Further Passivity Results for the Attitude Control Problem

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Abstract— **In this paper we establish passivity for the system which describes the attitude motion of a rigid body in terms of minimal three-dimensional kinematic parameters. In particular, we show that linear, asymptotically stabilizing controllers and control laws without angular velocity measurements follow naturally from these passivity properties. The results of this paper extend similar results for the case of the (nonminimal) Euler parameters.**

Keywords— **Attitude control, passivity, stabilization, Cayley-Rodrigues parameters, Modified Rodrigues parameters.**

I. Introduction

Recently it has been shown [1],[2],[3] that there exist *linear* asymptotically stabilizing control laws for the attitude motion of a rigid body using minimal, three-dimensional parameterizations for the kinematics. In [1] linear control laws were derived in terms of the classical Cayley-Rodrigues parameters [4] and the nonstandard Modified Rodrigues parameters [1],[4] In the present paper we show that the existence of linear asymptotically stabilizing controllers in terms of these parameters is intimately related to the passivity properties of the corresponding kinematic systems. We show these passivity properties by constructing the respective storage functions. Using these results, we also derive control laws which do not use angular velocity measurements.

The results in this paper complement and extend similar results published recently in terms of the (non-minimal) Euler parameter kinematic parameterizations [5],[6]. In particular, in [5] the authors establish the passivity between the angular velocity vector and the Euler parameter vector, as well as the Euler rotation vector. Adaptive control laws are then derived using these results. Reference [6] uses the same result to develop velocity-free controllers in terms of the Euler parameters. The approach in [6] is similar to the recent results of [7] and [8] on output stabilization of Euler-Lagrange systems, where it is shown that asymptotic stabilization for such systems may be possible without velocity measurements via the inclusion of a dynamic extension (lead filter) to the system. The so-called "dirty derivative" controllers of [8] provide the necessary damping for the global stabilization of the closed-loop system. The results of [7],[8] consider, however, vector second-order mechanical systems and thus, the derived PD controllers are in terms of generalized coordinates and their derivatives. Attitude control problems, on the other hand, are more conveniently and compactly described in terms of angular velocities which are not rates of any generalized coordinates. Thus, for attitude problems it is preferable to develop "PD" controllers in terms of the angular velocity vector and the kinematic parameters, instead. A comprehensive treatment of "PD" controllers for tracking of mechanical systems on Lie groups in a coordinate-free framework is given in [9].

II. System Equations

The dynamics for the attitude motion of a rigid body obey the differential equation

$$J\dot{\omega} = S(\omega)J\omega + u, \qquad \omega(0) = \omega_0 \tag{1}$$

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where $\omega \in \mathbb{R}^3$ denotes the angular velocity vector in a body-fixed frame, $u \in \mathbb{R}^3$ is the acting torque vector, and J is the symmetric inertia matrix. The matrix $S(\cdot)$ denotes a skew-symmetric matrix representing the cross product between two vectors, i.e., $S(v)w = -v \times w$.

In this paper, the orientation of the body with respect to the inertial frame will be described either in terms of the Rodrigues parameters, or in terms of the Modified Rodrigues parameters [4]. The kinematic equations in terms of the Rodrigues parameters take the form

$$\dot{\rho} = H(\rho)\omega, \qquad \rho(0) = \rho_0 \tag{2}$$

where

$$H(\rho) := \frac{1}{2}(I - S(\rho) + \rho\rho^T)$$

and I denotes the 3×3 identity matrix. The kinematic equations in terms of the Modified Rodrigues parameters take the form

$$\dot{\sigma} = G(\sigma)\omega, \qquad \sigma(0) = \sigma_0 \tag{3}$$

where

$$G(\sigma) := \frac{1}{2}\left(\frac{1-\sigma^T\sigma}{2} I - S(\sigma) + \sigma\sigma^T\right)$$

If $\hat{\mathbf{e}}$ and ϕ denote the principal axis and the principal angle [4] respectively, then the Cayley-Rodrigues parameters can be defined by $\rho = \hat{\mathbf{e}}\tan(\phi/2)$, whereas the Modified Rodrigues parameters can be defined by $\sigma = \hat{\mathbf{e}}\tan(\phi/4)$. The kinematic description using the Modified Rodrigues parameters has the advantage that remains valid for eigenaxis rotations up to 360 deg, whereas the Cayley-Rodrigues parameters cannot describe motions which correspond to eigenaxis rotations of more than 180 deg [1],[2],[4].

Direct manipulation shows that the matrices $H(\rho)$ and $G(\sigma)$ have the following properties [10]

$$\rho^T H(\rho) = \left(\frac{1+\rho^T\rho}{2}\right)\rho^T$$
$$H^T(\rho)(I+\rho\rho^T)^{-1}H(\rho) = \left(\frac{1+\rho^T\rho}{4}\right) I \tag{4}$$

for all $(\omega,\rho) \in \mathbb{R}^3 \times \mathbb{R}^3$, and

$$\sigma^T G(\sigma) = \left(\frac{1+\sigma^T\sigma}{4}\right)$$
$$G^T(\sigma)G(\sigma) = \left(\frac{1+\sigma^T\sigma}{4}\right)^2 I \tag{5}$$

for all $(\omega,\sigma) \in \mathbb{R}^3 \times \mathbb{R}^3$, respectively.

III. Passivity and Stabilizability

An important property of the system of equations (1) and (2), equivalently equations (1) and (3), is that they describe a system in *cascade interconnection*, of two *passive* systems. For the passivity definitions used in this work one may consult [11].

Proposition 1: (i) The system (1) with input u and output ω is passive.

(ii) The system (3) with input ω and output σ is passive.

(iii) The system (2) with input ω and output ρ is passive.

Proof: (i) Let the function $V_1(\omega) = \frac{1}{2}\omega^T J\omega$. Differentiation along the trajectories of (1) yields that $\dot{V}_1(\omega) = \omega^T u$, therefore integrating from 0 to T one obtains $\int_0^T \omega^T u\, dt = V_1(\omega(T)) - V_1(\omega_0)$ and since $V_1(\omega) \geq 0$ for all $\omega \in \mathbb{R}^3$ we have that $\int_0^T \omega^T u\, dt + V_1(\omega_0) \geq 0$ which establishes that the system is passive [11].

(ii) Let the function $V_2(\sigma) = 2\ln(1+\sigma^T\sigma)$. Differentiation along the trajectories of (3) and use of equations (4) and (5) yields that $\dot{V}_2(\sigma) = \sigma^T\omega$. Integrating from 0 to T, rearranging terms, and since $V_2(\sigma) \geq 0$ for all $\sigma \in \mathbb{R}^3$, we have that $\int_0^T \sigma^T\omega\, dt + V_2(\sigma_0) \geq 0$ which establishes that the system is passive [11].

(iii) The proof is identical to the case (ii), where we now use the positive definite function $V_3(\rho) = \ln(1+\rho^T\rho)$. ■

Remark 1: According to [11] the systems (1), (3) and (2) are passive with corresponding *storage functions* V_1, V_2 and V_3, respectively. Moreover, the proof of Proposition 1 shows that the systems (1), (3) and (2) are, in fact, *lossless* [12].

The passivity of system (1) is a well-known fact and has been used repeatedly in the past. The passivity of system (3) or of the system (2), however, is neither as a well-known nor as a frequently used result. Passivity in terms of the Euler parameter vector and the Euler rotation vector has been shown, however, in [5].

Stabilization of passive systems in cascade interconnection is a straightforward task. In particular, subject to some mild assumptions, one can stabilize such systems using linear feedback. In essence, the approach consists of using feedback to make the first subsystem strictly passive and then close the loop with the output of the second subsystem, to obtain a feedback interconnection of a strictly passive and a passive system.

Proposition 2: Consider the system (1) with the feedback control law

$$u = -k_1\omega + \nu \tag{6}$$

with $k_1 > 0$. Then the system with input ν and output ω is strictly passive.

Proof: Let $V_1(\omega) = \frac{1}{2}\omega^T J\omega$ as in the proof of Proposition 1. Differentiation of V_1 along the trajectories yields $\dot{V}_1 = \omega^T J\dot{\omega} = -k_1\|\omega\|^2 + \omega^T\nu$. Integrating from 0 to T the previous equation and rearranging terms one obtains $\int_0^T \omega^T\nu\, dt + V_1(\omega_0) \geq k_1\int_0^T \|\omega\|^2\, dt$. The last inequality establishes that the system from ν to ω is strictly passive with storage function V_1 and *dissipation rate* $k_1\|\omega\|^2$ [11]. ■

Theorem 1: (i) The linear control law

$$u = -k_1\omega - k_2\sigma \tag{7}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes the system (1) and (3) at the origin.

(ii) The linear control law

$$u = -k_1\omega - k_2\rho \tag{8}$$

with $k_1 > 0$ and $k_2 > 0$, globally asymptotically stabilizes the system (1) and (2) at the origin.

Proof: (i) The proof is easily obtained by constructing an appropriate Lyapunov function which is the sum of the storage functions of the passive interconnection. To this end, consider the positive definite, radially unbounded function $V(\omega,\sigma) = V_1(\omega)+k_2V_2(\sigma) = \frac{1}{2}\omega^T J\omega + 2k_2\ln(1+\sigma^T\sigma)$. Taking derivatives along closed-loop trajectories and using equation (5) one obtains

$$\begin{aligned}\dot{V} &= \omega^T J\dot{\omega} + 4k_2\frac{\sigma^T\dot{\sigma}}{1+\sigma^T\sigma} \\ &= \omega^T u + k_2\frac{\sigma^T}{1+\sigma^T\sigma}(1+\sigma^T\sigma)\omega = -k_1\|\omega\|^2 \leq 0 \end{aligned} \tag{9}$$

By LaSalle's invariance principle, the system is asymptotically stable in the (ω,ρ) space. Global asymptotic stability follows from the radial unboundedness of V [13].

(ii) Use the following positive definite, radially unbounded function $V(\omega,\sigma) = V_1(\omega) + k_2V_3(\rho) = \frac{1}{2}\omega^T J\omega + k_2\ln(1+\rho^T\rho)$ as a Lyapunov function for the closed-loop system. Asymptotic stability follows from a standard LaSalle argument as in part (i). ■

Remark 2: The previous results show global stability *in the* (ω,ρ) *and* (ω,σ) *spaces* using the linear control laws in equations (8) and (7). This implies asymptotic stability over an open and dense set in the configuration space of the attitude motion $SO(3)$. This term of stability is often coined *almost* global asymptotic stability [3].[1] The topological structure of $SO(3)$ (not a contractible space) does not allow for globally continuously stabilizing control laws. In practice, however, one can always modify these control laws (e.g., using an open-loop strategy applied over a finite and arbitrarily small interval) over a set of measure zero to get globally asymptotically stabilizing controls over the whole $SO(3)$.

Remark 3: The linear control laws (7) and (8) were initially developed in [1] using a Lyapunov approach. No passivity interpretation was given, however. Linear stabilizing controls in terms of the Cayley-Rodrigues parameters have also been used in stabilization of underwater vehicles in [3], also using Lyapunov theory[2].

IV. Velocity-Free Controllers

In this section we show that the linear control laws (7) and (8) can be implemented without angular velocity feedback and thus, one only needs orientation measurements. The methodology used in this section follows closely the one in [6] and uses the properties in equations (4) and (5).

Proposition 3: (i) The system (1) and (3) with control law

$$u = -k_2\sigma + v, \qquad (k_2 > 0) \tag{10}$$

and input v and output ω is passive.

(ii) The system (1) and (2) with control law

$$u = -k_2\rho + v, \qquad (k_2 > 0) \tag{11}$$

and input v and output ω is passive.

Proof: (i) Let the function $V(\omega,\sigma) = V_1(\omega) + k_2V_2(\sigma)$ where V_1 and V_2 as in Proposition 1. Differentiation along the trajectories of (1) and (3) yields that $\dot{V}(\omega,\sigma) = \omega^T u + k_2\sigma^T\omega$. Using (10) we get that $\dot{V}(\omega,\sigma) = \omega^T v$. The rest of the proof follows as in Proposition 1.

(ii) The proof is similar to part (i) and thus, omitted. ■

Notice that the kinematic equations (2) and (3) relate ω to the rates of the kinematic parameters through a matrix multiplication. One can use this result to establish input/output transformations for these systems which preserve passivity.

Proposition 4: (i) The system (1) and (3) with input $y = \left(\frac{4}{1+\sigma^T\sigma}\right)^2 G(\sigma)v$ and output $w = G(\sigma)\omega = \dot{\sigma}$ is passive.

(ii) The system (1) and (2) with input $y = \left(\frac{4}{1+\rho^T\rho}\right)(I + \rho\rho^T)^{-1}H(\rho)v$ and output $w = H(\rho)\omega = \dot{\rho}$ is passive.

Proof: (i) Using equation (5) we have that

$$\begin{aligned}\int_0^T w^T y\, dt &= \int_0^T \left(\frac{4}{1+\sigma^T\sigma}\right)^2 \omega^T G^T(\sigma)G(\sigma)v\, dt \\ &= \int_0^T \left(\frac{1+\sigma^T\sigma}{4}\right)^2 \left(\frac{4}{1+\sigma^T\sigma}\right)^2 \omega^T v\, dt\end{aligned}$$

[1] In fact, Ref. [3] reserves the term almost global stability to the case when the system is also defined over the complement of this set.

[2] We owe this observation to an anonymous reviewer.

$$= \int_0^T \omega^T v \, dt \tag{12}$$

Using now part (i) of Proposition 3 we establish the desired result.

(ii) The proof is similar to (i) and thus, omitted. ∎

Notice that if y is the new input as defined in Proposition 4 then v is given by $v = G^T(\sigma)y$ for the case of the Modified Rodrigues parameters or $v = H^T(\rho)y$ for the Cayley-Rodrigues parameters. Since by Propositions 4 the map from y to w is passive, one may explore the possibility of globally asymptotically stabilizing the system by choosing a feedback such that the map from w to y is strictly passive [11],[14],[6].

To this end, let A be any matrix which is Hurwitz, B any full column rank matrix, with the pair (A, B) controllable, and Q any positive definite matrix. Let also the matrix P be the solution of the Lyapunov equation $A^T P + PA = -Q$. Clearly then P is positive definite.

Theorem 2: Consider the system (1) and (3) and let the control law

$$u = -k_2\sigma - k_1 G^T(\sigma)y \tag{13}$$

with $k_1 > 0$, $k_2 > 0$, and where y is the output of the linear, time-invariant system

$$\dot{x} = Ax + B\sigma \tag{14a}$$

$$y = B^T PAx + B^T PB\sigma \tag{14b}$$

Then the closed-loop system is globally asymptotically stable. In particular, $\lim_{t\to\infty}(\omega(t), \sigma(t)) = 0$, for all initial conditions $(\omega_0, \sigma_0) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Proof: Consider the positive definite function

$$V(\omega, \sigma, x) = \frac{1}{2}\omega^T J\omega + 2k_2 \ln(1+\sigma^T\sigma) + \frac{k_1}{2}(Ax+B\sigma)^T P(Ax+B\sigma) \tag{15}$$

Noticing that the last term in (15) is just $\frac{k_1}{2}\dot{x}^T P\dot{x}$, the time derivative of V along the trajectories of the closed-loop system is

$$\begin{aligned}
\dot{V} &= \omega^T J\dot{\omega} + k_2\left(\frac{4}{1+\sigma^T\sigma}\right)\sigma^T G(\sigma)\omega + k_1\dot{x}^T P\ddot{x} \\
&= \omega^T(-k_2\sigma - k_1 G^T(\sigma)y) + k_2\sigma^T\omega + k_1\dot{x}^T PA\dot{x} \\
&\quad + k_1\dot{x}^T PBG(\sigma)\omega \\
&= \frac{k_1}{2}\dot{x}^T(PA + A^T P)\dot{x} = -\frac{k_1}{2}\dot{x}^T Q\dot{x} \le 0
\end{aligned} \tag{16}$$

First observe that since V is radially unbounded, all solutions are bounded. Consider now the set $\mathcal{E} = \{(\omega, \sigma, x) : \dot{V} = 0\}$. Trajectories in $\mathcal{E}$ satisfy $\dot{x} = 0$ and hence $x(t) = x_0$ for all $t \ge 0$ and from (14a) also $\sigma(t) = \sigma_0$ for all $t \ge 0$. Then $\dot{\sigma} = 0$ and from (3) also $\omega(t) = 0$ for all $t \ge 0$. Since $y = B^T P\dot{x}$ one has also that $y = 0$, and using (1) and (13) we have that $\omega = \dot{\omega} = 0$ and $y = 0$ implies that $\sigma = 0$. Since A is Hurwitz equation (14a) then also implies that $x = 0$. The largest invariant set in $\mathcal{E}$ is therefore the set $\mathcal{M} = \{(\omega, \sigma, x) \in \mathcal{E} : \omega = \sigma = x = 0\}$. By LaSalle's Theorem, and since V is radially unbounded, the system is globally asymptotically stable. In particular, all trajectories of the system asymptotically approach $\mathcal{M}$ thus $\lim_{t\to\infty}(\omega(t), \sigma(t)) = 0$, as claimed. ∎

Similarly, for the Cayley-Rodrigues parameters one obtains the following result.

Theorem 3: Consider the system (1) and (2) and let the control law

$$u = -k_2\rho - k_1 H^T(\rho)y \tag{17}$$

with $k_1 > 0$, $k_2 > 0$, and where y is the output of the linear, time-invariant system

$$\dot{x} = Ax + B\rho \tag{18a}$$

$$y = B^T PAx + B^T PB\rho \tag{18b}$$

Then the closed-loop system is globally asymptotically stable. In particular, $\lim_{t\to\infty}(\omega(t), \rho(t)) = 0$, for all initial conditions $(\omega_0, \rho_0) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Remark 4: The Cayley-Rodrigues and Modified Rodrigues parameters are of the "Euler-parameter" type in the sense that they can be represented by $\hat{\mathbf{e}}f(\phi)$ for some function $f(\phi)$ (see [15] for more details). It should be straightforward to extend the results of this paper to all "Euler-parameter" type attitude descriptions.

V. Concluding Remarks

In this paper we derive some additional passivity results for the attitude control problem when the kinematics are described in terms of minimal parameterizations and we provide the corresponding storage functions. We show that linear asymptotically stabilizing control laws in terms of the Cayley-Rodrigues and the Modified Rodrigues parameters follow directly from these passivity properties. Also, velocity-free controllers can be easily constructed. These results extend similar previous results in terms of the (non-minimal) Euler parameters.

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