Inverse Optimal Stabilization of a Rigid Spacecraft

Miroslav Krstić and Panagiotis Tsiotras

Abstract— **We present an approach for constructing optimal feedback control laws for regulation of a rotating rigid spacecraft. We employ the inverse optimal control approach which circumvents the task of solving a Hamilton-Jacobi equation and results in a controller optimal with respect to a meaningful cost functional. The inverse optimality approach requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. For the spacecraft problem, they are both constructed using the method of integrator backstepping. We give a characterization of (nonlinear) stability margins achieved with the inverse optimal control law.**

Keywords— **Attitude control, stabilization, inverse optimality, stability margins, backstepping.**

I. Introduction

Optimal control of rigid bodies has a long history stemming from interest in the control of rigid spacecraft and aircraft [1], [2], [3], [4], [5]. The main thrust of this research has been directed, however, towards the time-optimal and fuel-optimal control problems [6], [7], [8], [9], [10], [11]. The optimal regulation problem over a finite or infinite horizon has been treated in the past mainly for the angular velocity subsystem and for special quadratic costs [10], [12], [13], [14], [15], [16]. The case of general quadratic costs has also been addressed in [17]. Optimal control for the complete attitude problem, i.e., including the orientation equations is more difficult and has been addressed in terms of trajectory planning [18], [19], or in semi-feedback form [20]. The main obstruction in constructing feedback control laws in this case stems from the difficulty in solving the Hamilton-Jacobi equation, especially when the cost includes a penalty term on the control effort. In [21] the authors obtain closed-form optimal solutions for special cases of quadratic costs without penalty on the control effort. These control laws asymptotically recover the optimal cost for the kinematics but may lead to high-gain controllers. When a control penalty is included in the performance index, linear control laws have been constructed which provide an upper bound for a quadratic cost in some specified compact set of initial conditions. Suboptimal results can be obtained by minimizing this upper bound [21]. Alternatively, one can penalize only the high-gain portion of the control input. This approach is based on the optimality results of [22] and it has been used both for axi-symmetric [23] and non-symmetric bodies [24]. The most advanced efforts towards designing optimal feedback controllers have been made in [26], [27] in the framework on nonlinear $\mathcal{H}_\infty$ design. However, the authors in [27] solve the Hamilton-Jacobi-Isaacs *inequality* which, in general, only guarantees an upper bound of the cost for the zero-disturbance case.

In this paper we follow an alternative approach in order to derive optimal feedback control laws for the complete rigid body system. We employ the *inverse optimal* control approach which circumvents the task of solving a Hamilton-Jacobi equation and results in a controller optimal with respect to a meaningful cost functional. This approach, originated by Kalman to establish certain gain and phase margins of linear quadratic regulators [28], was introduced into nonlinear control in [29], and has been long dormant until it was recently revived in [30] to develop a methodology for design of *robust* nonlinear controllers. While [29] establishes a certain nonlinear "return difference" inequality which implies robustness to some input nonlinearities, the full analogy with the linear stability margins was only recently established in [31].

The inverse optimality approach used in this paper requires the knowledge of a control Lyapunov function and a stabilizing control law of a particular form. For the spacecraft problem, we construct them both using the method of integrator backstepping [32]. The resulting design includes a penalty on the angular velocity, orientation *and* the control torque. The weight in the penalty on the control depends on the current state and decreases for states away from the origin. We also present a result which puts a constant (identity) weight on control and possesses stability margins analogous to the infinite gain margin and the 60° phase margins for the linear quadratic regulators. It should be pointed out that global stabilizing controllers using the inverse optimality approach of [30] have also been presented in [33].

The paper is organized as follows. Section II reviews the basics of the inverse optimality approach and presents it in a format convenient for *design* of controllers. Section IV contains the main result—the construction of the inverse optimal feedback law for a rigid spacecraft, which is specialized in Section IV-B to the case of a symmetric spacecraft. A numerical example in Section V illustrates the theoretical result of the paper.

II. Inverse Optimal Control Approach

We consider nonlinear systems affine in the control variable

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth, vector- and matrix-valued functions respectively, with $f(0) = 0$. Moreover, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the state and control vectors, respectively.

Proposition II.1: **([29], [31])** Assume that the static, state-feedback control law

$$u = \kappa(x) := -R^{-1}(x) \left(\frac{\partial V}{\partial x} g(x)\right)^T \tag{2}$$

where $R : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a positive definite matrix-valued function (i.e., $R(x) = R^T(x) > 0$ for all $x \in \mathbb{R}^n$), stabilizes the system in Eq. (1) with respect to a positive definite radially unbounded Lyapunov function $V(x)$. Then the control law

$$u = \kappa^*(x) := \beta\, \kappa(x), \quad \beta \geq 2 \tag{3}$$

is optimal with respect to the cost

$$\mathcal{J} = \int_0^\infty \{l(x) + u^T R(x) u\}\, dt \tag{4}$$

where

$$\begin{aligned} l(x) \quad &= \quad -2\beta \frac{\partial V}{\partial x} \left(f(x) + g(x)\kappa(x)\right) \\ &\quad + \beta(\beta-2) \frac{\partial V}{\partial x} g(x) R^{-1}(x) \left(\frac{\partial V}{\partial x} g(x)\right)^T . \end{aligned} \tag{5}$$

Because $\partial V/\partial x(f(x)+g(x)\kappa(x)) < 0, \forall x \neq 0$, we have $l(x) > 0$ for all $x \neq 0$ and the performance index in Eq. (5) represents a meaningful cost, in the sense that it includes a positive penalty on the state and a positive penalty on the control for each x.

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The cost (5) depends on the particular system dynamics. This is understandable, since by requiring *closed-form solutions* to a nonlinear optimal feedback problem it is sensible to choose costs which are compliant with the system dynamics. In other words, the cost should reflect somehow, and take into account, the form of the nonlinearity of the system. This restricts of course the choice of performance indices. On the other hand, one avoids solving the often formidable Hamilton-Jacobi equation.

The result of Proposition II.1 was given in [31] for $\beta = 2$. The extension that we give here for $\beta \geq 2$ is straightforward and given without proof. However, this extension, already establishes an infinite gain margin of the inverse optimal controller, a well known property of linear quadratic regulators [28]. An equivalent of the phase margin was also given in [31] and it requires that the function $R^{-1}(x)$ be locally bounded. Under this condition, there exists a continuous positive function $\eta(\cdot)$ such that

$$R^{-1}(x) \leq \eta(V(x))I, \quad \forall x \in \mathbb{R}^n, \tag{6}$$

which follows from the radial unboundedness of $V(x)$. With this definition, we state the main result on robustness margins achievable using the inverse optimality approach. In the linear case, this result gives precisely the infinite gain margin[1] and the 60° phase margin.

Proposition II.2: (**[31]**) Under the conditions of Proposition II.1 and assuming that $R^{-1}(x)$ is locally bounded, the control law

$$v = \kappa^*(x) := -\beta\, \eta(V(x)) \left(\frac{\partial V}{\partial x} g(x)\right)^T, \quad \beta \geq 2 \tag{7}$$

is globally asymptotically stabilizing for the system (1) with the input dynamics $u = a(I + \mathcal{P})v$, where $a \geq 1/\beta$ is a constant and $\mathcal{P}$ is a strictly passive[2] (possibly nonlinear) system.

Note that the form of the control law (7) is

$$\kappa^*(x) := -\beta \left(\frac{\partial \hat{V}}{\partial x} g(x)\right)^T \tag{8}$$

where

$$\hat{V}(x) = \int_0^{V(x)} \eta(r)dr \tag{9}$$

is a positive definite and radially unbounded Lyapunov function. The control law (7) minimizes the cost functional

$$\mathcal{J} = \int_0^\infty \{\hat{l}(x) + u^T u\}\, dt \tag{10}$$

where $\hat{l}(x) \geq \eta(V) l(x)$ is positive definite.

III. The Rigid Body Model

In this section we use the inverse optimal results of Proposition II.1 in order to derive control laws which are optimal with respect to a cost which includes a penalty on the control input as well as the angular position and velocity of a rigid spinning spacecraft. The complete attitude motion of a rigid spacecraft can be described by the state equations [24], [25]

$$\dot{\omega} = J^{-1}S(\omega)J\omega + J^{-1}u \tag{11a}$$

$$\dot{\rho} = H(\rho)\omega \tag{11b}$$

where $\omega \in \mathbb{R}^3$ is the angular velocity vector in a body-fixed frame, $\rho \in \mathbb{R}^3$ is the Cayley-Rodrigues parameters vector [25] describing the body orientation, $u \in \mathbb{R}^3$ is the acting control torque, and J is the (positive definite) inertia matrix. The symbol $S(\cdot)$ denotes a 3×3 skew-symmetric matrix, that is,

$$S(\omega) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{12}$$

and the matrix-valued function $H : \mathbb{R}^3 \to \mathbb{R}^{3\times3}$ denotes the kinematics Jacobian matrix for the Cayley-Rodrigues parameters, given by

$$H(\rho) := \frac{1}{2}(I - S(\rho) + \rho\rho^T) \tag{13}$$

where I denotes the 3×3 identity matrix. The matrix $H(\rho)$ satisfies the following identity [24]

$$\rho^T H(\rho)\omega = \left(\frac{1 + \|\rho\|^2}{2}\right)\rho^T\omega, \tag{14}$$

for all $\omega, \rho \in \mathbb{R}^3$, where $\|\cdot\|$ denotes the euclidean norm, i.e., $\|x\|^2 = x^T x$, for $x \in \mathbb{R}^n$.

Observe that the system in Eqs. (11) is in cascade interconnection, that is, the kinematics subsystem (11b) is controlled only indirectly, through the angular velocity vector ω. Stabilizing control laws for systems in this hierarchical form can be efficiently designed using the method of *backstepping* [32]. According to this approach, one thinks of ω as the *virtual control* in Eq. (11b) and designs a control law, say $\omega_d(\rho)$, which stabilizes this system. Subsequently, one designs the actual control input u so as to stabilize the system in Eq. (11a) without destabilizing the system in Eq. (11b) by forcing, for example, $\omega \to \omega_d$. The main benefits of this methodology is that it is flexible, and lends itself to a systematic construction of stabilizing control laws along with the corresponding Lyapunov functions.

IV. Control Design

A. Backstepping

The first step for applying the results of Proposition II.1 is to construct a control-Lyapunov function for the system in Eq. (11). For systems with cascade interconnection structure, such as the rigid body equations, one can use the method of integrator backstepping to achieve this objective. Sontag and Sussmann were the first to notice this property for the rigid body in [36], where they used backstepping to design smooth feedback control laws for an underactuated rigid body. The same technique was also used in [37] for stabilization of an axisymmetric spacecraft using two control torques. Here we use backstepping in order to derive a control-Lyapunov function, along with a stabilizing controller of a particular form for the system in Eq. (11).

Control of the kinematic subsystem. Consider the kinematics subsystem in Eq. (11b) with ω promoted to a control input and let the control law

$$\omega_d = -k_1\rho, \quad k_1 > 0 \tag{15}$$

With this control law the closed-loop system becomes

$$\dot{\rho} = -k_1 H(\rho)\rho \tag{16}$$

The system in Eq. (16) is globally exponentially stable. To see this, consider the following Lyapunov function

$$V_1(\rho) = \frac{1}{2}\|\rho\|^2 \tag{17}$$

[1] See also [35] for a discussion on gain margins for nonlinear optimal regulators.
[2] In the sense of the definition in [34].

Using Eq. (14) the derivative of V_1 along the trajectories of Eq. (16) is given by

$$\dot{V}_1 = -\frac{k_1}{2}(1+\|\rho\|^2)\,\|\rho\|^2 \leq -k_1\, V_1 < 0, \quad \forall\, \rho \neq 0 \tag{18}$$

Global exponential stability with rate of decay $k_1/2$ follows.

Control of the full rigid body model. Consider now the error variable

$$z = \omega - \omega_d = \omega + k_1 \rho \tag{19}$$

The differential equation for the kinematics is written as

$$\dot{\rho} = -k_1 H(\rho)\rho + H(\rho) z \tag{20}$$

and, as shown above, it is globally exponentially stable for $z = 0$. The differential equation for z is

$$\begin{aligned} \dot{z} \;&=\; \left(J^{-1}S(\omega)J + k_1 H(\rho)\right)\, z \\ &\quad -k_1\, \left(J^{-1}S(\omega)J + k_1 H(\rho)\right)\rho + J^{-1}\,u \end{aligned} \tag{21}$$

We want to find $u = u(\rho, z)$ such that the system of Eqs. (20)-(21) is globally asymptotically stable. To this end, consider the following candidate Lyapunov function

$$V(\rho, z) = k_1^2 V_1(\rho) + \frac{1}{2}\|z\|^2 = \frac{k_1^2}{2}\|\rho\|^2 + \frac{1}{2}\|z\|^2 \tag{22}$$

In order to use the results of Proposition II.1 we need a stabilizing control law of the form in Eq. (2). Noticing that with V as in Eq. (22) one has

$$\frac{\partial V}{\partial z}\, J^{-1} = z^T\, J^{-1} \tag{23}$$

we are looking for a control law of the form

$$u = -R^{-1}(\rho, \omega) J^{-1}\, z \tag{24}$$

where $R(\rho, \omega) > 0$, $\forall\, \rho, \omega \in \mathbb{R}^3$. Taking the derivative of V along the trajectories of Eqs. (20)-(21) one obtains

$$\begin{aligned} \dot{V} \;&=\; -\frac{k_1^3}{2}(1+\|\rho\|^2)\,\|\rho\|^2 - k_1 z^T J^{-1} S(\omega) J\rho + z^T J^{-1} S(\omega) J z \\ &\quad+\; z^T\left(\frac{k_1}{2}(I + \rho\rho^T) z + J^{-1} u\right) \end{aligned} \tag{25}$$

and upon completion of squares,

$$\begin{aligned} \dot{V} \;&=\; -\frac{k_1^3}{4}(1+2\|\rho\|^2)\,\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2} J S(\omega) J^{-1} z\right\|^2 \\ &\quad -\frac{k_1}{4}\left\|\left(I + \frac{2}{k_1} J S(\omega) J^{-1}\right)\, z\right\|^2 \\ &\quad +z^T\left\{\left[\frac{k_1}{2}\left(\frac{3}{2}I + \rho\rho^T\right)\frac{2}{k_1} J^{-1} S(\omega)^T J^2 S(\omega) J^{-1}\right] z \right.\\ &\quad \left. +J^{-1}\,u\right\} \end{aligned} \tag{26}$$

Denote

$$\begin{aligned} R(\rho, \omega) \;&=\; J^{-1}\,\left[\left(k_2 + \frac{3}{4}k_1\right) I + \frac{k_1}{2}\rho\rho^T \right.\\ &\quad \left.+ \frac{2}{k_1}\left(S(\omega)J^{-1}\right)^T J^2 S(\omega) J^{-1}\right]^{-1} J^{-1} \end{aligned} \tag{27}$$

where $k_2 > 0$. Then (26) becomes

$$\begin{aligned} \dot{V} \;&=\; -\frac{k_1^3}{4}(1+2\|\rho\|^2)\,\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2} J S(\omega) J^{-1} z\right\|^2 \\ &\quad -\frac{k_1}{4}\left\|\left(I + \frac{2}{k_1} J S(\omega) J^{-1}\right)\, z\right\|^2 - k_2\,\|z\|^2 \\ &\quad +z^T J^{-1}\left\{R^{-1}(\rho, \omega) J^{-1}\, z + \,u\right\} \end{aligned} \tag{28}$$

With the choice of the feedback control law in Eqs. (24), (27), Eq. (26) yields

$$\begin{aligned} \dot{V} \;&=\; -\frac{k_1^3}{4}(1+2\|\rho\|^2)\,\|\rho\|^2 - \frac{k_1^3}{4}\left\|\rho - \frac{2}{k_1^2} J S(\omega) J^{-1} z\right\|^2 \\ &\quad -\; \frac{k_1}{4}\left\|\left(I + \frac{2}{k_1} J S(\omega) J^{-1}\right)\, z\right\|^2 - k_2\,\|z\|^2 \end{aligned} \tag{29}$$

and the equilibrium $\rho = \omega = 0$ is rendered globally asymptotically stable.

From Proposition II.1, for $\beta = 2$, we get the following result.

Theorem IV.1: The control law

$$u^* = -\,J\,\left[\left(2k_2 + \frac{3}{2}k_1\right) I + k_1\, \rho\rho^T \,+ \frac{4}{k_1} J^{-1} S(\omega)^T J^2 S(\omega) J^{-1}\right]\, z \tag{30}$$

minimizes the cost functional

$$\mathcal{J} = \int_0^\infty \{l(\rho, \omega) + u^T R(\rho, \omega) u\}\, dt \tag{31}$$

where

$$\begin{aligned} l(\rho, \omega) \;&=\; k_1^3(1 + 2\|\rho\|^2)\|\rho\|^2 + 4k_2\|\omega + k_1\rho\|^2 \\ &\quad+\; k_1^3\left\|\rho - \frac{2}{k_1^2} J S(\omega) J^{-1}(\omega + k_1\rho)\right\|^2 \\ &\quad+\; k_1\left\|\left(I + \frac{2}{k_1} J S(\omega) J^{-1}\right)(\omega + k_1\rho)\right\|^2 \end{aligned} \tag{32}$$

and $R(\rho, \omega)$ as in Eq. (27).

The performance index in Eq. (31) represents a meaningful cost since $l(\rho, \omega) > 0$ and $R(\rho, \omega) > 0$ for all $(\rho, \omega) \neq (0, 0)$, therefore it penalizes both the states ρ and ω, as well as the control effort u. As ρ and ω increase, the penalty on the control decreases. This is a desirable feature of the optimal control law, since it implies more aggressive control action far away from the equilibrium. Indeed, as the system state starts deviating from the intended operating point the controller allows for increasingly corrective action. For ρ and ω large we have

$$l(\rho, \omega) \;\sim\; 2k_1^3\|\rho\|^4 + \frac{8}{k_1}\left\|J S(\omega) J^{-1}(\omega + k_1\rho)\right\|^2 \tag{33a}$$

$$R(\rho, \omega) \;\sim\; \left[\frac{k_1}{2} J\rho\rho^T J \,+ \frac{2}{k_1} S(\omega)^T J^2 S(\omega)\right]^{-1} \tag{33b}$$

One can see that k_2 has no effect on the large-signal performance. In addition, larger values of k_1 tend to put more penalty on ρ while smaller values of k_1 tend to put more penalty on ω. At the same time, for ρ and ω small we have that

$$l(\rho, \omega) \;\sim\; 2k_1^3\,\|\rho\|^2 + (4k_2 + k_1)\,\|\omega + k_1\rho\|^2 \tag{34a}$$

$$R(\rho, \omega) \;\sim\; \left(k_2 + \frac{3}{4}k_1\right)^{-1} J^{-2} \tag{34b}$$

so, close to the origin, the control law reduces to an LQR-type linear control law. The control law in this case minimizes the LQR cost

$$\mathcal{J} = \int_0^\infty \{[\omega^T \rho^T]\, Q \begin{bmatrix} \omega \\ \rho \end{bmatrix} + u^T R u\}\, dt \tag{35}$$

where

$$Q = \begin{bmatrix} 4k_2 + k_1 & k_1(4k_2 + k_1) \\ k_1(4k_2 + k_1) & k_1^2(3k_1 + 4k_2) \end{bmatrix}, \quad R = \left(\frac{4}{4k_2 + 3k_1}\right) J^{-2} \tag{36}$$

It is important to realize that the optimal control law in Eq. (30) avoids the cancelation of the nonlinearities. Notice, for example, that from Eq. (25) one can globally asymptotically stabilize the system by choosing the control law

$$u = -k_2 J z - \frac{k_1}{2} J(I + \rho\rho^T)z - S(\omega)J\omega \qquad (37)$$

which renders

$$\dot{V} = -\frac{k_1^3}{2}(1 + \|\rho\|^2)\|\rho\|^2 - k_2\|z\|^2 < 0, \quad \forall\,(\rho, z) \neq (0,0) \quad (38)$$

There are no obvious optimality characteristics associated with this control law. In fact, as was pointed out in [31], [38] controllers which cancel nonlinearities are, in general, *non*optimal since the nonlinearity may be actually beneficial in meeting the stabilization and/or performance objectives.

An undesirable feature of the optimal control law in Eq. (30) is that it depends on the moment of inertia matrix J, which may not be always accurately known. The robustness properties of the optimal control law will be addressed in the future.

B. The symmetric case

When the rigid body is symmetric, its inertia matrix is a multiple of the identity matrix and

$$S(\omega)J\omega \equiv 0, \quad \forall\,\omega \in \mathbb{R}^3 \qquad (39)$$

In this case the optimal control law simplifies to

$$u^* = -\,J\left[(2k_2 + k_1)I + k_1\,\rho\rho^T\right] z \qquad (40)$$

which minimizes the cost in Eq. (4) where

$$l(\omega, \rho) \;=\; 2\,k_1^3(1 + \|\rho\|^2)\|\rho\|^2 + 4\,k_2\|\omega + k_1\rho\|^2 \quad (41a)$$

$$R(\omega, \rho) \;=\; J^{-1}\left[(k_2 + \frac{k_1}{2})I + \frac{k_1}{2}\rho\rho^T\right]^{-1} J^{-1} \quad (41b)$$

This control law reduces to an LQR-type feedback control law close to the origin, with

$$Q = \begin{bmatrix} 4k_2 & 4k_1k_2 \\ 4k_1k_2 & 2k_1^2(k_1 + 2k_2) \end{bmatrix} \quad \text{and} \quad R = \left(\frac{2}{2k_2 + k_1}\right) J^{-2} \qquad (42)$$

We note that the symmetric case has been previously addressed by Wie *et al.* [39], where an Euler parameter description for the kinematics was used.

C. A controller with stability margins

We now set out to derive a control law that has stability margins described in Proposition II.2. Lengthy calculations show that

$$R^{-1}(\rho, \omega) \leq \lambda_{\max}^2(J)\,\left[k_2 + \frac{3}{4}k_1 + \frac{9}{k_1}V(\rho, z)\right] I, \quad \forall\,\rho, \omega \in \mathbb{R}^3 \qquad (43)$$

By Proposition II.2, the control law

$$u^* \;=\; -\lambda_{\max}^2(J)\,\left[k_2 + \frac{3}{4}k_1 + \frac{9}{2k_1}\left(k_1^2\|\rho\|^2 + \|\omega + k_1\rho\|^2\right)\right] \times J^{-1}(\omega + k_1\rho) \qquad (44)$$

where $\lambda_{\max}(J)$ is the maximum eigenvalue of the matrix J, is robust to the input dynamics $a(I + \mathcal{P})$, where $a \geq 1/2$ is a constant and $\mathcal{P}$ is a strictly passive (possibly nonlinear) system. For example, the controller (44) will be stabilizing when passed through linear input dynamics $a(s+z)/(s+p)$ for any $z \geq p > 0$ and any $a \geq 1/2$ because the transfer function $(z-p)/(s+p)$ is strictly positive real.

V. Numerical Example

Numerical simulations were performed to establish the validity of the theory. We assume a rigid spacecraft with inertia matrix $J = diag(10, 15, 20)\ kg\,m$. A rest-to-rest maneuver is considered, thus $\omega(0) = 0$. First, we consider the kinematics subsystem in Eq. (11b) with ω regarded as the control input. Let the initial conditions $\rho(0) = [1.4735, 0.6115, 2.5521]^T$ in terms of the Cayley-Rodrigues parameters. These initial conditions correspond to a principal axis/angle pair $\hat{\mathbf{e}} = [0.4896, 0.2032, 0.8480]^T$ and $\Phi = 2.5\ rad$ and describe an almost "upside-down" initial orientation. The trajectories of the system with the control law in Eq. (15) with $k_1 = 0.5$ are shown in Figs. (1) and (2). The exponential stability of the closed-loop system is evident from these figures. At this step the choice of k_1 is basically dictated by the required speed for the completion of the rest-to-rest maneuver.

For the stabilization of the complete system we use the control law in Eq. (30). The state trajectories for different values of the gain k_2 are depicted in Figs. (3) and (4). The optimal trajectories have a very uniform behavior which is essentially independent of the value of k_2 and they follow very closely the corresponding trajectories for the kinematics subsystem. The control action varies a great deal, however, with k_2. The initial control action consists, essentially, in making $\omega \to -k_1\rho$. This is clearly shown in Fig. (3).

Finally, Fig. (6) shows the time history of the Frobenious norm of the control penalty matrix $R(\omega, \rho)$. The control penalty is decreased rapidly at the initial portion of the trajectory when increased control action is necessary in order to "match" ω with ω_d within a short period of time.

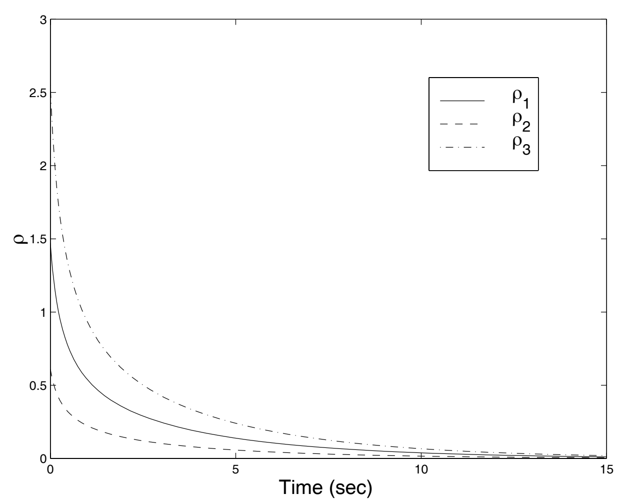

Fig. 1. Orientation parameters for the kinematics.

VI. Conclusions

Due to the difficulty in obtaining closed-form solutions to the Hamilton-Jacobi-Bellman equation, the *direct* optimal control problem for nonlinear systems remains open. However, the knowledge of a control Lyapunov function allows us to solve the *inverse* optimal control problem, i.e., find a controller which is optimal with respect to a meaningful cost. The inverse optimal stabilization design for a rigid spacecraft in this paper is, to the authors' knowledge, the first feedback control law that minimizes a cost that incorporates penalty on both the state (angular velocity and orientation) and the control effort (torque).

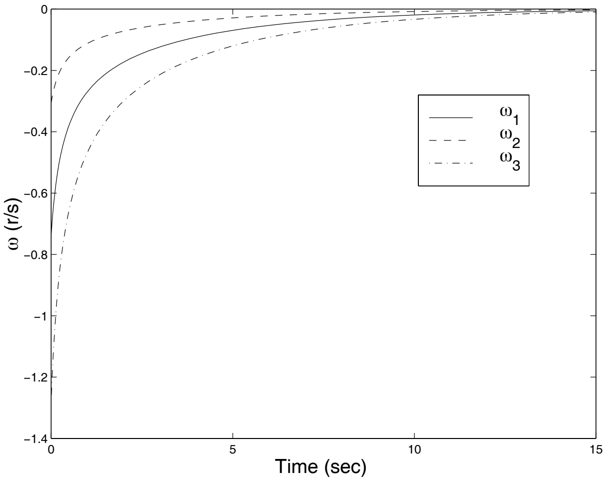

Fig. 2. Angular velocity for the kinematics.

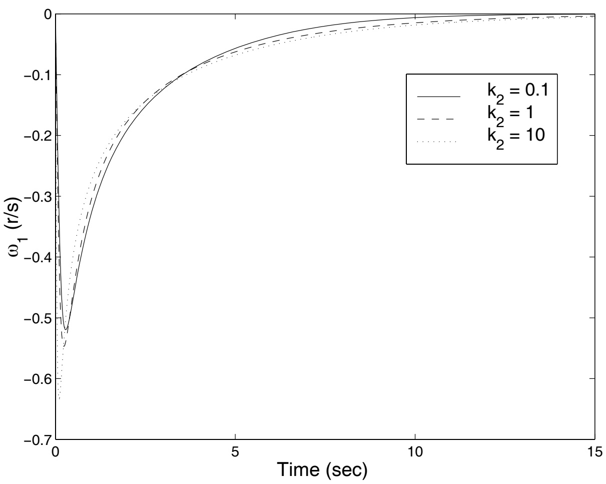

Fig. 3. Angular velocity ω_1.

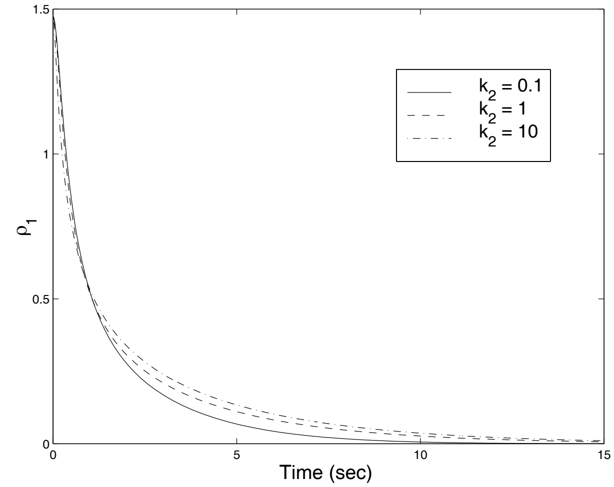

Fig. 4. Orientation parameter ρ_1.

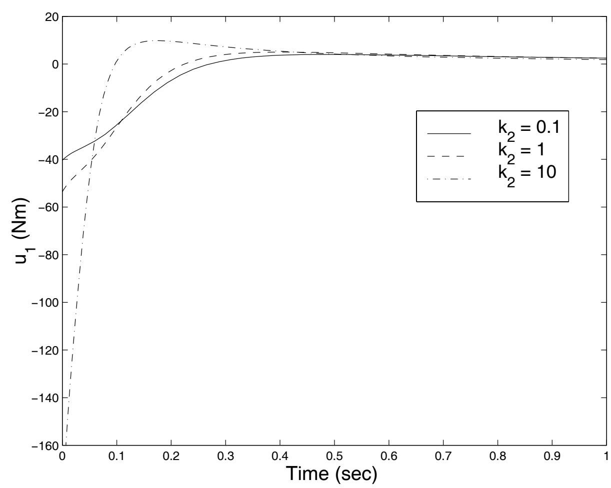

Fig. 5. Control input u_1.

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References

[1] A. S. Debs and M. Athans, "On the optimal angular velocity control of asymmetrical space vehicles," *IEEE Transactions on Automatic Control*, vol. 14, pp. 80–83, Feb. 1969.

[2] T. E. Dabbous and N. U. Ahmed, "Nonlinear optimal feedback regulation of satellite angular momenta," *IEEE Trans. Aerosp. Elec. Sys.*, vol. 18, no. 1, pp. 2–10, 1982.

[3] G. Porcelli, "Optimal attitude control of a dual-spin satellite," *Journal of Spacecraft and Rockets*, vol. 5, no. 8, pp. 881–888, 1968.

[4] V. S. Sohoni and D. R. Guild, "Optimal attitude control systems for spinning aerospace vehicles," *Journal of the Astronautical Sciences*, vol. 18, no. 2, pp. 86–100, 1970.

[5] S. R. Vadali, L. G. Kraige, and J. L. Junkins, "New results on the optimal spacecraft attitude maneuver problem," *Journal of Guidance, Control, and Dynamics*, vol. 7, no. 3, pp. 378–380, 1984.

[6] S. L. Scrivener and R. C. Thomson, "Survey of time-optimal attitude maneuvers," *Journal of Guidance, Control, and Dynamics*, vol. 17, no. 2, pp. 225–233, 1994.

[7] J. L. Junkins, C. K. Carrington, and C. E. Williams, "Time-optimal magnetic attitude maneuvers," *Journal of Guidance, Control, and Dynamics*, vol. 4, no. 4, pp. 363–368, 1981.

[8] R. M. Byers and S. R. Vadali, "Quasi closed-form solution to the time-optimal rigid spacecraft reorientation problem," 1991. AAS Paper 91-120.

[9] J. R. Etter, "A solution of the time-optimal Euler rotation problem," 1989. AIAA Paper 89-3601-CP.

[10] M. Athans, P. L. Falb, and R. T. Lacoss, "Time-, fuel-, and energy-optimal control of nonlinear norm-invariant systems," *IRE Transactions on Automatic Control*, vol. 8, pp. 196–202, July 1963.

[11] M. V. Dixon, T. Edelbaum, J. E. Potter, and W. E. Vandervelde, "Fuel optimal reorientation of axisymmetric spacecraft," *J. Spacecraft*, vol. 7, pp. 1345–1351, 1970.

[12] T. G. Windeknecht, "Optimal stabilization of rigid body attitude," *Journal of Mathematical Analysis and Applications*, vol. 6, pp. 325–335, 1963.

[13] H. Bourdache-Siguerdidjane, "Further results on the optimal regulation of spacecraft angular momentum," *Optimization, Control, Applications and Methods*, vol. 12, pp. 273–278, 1991.

[14] K. S. P. Kumar, "Stabilization of a satellite via specific optimum control," *IEEE Transactions on Aerospace Electronic Systems*, vol. 2, pp. 446–449, 1966.

[15] K. S. P. Kumar, "On the optimum stabilization of a satellite," *IEEE Transactions on Aerospace Electronic Systems*, vol. 1, no. 2, pp. 82–83, 1965.

[16] T. A. W. Dwyer, M. S. Fadali, and N. Chen, "Single step optimization of feedback-decoupled spacecraft attitude maneuvers," in *Proceedings of the 24th Conference on Decision and Control*, pp. 669–671, 1985. Ft. Lauderdale, FL.

[17] M. Tsiotras, P. Corless and M. Rotea, "Optimal control of rigid body angular velocity with quadratic cost," in *Proceedings of the 35th Conference on Decision and Control*, 1996. Kobe, Japan; also *Journal of Optimization Theory and Applications*, (to appear).

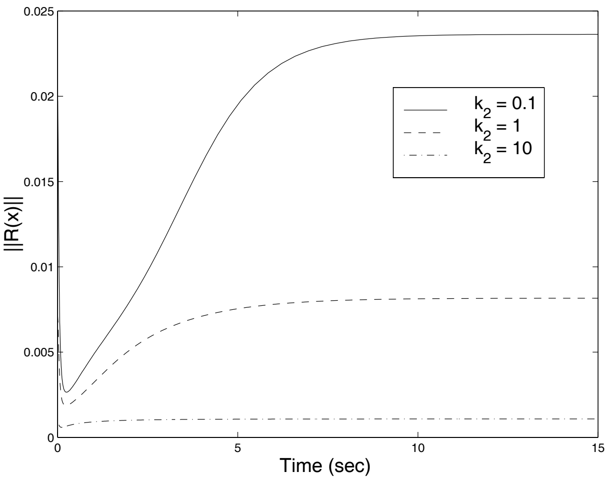

Fig. 6. Norm of $R(\omega, \rho)$.

[18] S. R. Vadali and J. L. Junkins, "Optimal open-loop and stable feedback control of rigid spacecraft attitude maneuvers," *Journal of the Astronautical Sciences*, vol. 32, no. 2, pp. 105–122, 1984.

[19] Y. Y. Lin and L. G. Kraige, "Enhanced techniques for solving the two-point boundary-value problem associated with the optimal attitude control of spacecraft," *Journal of the Astronautical Sciences*, vol. 37, no. 1, pp. 1–15, 1989.

[20] C. K. Carrington and J. L. Junkins, "Optimal nonlinear feedback control for spacecraft attitude maneuvers," *Journal of Guidance, Control, and Dynamics*, vol. 9, no. 1, pp. 99–107, 1986.

[21] M. Rotea, P. Tsiotras, and M. Corless, "Suboptimal control of rigid body motion with a quadratic cost," in *Third IFAC Nonlinear Symposium on Nonlinear Control Systems Design*, June 26-28, 1995. Tahoe City, CA; also *Dynamics and Control*, (to appear).

[22] C. J. Wan and D. S. Bernstein, "Nonlinear feedback control with global stabilization," *Dynamics and Control*, vol. 5, pp. 321–346, 1995.

[23] P. Tsiotras, "On the optimal regulation of an axi-symmetric rigid body with two controls," in *AIAA Guidance, Navigation, and Control Conference*, (San Diego, CA), July 29-31, 1996. Paper AIAA 96-3791.

[24] P. Tsiotras, "Stabilization and optimality results for the attitude control problem," *Journal of Guidance, Control, and Dynamics*, vol. 19, no. 4, pp. 772–779, 1996.

[25] M. D., Shuster, "A survey of attitude representations," *Journal of the Astronautical Sciences*, vol. 41, no. 4, pp. 439–517, 1993.

[26] W. Kang, "Nonlinear $\mathcal{H}_\infty$ control and application to rigid spacecraft," *IEEE Transactions on Automatic Control*, vol. 40, pp. 1281–1285, 1995.

[27] M. Dalsmo and O. Egeland, "State feedback $\mathcal{H}_\infty$ suboptimal control of a rigid spacecraft," *IEEE Transactions on Automatic Control*, vol. 42, pp. 1186–1189, 1997.

[28] B. D. O. Anderson and J. B. Moore, *Optimal Control: Linear Quadratic Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1990.

[29] P. J. Moylan and B. D. O. Anderson, "Nonlinear regulator theory and an inverse optimal control problem," *IEEE Transactions on Automatic Control*, vol. 18, no. 5, pp. 460–465, 1973.

[30] R. A. Freeman and P. Kokotović, *Robust Nonlinear Control Design : State-Space and Lyapunov Techniques.* Boston: Birkäuser, 1996.

[31] R. Sepulchre, M. Janković and P. V. Kokotović, *Constructive Nonlinear Control,* New York: Springer-Verlag, 1997.

[32] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design.* New York: Wiley and Sons, 1995.

[33] M. Osipchuk, S. Bharadwaj, and K. Mease, "Achieving good performance in global attitude stabilization," in *Proceedings of the American Control Conference*, pp. 403–407, 1997. Albuquerque, NM.

[34] C.I. Byrnes, A. Isidori, and J.C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 11, pp. 1228–1240, 1991.

[35] S. T. Glad, "On the gain margin of nonlinear and optimal regulators," *IEEE Transactions on Automatic Control*, vol. 29, no. 7, pp. 615–620, 1984.

[36] E. Sontag and H. Sussmann, "Further comments of the stabilizability of the angular velocity of a rigid body," *Systems and Control Letters*, vol. 12, pp. 213–217, 1988.

[37] P. Tsiotras and J. M. Longuski, "Spin-axis stabilization of symmetric spacecraft with two control torques," *Systems and Control Letters*, vol. 23, pp. 395–402, 1994.

[38] R. A. Freeman and P. V. Kokotović, "Optimal nonlinear controllers for feedback linearizable systems," in *Proceedings of the American Control Conference*, pp. 2722–2726, 1995. Seattle, WA.

[39] B. Wie, H. Weiss and A. Arapostathis, "Quaternion feedback regulator for spacecraft eigenaxis rotation," *Journal of Guidance, Control, and Dynamics*, vol. 12, pp. 375–380, 1989.