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ON THE LARGE ANGLE PROBLEM IN RIGID BODY ATTITUDE DYNAMICS

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Abstract

For many problems involving rotating rigid bodies (e.g. spin-stabilized satellites in space) one usually linearizes the equations of motion for the Eulerian angles to obtain tractable analytic solutions. However, these methods -- based on a small angle assumption -- have failed to provide a comprehensive treatment of the behavior of a rigid body during large angular motions that occur, for example, during despinning. For such cases the nonlinear effects dictate a more sophisticated theory for problem analysis. In this paper we discuss three different approaches for this class of problems.

Introduction

Rockets and spacecraft are often spun up to provide stability. When for some reason the spin rate decreases (e.g. thruster failure or spin-down maneuver), this stabilizing effect diminishes, and in the presence of transverse torques, the vehicle is subject to large angular displacements. In such a case, attitude solutions based on small angle assumptions are no longer valid. For large angular displacements the nonlinearities play the predominant role, and a more sophisticated theory needs to be developed to handle such cases. In this paper three possible methods are presented to tackle the problem of large angles during despinning of a rotating rigid body.

The first method, based on Euler angle formulation, uses a nonlinear transformation that enables one to reformulate the problem in another set of Euler angles. This reformulation allows one to reinterpret the previous solutions, so that one can extend their validity to moderately large angle regimes. This method has the additional advantage that one can extend previous analytic solutions[1] *without* the need to solve any additional differential equations. Nevertheless, because the small angle assumption has not been removed, these transformations simply extend the linearized solutions. When the small angle assumption is removed, the transformation yields a system of two nonlinear differential equations that give the *exact* solution to the attitude problem. The advantage of the transformations in this setting lies in the fact that the nonlinearities of the transformed equations are polynomial in nature, whereas the original equations involve trigonometric nonlinearities. As a result, the transformed equations are more tractable using classical series and/or perturbation techniques.

The second method uses an approach based on quaternions and their counterpart, the Euler parameters. It is well-known that the quaternion formulation leads to a set of linear differential equations, but because of the time-varying nature of the coefficient matrix, analytic procedures do not fair well. A *semi-analytic* solution based on Picard's method of the *product integral* (also called the *time-ordered exponential*) is presented, that allows one to find approximate solutions to this system of linear time-varying differential equations. The methodology for the solution works for every linear time-varying system of differential equations, but is especially suited for problems of rotational kinematics, where the special structure of the state matrix allows the computation of its exponential in *closed form*. (This is not necessarily true for general time-varying state equation matrices). Due to the iterative nature of the solution however, this approach has short term validity. The applicability of the Baker-Campbell-Hausdorff formula -- appearing in the theory of infinitesimal generators of one-parameter subgroups of Lie groups -- is discussed as an approach to overcome this limitation and to extend the validity of the approximation over longer time intervals.

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Finally, an old but relatively unknown method due to Darboux, is used to reformulate the attitude problem as the solution of a single but complex Riccati equation that governs the attitude of a rotating body in space. Although equations of this form are often encountered in classical differential geometry to describe the orientation of a moving trihedral along a rectifiable curve (in terms of the Frenet formulas and direction cosines) nevertheless, its use in the rotating rigid body problem, with a few notable exceptions, has been extremely limited. Moreover, no solutions of this equation have been reported in the literature, as far as the authors know.

Parametrizations of the Rotation Group

The set of matrices that relate two arbitrary reference frames form what is commonly known as the (three-dimensional) rotation group. This group consists of all the matrices that are orthogonal and have determinant +1. This group is also known as the (three-dimensional) *special orthogonal group* and it is commonly denoted by SO(3). In mathematical language we therefore write that $SO(3) = \{M \in GL(3) : M^T M = I$ and $\det(M)=+1\}$, where GL(3) is the group of all nonsingular 3x3 matrices. In this section we will concentrate on rotation matrices that describe the orientation of the body-fixed reference frame described by the unit vectors $\{\hat{b}\} \triangleq \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$ with respect to the inertial reference frame described by the unit vectors $\{\hat{n}\} \triangleq \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$, that is, for some $R \in SO(3)$ we have that $\{\hat{b}\} = R\,\{\hat{n}\}$. The matrix R therefore describes the relative orientation between the reference frames $\{\hat{b}\}$ and $\{\hat{n}\}$ and is varying with time since it depends on the angular velocity vector between the two reference frames. Each possible orientation corresponds to an element of the rotation group SO(3), which we may view as a configuration space for all non-trivial rotational motions of the body. Henceforth, we will refer to SO(3) simply as the rotation group. In fact, it is well known[2] that SO(3) is more than simply a group, but carries an inherent smooth manifold structure, and thus, forms a (continuous) Lie group. We will not exploit the Lie group structure of the rotation group until later on, when we discuss the applicability of the Baker-Campbell-Hausdorff formula for approximating elements of a Lie group by the exponential map.

There is more than one way to parametrize the rotation group, i.e., to specify a set of parameters such that an element in SO(3) is uniquely and unambiguously determined. Different parametrizations of the rotation group correspond to the well-known alternatives of solving for the relative attitude history between two reference frames: direction cosines, Euler parameters, Eulerian angles, etc. Although Hopf showed that five is the minimum number of parameters which suffices to represent the rotation group in a 1-1 global manner,[3] the so-called "quaternion method" (to be discussed later in this section) of parametrizing the group in a 1-2 way, using four parameters, is sufficient for practical purposes. This 4-dimensional parametrization is the lowest order singularity-free parametrization of SO(3). It is well known that the commonly used 3-dimensional parametrization of the Eulerian angles leads to singular points for the rotation group, that is, equations that exhibit singularities for certain orientations. Nevertheless, the use of Eulerian angles has survived until today, mainly because they represent physical quantities that are amenable to engineering insight. That is, the Euler angles themselves provide a useful output, whereas with the quaternion method it is necessary to transform the solution to the rotation group after integrating. In this research we are interested in solving the kinematic equations associated with the 3-dimensional Eulerian angle, and the 4-dimensional quaternion parametrizations of the rotation group SO(3). For a concrete exposition on the complete parametrization of SO(3) one may consult Stuelpnagel.[4]

Transformation Techniques for Eulerian Angles

For many problems involving rotating rigid bodies (e.g. spin-stabilized satellites in space), one often makes the assumption that the body spin-axis does not deviate much from its original direction. In such cases, and for an appropriately chosen set of three Euler angles, to be defined shortly, one can simplify the kinematic equations relating the three Eulerian angles with the components of the angular velocity vector. One thus obtains a simplified system of differential equations that can be used for analytic studies.[5–6] For example, if one wants to analyze the motion of a spin-stabilized body about its z-axis, then for a 3-1-2 Eulerian angle sequence, the angles β_x and β_y describe the attitude deviation of the spin axis from its initial orientation (assumed to be the inertial Z-axis). These angles represent unwanted deviation of the spin axis caused by application of disturbances and are typically small (see Fig. 1). In fact, the complex angle $\beta \triangleq \beta_x + i\,\beta_y$ represents a measure of the total deviation of the spin axis and is often referred to as the attitude "error angle." According to the previous discussion, a small angle assumption for β_x and β_y is quite reasonable for this particular problem and, thus, can be used to simplify the equations.

Recall that there are 12 different sets of angles that can be used to describe the orientation of a rigid body. Not all of the choices are equivalent for analytic representations of solutions, and the choice of the particular set of angles should be decided according to the relevance to the problem at hand.

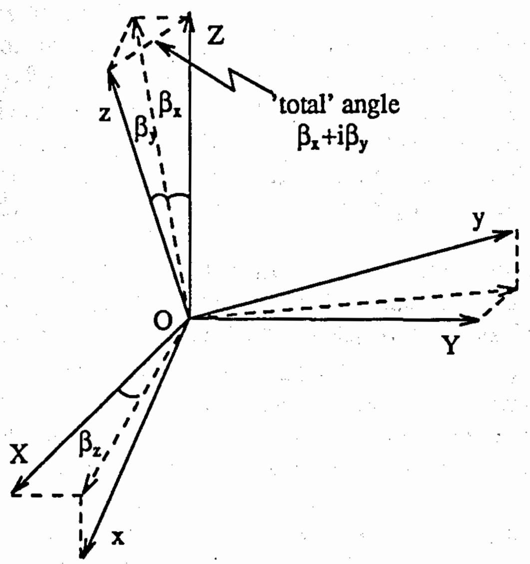

Fig. 1 Euler sequence 3-1-2 and attitude error components

For a spin-stabilized vehicle for instance, the 3-1-2 system is different from the 3-1-3 system of Eulerian angles in the sense that the first set is composed of two small and one large rotation, whereas the second set is composed of two large rotations and a small one (Fig. 2).

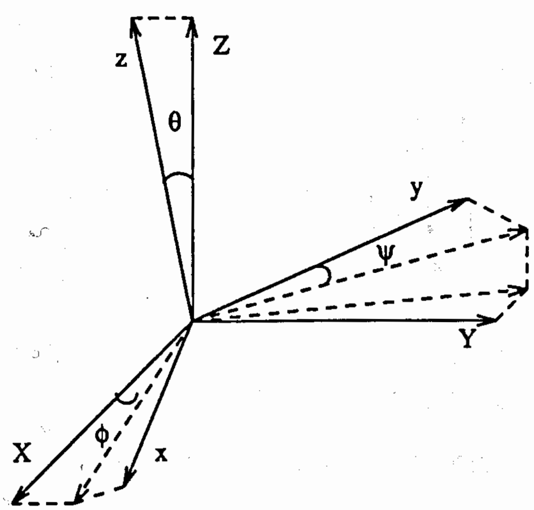

Fig. 2 Euler sequence 3-1-3

Therefore, if one approximates the true motion by linearization,[1,5-6] as is often the case, the resulting equations derived are far more simple using the 3-1-2 system than using the 3-1-3 system for this particular problem. This does not imply however that a description of the kinematic equations by an alternative set of Eulerian angles is fruitless. In fact, as we will show, one can use the interplay between different sets of Euler angles to achieve increased accuracy of the linearized solutions, or even better, (when possible) to derive directly (probably approximate) analytic solutions of the exact equations, that will remain valid for a large number of applications.

Kinematic equations

The kinematic equations for a 3-1-2 Eulerian angle sequence that relate the Eulerian angles and their rates to the components of the angular velocity vector, expressed in a body-fixed frame, are given by

$$\dot{\beta}_x = \omega_x \cos\beta_y + \omega_z \sin\beta_y \tag{1a}$$
$$\dot{\beta}_y = \omega_y - (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\tan\beta_x \tag{1b}$$
$$\dot{\beta}_z = (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\sec\beta_x \tag{1c}$$

Any attempt to solve these equations directly, for arbitrary ω_x, ω_y and ω_z is futile. It is clear however from (1) that β_z is an ignorable variable. The decoupling of β_z from β_x and β_y means that if one knows the solution for the latter two, one can immediately compute the solution for the former in terms of a simple quadrature by

$$\beta_z(t) = \int_0^t \{[\omega_z(\tau)\cos(\beta_y(\tau)) - \omega_x(\tau)\sin(\beta_y(\tau))] \times \sec(\beta_x(\tau))\}\, d\tau \tag{2}$$

Therefore, one can merely concentrate on solving for the Eulerian angles β_x and β_y from equations (1a) and (1b), keeping in mind that the solution of β_z can be obtained then by equation (2). According to the previous discussion, a small angle approximation for β_x, β_y is quite reasonable if equations (1) describe the attitude evolution of a spin-stabilized (about its z-axis) rigid body, and therefore, together with the assumption that the product $\beta_y\omega_x$ in Eq. (1c) is small compared to ω_z (as is usually the case for spin-stabilized bodies), the system of equations (1) reduces to

$$\dot{\hat{\beta}}_x = \omega_x + \hat{\beta}_y \omega_z \tag{3a}$$
$$\dot{\hat{\beta}}_y = \omega_y - \hat{\beta}_x \omega_z \tag{3b}$$
$$\dot{\hat{\beta}}_z = \omega_z \tag{3c}$$

The caret denotes the solution to the linear problem (3), in order to distinguish from the exact solution given by the system of equations (1). Again, because of the decoupling of $\hat{\beta}_z$ from $\hat{\beta}_x$ and $\hat{\beta}_y$, one can merely concentrate on solving (3a) and (3b). Using the complex notation introduced by Tsiotras and Longuski,[1] one writes these two equations in the following single complex scalar equation for the transverse Eulerian angles

$$\dot{\hat{\beta}} + i\,\omega_z\,\hat{\beta} = \omega \tag{4}$$

where $\hat{\beta} \triangleq \hat{\beta}_x + i\,\hat{\beta}_y$ and $\omega \triangleq \omega_x + i\,\omega_y$. Notice that (4) is a linear differential equation, the solution of which can be written immediately in terms of a quadrature. The error between the linearized and original equations $\beta_e(t) \triangleq \beta(t) - \hat{\beta}(t)$ will be of course relatively small, as long as the angles β_x and β_y remain within the realm of the linear approximation. As mentioned earlier, this error is surely small for the case of a spin-stabilized body, thus justifying its terminology. It is not necessarily so, however, when for some reason the stabilizing effect of the axial torque ceases to exist (during a spin-down maneuver, for example), and as a result the body z-axis tends to depart from its initial orientation, giving rise to large values for the angles $\hat{\beta}_x$ and $\hat{\beta}_y$. The problem has entered the region of nonlinearity, as is vividly demonstrated in Figs. 3, and a more comprehensive method is needed to solve for the true attitude motion of the body. Figures 3 show the results of a spin-down maneuver through zero spin rate, in the presence of transverse constant body fixed torques, for a typical spacecraft.[1] As a first step to alleviate this problem, notice that $\hat{\beta}(t)$ denotes the solution to the linearized equations (3), or equivalently (4), whereas what we actually want is the solution $\beta(t)$ of the original equations (1). Of course there is no way we can recover the exact solution from the linearized one, but one can improve the accuracy of the linearized solution by just *reinterpreting* this solution.

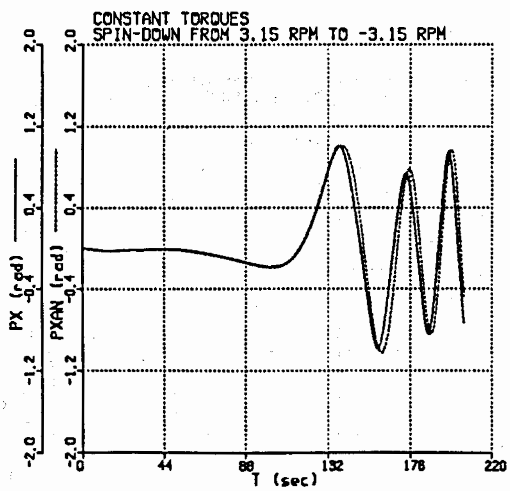

Fig. 3a Exact and analytic solution for β_x

The method of doing that involves an intermediate transformation to another set of Eulerian angles, and will be presented next.

For reasons that will become clear later, the appropriate new set of Eulerian angles, in order to perform the transformation for our problem, is the 3-1-3 set which obeys the following set of differential equations:

$$\omega_x = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{5a}$$

$$\omega_y = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{5b}$$

$$\omega_z = \dot{\psi} + \dot{\phi}\cos\theta \tag{5c}$$

From (5a) and (5b) we get that

$$\omega = (\dot{\theta} + i\,\dot{\phi}\sin\theta)\,e^{-i\psi} \tag{6}$$

where, as before, $\omega = \omega_x + i\,\omega_y$. Eliminating $\dot{\phi}$ from (6) with the help of (5c) and assuming that θ is small, so that we can set $\theta \approx \tan\theta$, the previous equation reduces to

$$\omega = [\dot{\theta} + i\,(\omega_z - \dot{\psi})\,\theta]\,e^{-i\psi} \tag{7}$$

If we now define the complex quantity

$$\alpha \triangleq \theta\,e^{-i\psi} \tag{8}$$

it is not difficult to verify that α obeys the following differential equation

$$\dot{\alpha} + i\,\omega_z\,\alpha = \omega \tag{9}$$

Comparing (4) and (9), one immediately sees that the following equation holds

$$\theta\,e^{-i\psi} = \hat{\beta}_x + i\,\hat{\beta}_y \tag{10}$$

where $\hat{\beta}_x(t)$ and $\hat{\beta}_y(t)$ are known from the solution of equation (4).

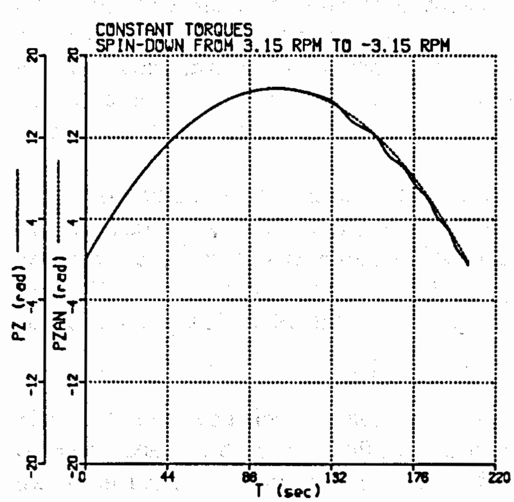

Fig. 3b Exact and analytic solution for β_z

Equation (10) is the basic relationship between the two sets of Eulerian angles that allows us to reinterpret the linearized solutions $\hat{\beta}_x$ and $\hat{\beta}_y$. This is done as follows. Recall that the parametrization of the rotation group for the two sets of Eulerian angles gives the following expressions for a typical element of SO(3):

$$\begin{bmatrix} -s\beta_z s\beta_x s\beta_y + c\beta_z c\beta_y & c\beta_z s\beta_x s\beta_y + s\beta_z c\beta_y & -c\beta_x s\beta_y \\ -s\beta_z c\beta_x & c\beta_z c\beta_x & s\beta_x \\ s\beta_z s\beta_x c\beta_y + c\beta_z s\beta_y & -c\beta_z s\beta_x c\beta_y + s\beta_z s\beta_y & c\beta_x c\beta_y \end{bmatrix} \tag{11a}$$

and

$$\begin{bmatrix} c\psi\, c\phi - s\phi\, c\theta s\psi & c\psi\, s\phi + c\phi\, c\theta\, s\psi & s\theta\, s\psi \\ -s\psi\, c\phi - s\phi\, c\theta\, c\psi & -s\psi\, s\phi + c\phi\, c\theta\, c\psi & s\theta\, c\psi \\ s\phi\, s\theta & -c\phi\, s\theta & c\theta \end{bmatrix} \tag{11b}$$

where c denotes cos and s denotes sin, respectively. Since (11a) and (11b) are different parametrizations of the same element of SO(3) we have, by comparing corresponding entries of the matrices, that one must have

$$\sin\beta_x = \sin\theta \cos\psi \qquad \tan\beta_y = -\sin\psi \tan\theta$$

$$\tan\beta_z = \frac{\sin\phi \cos\theta \cos\psi + \sin\psi \cos\phi}{\cos\phi \cos\theta \cos\psi - \sin\psi \sin\phi} \tag{12}$$

The previous equations are the standard relationships that provide the exact transformations between the 3-1-2 and 3-1-3 sets of Eulerian angles.[7] Using now the fact that $\hat{\beta}_x = \theta \cos\psi$ and $\hat{\beta}_y = -\theta \sin\psi$ from (10), we get the relationship between the computed quantities $\hat{\beta}_x$ and $\hat{\beta}_y$ and the corresponding required exact quantities β_x and β_y. The third of equations (12) especially captures the behavior of β_z; see Fig. 4 and compare with Fig. 3b.

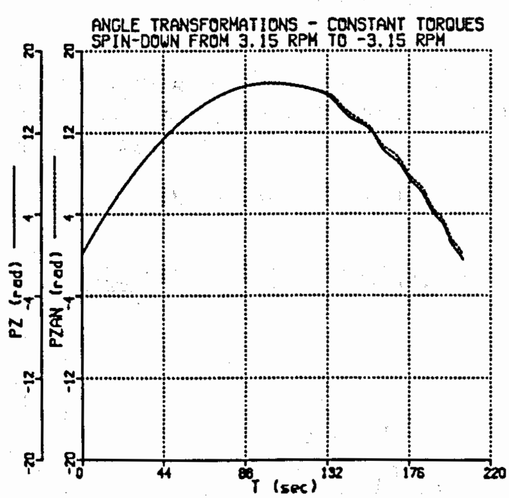

Fig. 4 Exact and analytic solution, β_z

The previously described methodology can be used to obtain improved solutions for the linearized equations, and as such, will be valid for moderately large angles only. This limitation stems from the assumption $\theta \approx \tan\theta$ made in deriving the transformation (10). If one drops this restriction, and one lets

$$\alpha \triangleq \tan\theta\, e^{-i\psi} \tag{13}$$

then by tracing the same steps as before, one readily finds that α satisfies the following differential equation

$$\dot{\alpha} + i\,\omega_z\,\alpha = \omega + \mathrm{Re}(\bar{\omega}\alpha)\,\alpha \tag{14}$$

where the bar denotes complex conjugate. This equation is a nonlinear differential equation for α which, in contrast to (9), is *exact*. That is, the solution of (14) along with (13) and (12) give the exact solution to the differential equations (1a) and (1b). In some sense we have traded the two nonlinear differential equations (1a) and (1b) for the scalar, but complex, differential equation (14), which is probably as difficult to solve as the original equations. However, (14) has a more suitable form for analytical studies. To see this, by letting $\alpha \triangleq \alpha_x + i\,\alpha_y$ one gets the following differential equations for the real and imaginary parts of (14)

$$\dot{\alpha}_x = \omega_z\alpha_y + \omega_x + \omega_x\alpha_x^2 + \omega_y\alpha_x\alpha_y \tag{15a}$$
$$\dot{\alpha}_y = -\omega_z\alpha_x + \omega_y + \omega_y\alpha_y^2 + \omega_x\alpha_x\alpha_y \tag{15b}$$

If we drop the nonlinear terms in the equations we get a system of equations that is basically Eq. (9). At first glance, it seems that no great improvement has been achieved by transforming to the new set of differential equations (15). However this is not so, because the system of equations (15) contains (up to quadratic) polynomial nonlinearities, whereas the original system of equations (1) contains trigonometric nonlinearities. As such, the system of differential equations (15), or equivalently (14), is suited for analytic treatments using series expansions, whereas the original system of equations, in terms of the Eulerian angles, is not directly amenable to such techniques.

Note that using the natural identification between α_x, α_y and θ, ψ from (13), and the identification between the two parametrizations of the rotation group from (11), one establishes the following relations between α_x, α_y and the Eulerian angles β_x and β_y.

$$\tan\beta_y = \alpha_y \qquad \tan\beta_x = \alpha_x \cos\beta_y \tag{16}$$

The last two equations along with the equations $\alpha_x = \tan\theta \cos\psi$ and $\alpha_y = -\tan\theta \sin\psi$ can be used to transform back and forth between the different sets of parameters.

Method of Solution

We briefly discuss now a procedure that will allow for approximate solutions of the nonlinear differential equation (14). Equation (9) is the linear part of equation (14), and its solution has already been established. Let the linearized solution of equation (14), or equivalently the solution of Eq. (9), be denoted by $\alpha_0 \triangleq \alpha_{x0} + i\,\alpha_{y0}$, that is, let

$$\dot{\alpha}_0 + i\,\omega_z\,\alpha_0 = \omega \tag{17}$$

or equivalently,

$$\dot{\alpha}_{x0} = \omega_z \alpha_{y0} + \omega_x \qquad (18a)$$
$$\dot{\alpha}_{y0} = -\omega_z \alpha_{x0} + \omega_y \qquad (18b)$$

This is the zero-order (linear) approximation to the exact solution of (14). A solution to this equation is given by

$$\alpha_0(t) = \alpha_0(0) \exp[-i \int_0^t \omega_z(u)\, du] + \exp[-i \int_0^t \omega_z(u)\, du] \int_0^t \omega(u) \exp[i \int_0^u \omega_z(v)\, dv]\, du \qquad (19)$$

Using this solution for $\alpha_0(t)$, one can then obtain the first-order approximation via the zero-order solution by solving

$$\dot{\alpha} + i\, \omega_z\, \alpha = \omega + \mathrm{Re}(\bar{\omega}\alpha_0)\, \alpha_0 \qquad (20)$$

or equivalently,

$$\dot{\alpha}_x = \omega_z \alpha_y + \omega_x + \omega_x \alpha_{x0}^2 + \omega_y \alpha_{x0} \alpha_{y0} \qquad (21a)$$
$$\dot{\alpha}_y = -\omega_z \alpha_x + \omega_y + \omega_y \alpha_{y0}^2 + \omega_x \alpha_{x0} \alpha_{y0} \qquad (21b)$$

Equation (20) is a linear differential equation that can be solved in terms of quadratures. Consulting (19) and (20), one in fact has that the first-order approximation to the solution of (14) is given by

$$\alpha(t) = \alpha(0) \exp[-i \int_0^t \omega_z(u)\, du] + \exp[-i \int_0^t \omega_z(u)\, du] \int_0^t \tilde{\omega}(u) \exp[i \int_0^u \omega_z(v)\, dv]\, du \qquad (22)$$

where $\tilde{\omega} \triangleq \omega + \mathrm{Re}(\bar{\omega}\alpha_0)\alpha_0$. Similarly, one can solve the zero-order solution and substitute into the first-order solution as follows

$$\dot{\alpha} + i\, [\omega_z + i\, \mathrm{Re}(\bar{\omega}\alpha_0)]\, \alpha = \omega \qquad (23)$$

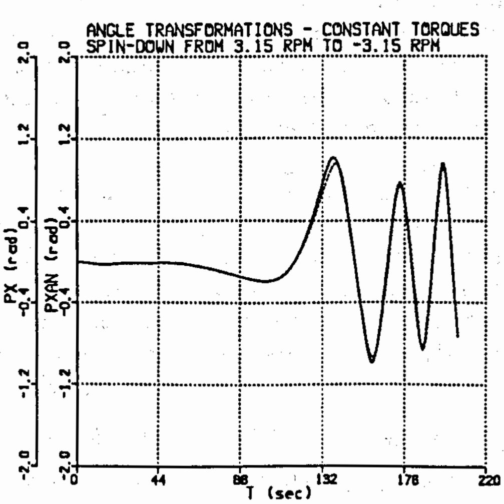

Fig. 5a Exact and analytic solution, β_x

That is, the first-order solution has the effect of altering the time-varying coefficient (frequency) of the zero-order (linear) equation, to the complex quantity $\tilde{\omega}_z \triangleq \omega_z + i\, \mathrm{Re}(\bar{\omega}\alpha_0)$. Preliminary simulations of the above two procedures indicate that these methods yield very accurate solutions of (14) and, in fact, capture the phase-shift error effect created by the zero-order (linearized) solution, a error which dictated the development of a large angle theory in the first place. Figures 5 show the results obtained for the solution for β_x using Eq. (20), for spin-down under constant torques; compare Fig. 5a with Fig. 3a.

Quaternion Formulation

As mentioned earlier, the parametrization of the rotation group with three Eulerian angles, in addition to the nonlinearity that it introduces in the kinematical equations, also has the disadvantage that it introduces singularities, i.e., points at which the parametrization is not defined. If one needs to avoid the singular points, one has to switch to another set of Eulerian angles. It is possible to circumvent this difficulty and introduce a parametrization that is globally valid, but this will imply, necessarily, the introduction of redundant parameters. The most often used global parametrization of the rotation group involves the introduction of *one* additional redundant parameter, and is called the *quaternion method*,[8] first introduced by Lord Hamilton. The parameters are then called quaternions and in fact, when the kinematic equations are expressed in terms of quaternions, consist of a system of *linear* (although time-varying) differential equations. The linear nature of the kinematic equations, when expressed in terms of quaternions is considered the most significant advantage of the 4-dimensional parametrization of SO(3), limited

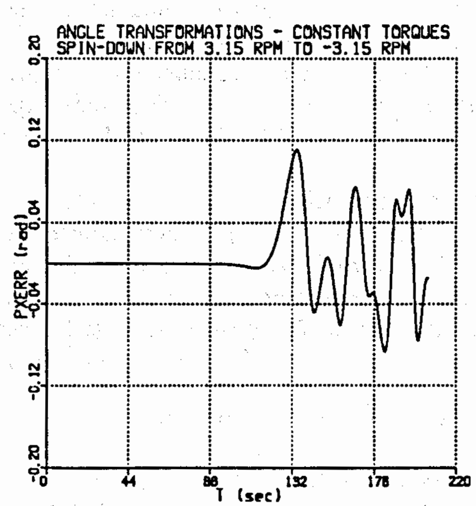

Fig. 5b Exact minus analytic solution, β_x

however by the fact that, in general, no explicit formula for the solution of a system of linear *time-varying* differential equations is known to exist. It is true of course, that the solution to a system of linear differential equations is given in terms of the fundamental (or state transition) matrix, but for the time-varying case, no general method exists for computing this matrix, and one often has to resort to numerical simulations.

Next, we will show how one can apply a method, initially due to Picard,[9] to approximate the solution to a linear, time-varying system of differential equations, as accurately as one desires, using the notion of the *product integral*. The methodology in essence seeks to approximate the state transition matrix and is *semi-analytic* in nature, since it is confined to small time steps. Picard's method can, in principle, be applied for all time-varying linear systems, but is especially convenient for the systems with the special form of skew-symmetric state matrices that appear in the kinematics of rotating bodies, since then the matrix exponentials can be computed in *closed form*.

Kinematic Equations

Recall that the vector of quaternions $q \triangleq q_0 + q_1\,\hat{i} + q_2\,\hat{j} + q_3\,\hat{k}$, evolves in time by the linear system of differential equations

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{24}$$

where q_0, q_1, q_2, q_3 are the *Euler parameters.* Although the kinematic equations in this form are linear, closed form solutions are extremely difficult to obtain, due to the time varying nature of the differential equations. Analytic solutions of (24) have been constructed for the special case of a torque-free rotating body.[10] Kane[11] has also obtained approximate solutions to (24) for an axisymmetric rigid body subject to body-fixed transverse torques of constant magnitude, employing an averaging technique. Similar approximate solutions have also been reported by Kane and Levinson.[12] As often occurs in practice, rotating rigid bodies have an axis of symmetry, which is also usually the spin axis. If this is the case, then it is advantageous to introduce, in place of the quaternions, the parameters

$$\rho \triangleq q_0 + i\,q_3 \qquad \text{and} \qquad \sigma \triangleq q_1 + i\,q_2 \tag{25}$$

because then, using (25), equation (24) can be reduced to the compact form

$$\begin{bmatrix} \dot{\rho} \\ \dot{\sigma} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} i\,\omega_z & -\bar{\omega} \\ \omega & -i\,\omega_z \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \end{bmatrix} \tag{26}$$

The reason we prefer to work with (26) rather than with (24) is that for the case of spinning (usually near-symmetric) rigid bodies, one often takes advantage of the special symmetry (or even better, skew-symmetry) of the equations to obtain the solution of the transverse components of the angular velocity ω_x and ω_y in the compact complex form of $\omega = \omega_x + i\,\omega_y$. Van der Ha[13] attempted to obtain approximate solutions to equations of the form (26) using a perturbation scheme. Perturbation methods have also been used to obtain approximate solutions to the original form of the equations (24) by Kraige and Junkins.[14]

Notice that equation (26) is of the form

$$\dot{\xi} = A(t)\,\xi(t) \tag{27}$$

which is a linear, time-varying differential equation in vector format. The solution of the previous differential equation is given by

$$\xi(t) = \Phi(t,0)\,\xi(0) \tag{28}$$

where $\Phi(t,0)$ is the *product integral* (state transition matrix) satisfying the matrix differential equation

$$\dot{\Phi}(t,0) = A(t)\,\Phi(t,0)\,, \qquad \Phi(0,0) \triangleq I \tag{29}$$

Following Nelson,[9] the solution of the previous equation is approximated by

$$\Phi(t,0) = \exp(A_n\Delta t_n)\cdots\exp(A_1\Delta t_1)$$

$$A_j = A(t) \quad \text{for} \quad t_{j-1} < t < t_j \tag{30}$$

$$\text{and} \quad \Delta t_j = t_j - t_{j-1} \qquad 0 = t_0 < t_1 < \cdots < t_n = t$$

Notice that in (30) operators with the smallest value of the time parameter operate first. This is very important, because commutativity does not hold in general between matrix exponentials. For $\Delta t_j \to 0$ $(j = 1,2,\ldots,n)$, expression (30) gives the exact solution to the differential equation for $\Phi(t,0)$.

The closed-form calculation of the matrix exponential $\exp[A(t)]$ for a time-varying $A(t)$ is, in general, a formidable task. However, for the special structure of the matrices that appear in equation (24), or equivalently in equation (26), one can immediately verify[11] that

$$\exp(A) = I\cos(w) + \frac{A}{w}\sin(w) \tag{31}$$

where $w^2 \triangleq \det(A) = \omega\,\bar{\omega} + \omega_z^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ and I is the 2x2 identity matrix. This formula holds for all skew-hermitian matrices A, for which $A^2 = -\det(A)\,I$. One can verify easily this property for the matrix in (26). Use of the formula (31) allows the (approximate) evaluation of the exponentials in the equation (30) for $\Phi(t,0)$. However, an accurate calculation for $\Phi(t,0)$ will require very small time intervals Δt_j. We can circumvent this difficulty, and extend the solution to larger time steps, but we require first some results from the theory of Lie groups and their associated Lie algebras.

Generalized Baker–Campbell–Hausdorff formula

Because of the special structure of the state matrix A in (27), it is known that the state transition matrix $\Phi(t,0)$ is a unitary matrix, and as such, it is given by the exponential of some skew-hermitian matrix $W(t)$, i.e., $\Phi(t,0) = \exp[W(t)]$ for all t. We want to find the matrix $W(t)$, starting from Eq. (30), namely, to combine the product of exponentials into a single exponential (that of the matrix W). Recall that if X,Y are nxn matrices, then $\exp(X)\exp(Y) \neq \exp(X+Y)$, in general, unless XY=YX, i.e., unless the matrices X and Y commute. However, the following result from the Lie group theory[15] states that if X and Y are sufficiently near the zero matrix, there exists a matrix Z in the Lie algebra generated by {X,Y} that satisfies

$$e^X e^Y = e^Z \tag{32a}$$

Specifically, Z is given by the expansion

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[[X,Y],Y] + \cdots \tag{32b}$$

where [.,.] denotes the Lie bracket (commutator), defined by $[X,Y] \triangleq XY-YX$. Equation (32) is called the Baker-Campbell-Hausdorff formula. Applying this formula to Eq. (30), starting from the left, and keeping terms only up to $O(\Delta t^2)$, we get that

$$\begin{aligned}\Phi(t,0) &= \exp(A_n\Delta t_n)\cdots\exp(A_2\Delta t_2)\exp(A_1\Delta t_1)\\ &= \exp(A_n\Delta t_n)\cdots\exp(A_3\Delta t_3)\exp(\hat{A}_2)\end{aligned} \tag{33a}$$

where

$$\hat{A}_2 = A_2\Delta t_2 + A_1\Delta t_1 + \frac{1}{2}[A_2\Delta t_2, A_1\Delta t_1] + O(\Delta t^3) \tag{33b}$$

The next application of the BCH formula to the exponentials $\exp(A_3\Delta t_3)\exp(\hat{A}_2)$, keeping again only terms up to $O(\Delta t^2)$, gives

$$\Phi(t,0) = \exp(A_n\Delta t_n)\cdots\exp(A_4\Delta t_4)\exp(\hat{A}_3) \tag{34a}$$

where

$$\hat{A}_3 = A_3\Delta t_3 + A_2\Delta t_2 + A_1\Delta t_1 + \frac{1}{2}[A_2\Delta t_2, A_1\Delta t_1] + \frac{1}{2}[A_3\Delta t_3, A_2\Delta t_2 + A_1\Delta t_1] + O(\Delta t^3) \tag{34b}$$

Continuing the same way, one obtains the following approximation of $\Phi(t,0)$ to order $O(\Delta t^3)$:

$$\Phi(t,0) = \exp\{\sum_{j=1}^{n} A_j\Delta t_j + \frac{1}{2}\sum_{j=1}^{n}[A_j\Delta t_j, \sum_{i=1}^{n-j} A_i\Delta t_i] + O(\Delta t^3)\} \tag{35}$$

Taking limits for $n \to \infty$, or $\Delta t \to 0$, one easily gets from the Riemann sums of (35) that

$$\Phi(t,0) = \exp[W(t)]$$
$$W(t) \triangleq \int_0^t A(\tau)d\tau + \frac{1}{2}\int_0^t [A(\tau)d\tau, \int_0^\tau A(\sigma)d\sigma] + \cdots \tag{36}$$

Equation (36) gives the expression for the state transition matrix $\Phi(t,0)$ required for the solution of (27). It can be easily verified that the matrix W is skew-hermitian with $W^2 = -\det(W)\,I$ so that it has the form required, in order to compute its exponential from equation (31), for all t. The calculation of $W(t)$ from (36) can be performed easily, by direct integrations. From (26)

$$\int_0^t A(\tau)\,d\tau = \frac{1}{2}\begin{bmatrix} i\int_0^t \omega_z(\tau)\,d\tau & -\int_0^t \bar{\omega}(\tau)\,d\tau \\ \int_0^t \omega(\tau)\,d\tau & -i\int_0^t \omega_z(\tau)\,d\tau \end{bmatrix} \tag{37}$$

Because of the skew-hermitian structure of the matrix A, one needs to calculate only two of the above integrals. The second term of $W(t)$ requires the evaluation of

$$\hat{A}(t) \triangleq \int_0^t [A(\tau), \int_0^\tau A(\sigma)\,d\sigma]\,d\tau \tag{38}$$

Carrying out the algebra, it can be immediately shown that $[A(\tau), \int_0^\tau A(\sigma)d\sigma]$ takes the form

$$\begin{bmatrix} \omega\int_0^\tau\bar{\omega}d\tau - \bar{\omega}\int_0^\tau\omega d\tau & 2i[\bar{\omega}\int_0^\tau\omega_z d\tau - \omega_z\int_0^\tau\bar{\omega}d\tau \\ 2i[\omega\int_0^\tau\omega_z d\tau - \omega_z\int_0^\tau\omega d\tau] & \bar{\omega}\int_0^\tau\omega d\tau - \omega\int_0^\tau\bar{\omega}d\tau] \end{bmatrix} \tag{39}$$

Again, because of the special skew-hermitian structure of the matrix $\hat{A}$, we need to evaluate the integrals of only two of the entries of $\hat{A}$, say

$$\hat{A}_{11}(t) = 2\,i\int_0^t \mathrm{Im}[\omega(t)\int_0^\tau \bar{\omega}(\sigma)d\sigma]d\tau \tag{40a}$$

$$\hat{A}_{21}(t) = 2\,i\int_0^t [\omega(\tau)\int_0^\tau \omega_z(\sigma)d\sigma - \omega_z(\tau)\int_0^\tau \omega(\sigma)d\sigma]d\tau \tag{40b}$$

Of course, the calculation of the integrals of these quantities becomes very involved. For simple enough expressions for the angular velocities ω_x, ω_y, and ω_z, symbolic language manipulation routines can be used to alleviate the effort.

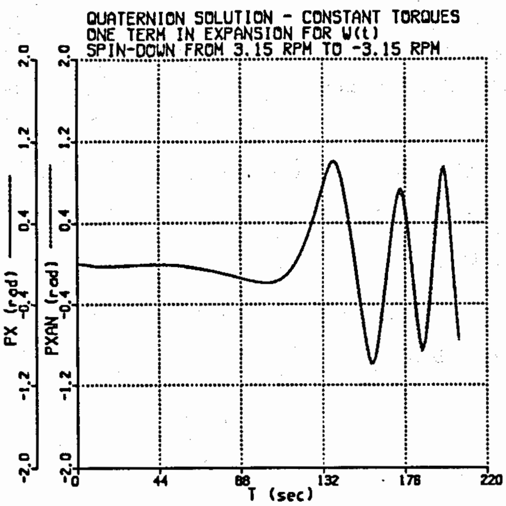

Fig. 6a Exact and analytic solution, β_x

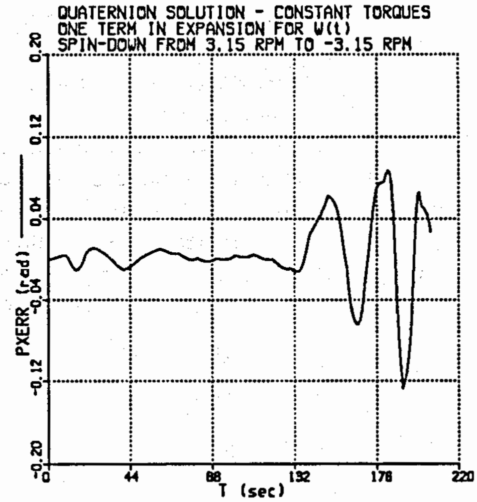

Fig. 6b Exact minus analytic solution, β_x

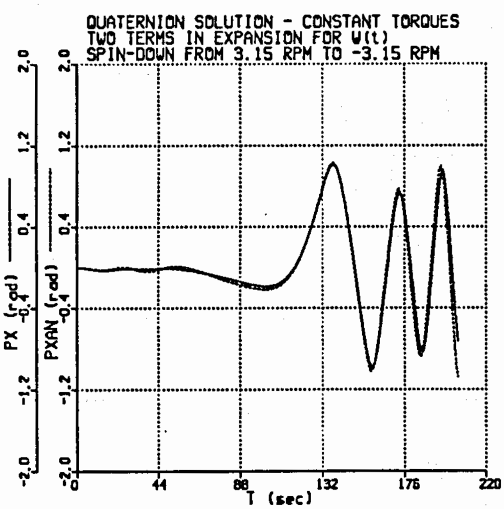

Fig. 7a Exact and analytic solution, β_x

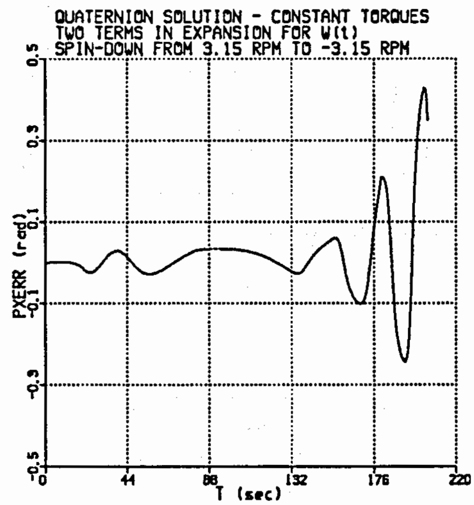

Fig. 7b Exact minus analytic solution, β_x

The BCH Theorem states that (32) holds for some matrices X and Y "close" to the zero matrix. That is, X and Y should be "small" with respect to a norm $\|.\|$ that is *compatible* with the operation of the Lie bracket, i.e., a norm such that $\|[X,Y]\| \le \|X\|\,\|Y\|$. Thus, the expansion (32b) is only locally convergent, so (32a) can be used to determine the existence of Z in the Lie algebra generated by X and Y when the norm of [X,Y] is sufficiently small. The applicability of the BCH formula is thus limited inside a ball of unit radius (with respect to a compatible norm). This local convergence of the BCH formula restricts the validity of (36) to the neighborhood of $t=0 \triangleq T_0$. One can circumvent this problem, by redefining the initial condition in regular time intervals as follows: Choose a time T_1 such that the series in (36) converges. Then the solution is given by

$$\xi(t)=\Phi(t,T_0)\,\xi(T_0), \qquad \text{for} \qquad T_0 \le t < T_1 \qquad (41)$$

Then choose a time T_2 such that the series expansion starting from T_1 converges. Then the solution is given by

$$\xi(t)=\Phi(t,T_1)\,\xi(T_1),$$

$$T_1 \le t < T_2, \qquad \text{and} \qquad \xi(T_1)=\Phi(T_1,T_0)\,\xi(T_0) \qquad (42)$$

In practice one usually chooses $T_{j+1}-T_j=T$, $j=0, 1, \cdots n-1$. Redefining thus the initial condition every T seconds, one can keep the norm of the matrices small, keep the convergence of the BCH formula under control. The results for the solution of (26) using $\Phi(t,0)$ from (36) with only the first term, and with reinitialization every 10 seconds, are shown in Figs. 6. The time interval that one needs to reinitialize in order to keep the norm small can be extended by including more terms in the expression of W(t). The results for the solution of (26) using $\Phi(t,0)$ from (36) with the first two terms, and with reinitialization every 20 seconds, are shown in Figs. 7.

Once we know q_0, q_1, q_2, q_3, the Eulerian angles $\beta_x, \beta_y, \beta_z$ are given by comparing (11a) with the corresponding typical element of SO(3), when expressed in terms of the Euler parameters. For such a parametrization[7] we have

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix} \quad (43)$$

One easily sees then that $\beta_x, \beta_y, \beta_z$ are given by

$$\sin\beta_x = 2(q_2q_3+q_0q_1) \qquad \tan\beta_y = -\frac{2(q_1q_3-q_0q_2)}{q_0^2-q_1^2-q_2^2+q_3^2}$$
$$\tan\beta_z = -\frac{2\,(q_1q_2-q_0q_3)}{q_0^2-q_1^2+q_2^2-q_3^2} \quad (44)$$

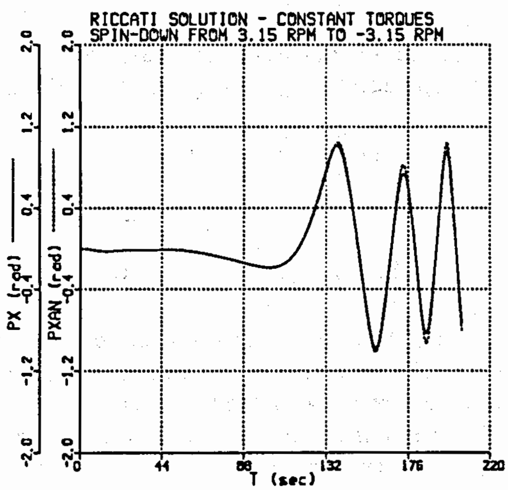

Fig. 8a Exact and analytic solution, β_x.

Direction Cosines

Each element of the rotation group SO(3) describes the orientation of two given sets of mutually orthogonal unit vectors (frames), the first of which is attached and moving with the rotating body, while the other one is constant. Both frames coincide at time zero. The attitude history of the moving reference frame with respect to the constant (inertial) reference frame can then be described by a curve traced by the corresponding rotation R(t) matrix in SO(3). The differential equation satisfied while R(t) is moving along this trajectory is given by

$$\dot{R}(t) = \tilde{\Omega}(t)\, R(t) \quad (45)$$

where

$$\tilde{\Omega} \triangleq \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \quad (46)$$

This matrix differential equation involves nine parameters (the direction cosines of the corresponding frames), however because of the constraint $RR^T = I$ imposed on the elements of SO(3), there are actually only *three* free parameters involved in the system of equations (45). These three parameters actually provide another 3-dimensional parametrization of the rotation group. Now let $[a,b,c]^T$ denote a column vector of the matrix representation of R having entries r_{ij}, for i,j=1,2,3. That is, $[a,b,c]^T = [r_{1j}, r_{2j}, r_{3j}]^T$ for some j=1,2,3. Clearly,

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (47)$$

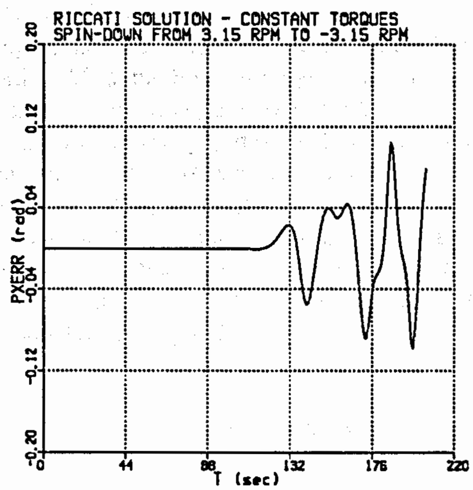

Fig. 8b Exact minus analytic solution, β_x

Because of the constraint $a^2+b^2+c^2 = 1$ we can eliminate one of the three parameters a, b, c, to get a system of *two* first order differential equations. The most natural and elegant way to reduce the third order system (47) to a second order system is by the use of *stereographic projection*. That is, if we let a, b, and c represent the coordinates on the unit sphere $S^1 = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, then, for $(a,b,c) \in S^1$, the stereographic projection $(a,b,c) \rightarrow w$, with $w \in \mathbb{C}$, is given by

$$w \triangleq \frac{b-ia}{1+c} = \frac{1-c}{b+ia} \quad (48)$$

Then in terms of the complex quantity w, the system of differential equations (48) can be combined to the single differential equation

$$\dot{w} + i\omega_z w = \frac{\omega}{2} + \frac{\overline{\omega}}{2} w^2 \quad (49)$$

The inverse transformation $w \rightarrow (a,b,c)$ is given by

$$a = \frac{i(w-\bar{w})}{|w|^2+1} \quad b = \frac{w+\bar{w}}{|w|^2+1} \quad c = -\frac{|w|^2-1}{|w|^2+1} \tag{50}$$

and can be used to find a, b, c once w is known. The real and imaginary parts of $w \triangleq w_1 + i\,w_2$ satisfy the differential equations

$$\dot{w}_1 = \omega_z w_2 + \omega_y w_1 w_2 + \frac{\omega_x}{2}(1+w_1^2-w_2^2) \tag{51a}$$

$$\dot{w}_2 = -\omega_z w_1 + \omega_x w_1 w_2 + \frac{\omega_y}{2}(1+w_2^2-w_1^2) \tag{51b}$$

Equation (49) is a Riccati equation with *time-varying* coefficients, the solution of which is very hard to establish. An approximate solution can be obtained however as follows. One can obtain the zero-order (linear) solution of (49) by solving the equation

$$\dot{w}_0 + i\omega_z w_0 = \frac{\omega}{2} \tag{52}$$

Then the first-order approximation of the solution of (49) can be obtained by solving the linear equation

$$\dot{w} + i\omega_z w = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w_0^2 \tag{53}$$

or the linear equation

$$\dot{w} + i(\omega_z + i\,\frac{\bar{\omega}}{2} w_0)\, w = \frac{\omega}{2} \tag{54}$$

Both of these equations are linear, and their solutions can be given in terms of quadratures. The difference between these two methods of solution lies in the fact that in Eq. (53) the zero-order solution acts in such a way as to change the forcing term, whereas in Eq. (54) it acts in such a way as to change the time-varying coefficient. Both equations retain the same form as the zero-order equation. This is easy to see by rewriting (53) and (54) in the form

$$\dot{w} + i\,\omega_z\, w = \frac{\tilde{\omega}}{2} \quad \text{with} \quad \tilde{\omega} \triangleq \omega + \bar{\omega}\, w_0^2 \tag{55}$$

and

$$\dot{w} + i\,\tilde{\omega}_z\, w = \frac{\omega}{2} \quad \text{with} \quad \tilde{\omega}_z \triangleq \omega_z + i\,\frac{\bar{\omega}}{2} w_0 \tag{56}$$

The results for the solution of (49), using equations (52) and (53), are shown in Figs. 8. Notice the resemblance of equations (49), (53) and (54) to the equations (14), (20) and (23) respectively. We note in passing that the solution to the linear equation (52), or the linear equation (17), is easy to establish. In fact, by simple comparison, one sees that the solutions to (52) and (17) are the same as the solution to (4), of the linearized 3-1-2 Eulerian angle problem which has been solved in Tsiotras and Longuski.[1]

Conclusions

Solutions for the kinematic equations of a rotating rigid body, that are *not* limited to small angular displacements, have been derived using three different approaches. A typical example of practical interest, for which such solutions have proved to be very useful, is the despinning of an (initially) spin-stabilized vehicle in the presence of large transverse torques. Two of the methodologies developed, one using a reformulation of the Eulerian angle kinematics and the other using direction cosines, will give very accurate results, but require that the linerized (zero-order) solution be reasonably good. In fact, as will be shown in a forthcoming paper these two methodologies are very similar, and the solutions obtained by equations (14) and (49) are related by a simple nonlinear transformation. The third method, using Euler parameters, does not require any such assumption, and therefore is globally valid. This method employs a generalization of the Baker-Campbell-Hausdorff formula, encountered in the theory of Lie groups, in order to find explicitly the state transition matrix as the exponential of a known matrix. As such, it is applicable to any linear-time varying system of differential equations, but its applicability is especially suited for rotational dynamics problems, because then one can evaluate the arising matrix exponentials in closed form.

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