(A + A_d) + Q_1 + Q_2$. Now, letting $Q_1 = \bar{\tau} \beta_1^{-1} X$ and $Q_2 = \bar{\tau} \beta_2^{-1} X$, where constants $\beta_1 > 0$ and $\beta_2 > 0$, (12) is reduced to (7).

Finally, consider the general case of (3) and rewrite it as the following:

$$\begin{aligned} \dot{x} &= (A + MA_d)x + (I - M)A_d u_2 + \bar{\tau} M u_1 \\ y_1 &= A_d A x + A_d A_d u_2 \\ y_2 &= x \\ u_1 &= \Delta_2[y_1](t) \\ u_2 &= \Delta_1[y_2](t). \end{aligned} \qquad (13)$$

Therefore, applying Lemma 2 with

$$G(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

where $\hat{A} = A + MA_d$, $\hat{B} = [\bar{\tau} M \quad (I - M)A_d]$, $\hat{C} = [A^T A_d^T \quad I]^T$, and $\hat{D} = \left[\begin{smallmatrix} 0 & A_d A_d \\ 0 & 0 \end{smallmatrix}\right]$, the system (1) is asymptotically stable if there exist $X > 0$ and $Q = \mathrm{diag}[V,\ U] > 0$ satisfying

$$\begin{bmatrix} \hat{A}^T X + X\hat{A} & X\hat{B} & \hat{C}^T Q \\ \hat{B}^T X & -Q & \hat{D}^T Q \\ Q\hat{C} & Q\hat{D} & -Q \end{bmatrix} < 0. \qquad (14)$$

Using Schur complement, (14) is equivalent to

$$\begin{bmatrix} \Omega_1 & \bar{\tau} X M & \Omega_2 & A^T A_d^T V \\ \bar{\tau} M^T X & -V & 0 & 0 \\ \Omega_2^T & 0 & -U & A_d^T A_d^T V \\ VA_d A & 0 & VA_d A_d & -V \end{bmatrix} < 0$$

where

$$\begin{aligned} \Omega_1 &= (A + MA_d)^T X + X(A + MA_d) + U \\ \Omega_2 &= X(I - M)A_d. \end{aligned}$$

Defining $W = X(M - I)$, it follows immediately that above condition is equivalent to (9). ∎

It should be noted that while all the Lyapunov-based conditions discussed may be obtained from a single comparison system, the realization of this system used in each condition may be different.

An implication of the equivalence between the Lyapunov-based results and the comparison system robust stability analysis is that the μ framework may be used for analysis (and, more importantly, controller synthesis) of uncertain time-delay systems without incurring any penalty vis-à-vis known Lyapunov-based approaches. Furthermore, the μ framework offers the advantage that robustness analysis with respect

[2]Similar observations can also be found, for example, in [26] and [4].

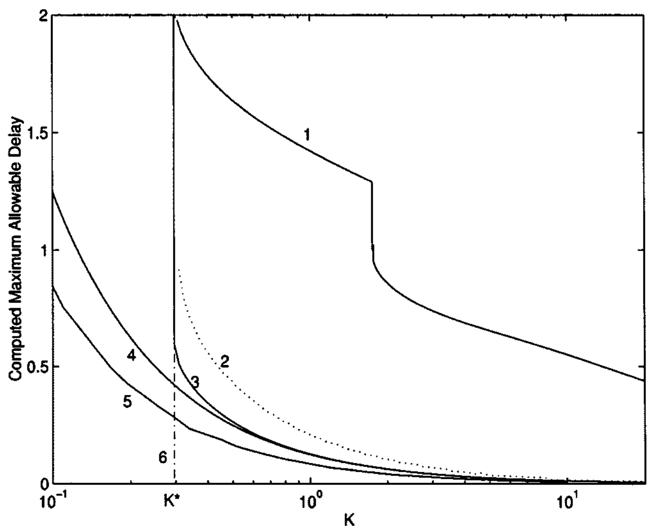

Fig. 2. Delay margin versus K. (1) Nyquist Criterion. (2) μ upper bound with frequency-dependent D scaling. (3) Condition of [19]. (4) Condition of [13]. (5) Condition of [16]. (6) Condition of [25], [26] for $K < K^*$, the stability is delay independent.

to LTI dynamic or parametric uncertainties in the time-delay system can be accomplished via the introduction of these uncertainties into the model description.

V. Conservatism of Existing Analysis Results

We now turn our attention to the conservatism of these results and what insights can be gained from the scaled small-gain interpretation. To illustrate our points, we will examine the following example motivated by the dynamics of machining chatter [24]:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10.0+K) & 10.0 & 0 & 0 \\ 5.0 & -15.0 & 0 & -0.25 \end{bmatrix}$$

$$A_d = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (15)$$

For this case, the generalization of the Nyquist criterion to time-delayed systems [27], [5] can be used to obtain the exact stability delay margin. The delay margins based on the criteria discussed above are shown in Fig. 2 as a function of K. We see that (9) generalizes the delay-independent condition (6), and the delay-dependent condition (8) and, thus, it is less conservative. However, condition (9) still provides a conservative delay margin compared with the exact values calculated from the Nyquist criterion. There are three possible sources of this conservatism.

1) Condition (5) is equivalent to applying the small μ theorem with the μ upper bound computed using constant real D scales. However, constant D scaling is well known to provide a more conservative result than frequency-dependent D scaling for dynamic uncertainty. In fact, constant scales are typically used for linear, time-varying uncertainties. Calculating the μ upper-bound via frequency sweep using frequency-dependent scales leads to the delay margin shown as the dotted line in Fig. 2. Here, the same M as employed implicitly by the LMI (9) was used for each value of K tested.
2) Since the delay "uncertainties" $e^{-\tau s}$ and $(e^{-\tau s} - 1)/(\overline{\tau} s)$ are covered with unit disks in the comparison system, all of the phase information and some of the gain information inherent in these elements is lost.
3) The μ upper bound used in (4) is guaranteed to be equal to μ only when $2S + F \leq 3$ [18], where S and F are the number of repeated complex scalar blocks and the number of full complex blocks, respectively. For the delay-dependent conditions examined, $S = 2$ and $F = 0$. Thus, some conservatism may result from the gap between μ and its upper bound. For this example, the μ lower and upper bounds are nearly identical, indicating that the actual value of μ is very close to the μ upper bound. Thus, this source is not a significant contributor to the conservatism for the example considered.

It is apparent that, by far, the largest source of conservatism for this example problem is the manner in which the time-delay elements are eliminated by covering their value sets with unit disks. This is hidden in the Lyapunov framework of the problem, but can be clearly seen in the scaled small-gain formulation. This insight has led the authors to develop less conservative analysis techniques for LTDS [29]–[31].

VI. Conclusion

It has been demonstrated that several recent results in the analysis of the stability of linear time-delay systems are, in fact, equivalent to robust stability analysis of a linear uncertain delay-free comparison system via the scaled small-gain LMI. This result unifies several previous criteria, all of which were originally derived via Lyapunov's Second Method.

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Bounded Stochastic Distributions Control for Pseudo-ARMAX Stochastic Systems

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***Abstract*—Following the recently developed algorithms for the control of the shape of the output probability density functions for general dynamic stochastic systems [6]–[8], this note presents the modeling and control algorithms for pseudo-ARMAX systems, where, different from all the existing ARMAX systems, the considered system is subjected to any arbitrary bounded random input and the purpose of the control input design is to make the output probability density function of the system output as close as possible to a given distribution function. At first, the relationship between the input noise distribution and the output distribution is established. This is then followed by the description on the control algorithm design. A simulated example is used to demonstrate the use of the algorithm and encouraging results have been obtained.**

***Index Terms*—B-splines approximations, dynamic stochastic system, papermaking systems, probability density function.**

II. Introduction

Instead of only controlling the mean and the variance of stochastic systems [1], since 1996 a group of control algorithms [6]–[8] have been developed for the control of the shape of the output probability density functions for general stochastic systems. This is based upon the requirement of probability density function control of system variables in a number of industrial processes [4], [5], where the stochastic system considered is subjected to arbitrary bounded random inputs, rather than standard Gaussian noises. The purpose of the controller design is to select a deterministic control input so as to make the shape of the output probability density functions of the stochastic system as close as possible to a given (desired) distribution [6]–[8].

To avoid the use of partial differential equations, the following decoupled expression [6]–[8] is obtained by using a B-spline to approximate the output probability density function

$$V(k+1) = AV(k) + Bu_k$$
$$\gamma(y,\, u) = \sum_{i=1}^{M} v_i(k)B_i(y) y \in [a,\, b] \tag{1}$$

where

u_k	control input;
$\gamma(y,\, u)$	measured probability density function of the system output;
$V(k)$	$= (v_1,\, v_2,\, \ldots,\, v_M)^T$, weight vector;
$B_i(y)$	pre-specified basis functions for the approximation of $\gamma(y,\, u)$ [2];
A and B	constant matrices.

Although there are several advantages in using this type of model to design the required control algorithm, it is difficult to link such a model structure to a physical system. In particular, the key assumption that the control input only affects the weights of the output probability density function is strict for some applications. As such, it would be ideal if a

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