Rigid Body Motion Tracking Without Linear and Angular Velocity Feedback Using Dual Quaternions*

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Abstract— **This paper takes advantage of a new, recently proposed representation of the combined translational and rotational dynamic equations of motion of a rigid body in terms of dual quaternions. We show that combined position and attitude tracking controllers based on dual quaternions can be developed with relatively low effort from existing attitude-only tracking controllers based on quaternions. We show this by developing an *almost* globally asymptotically stable nonlinear controller capable of simultaneously following time-varying position and attitude profiles without linear and angular velocity feedback based on an existing attitude-only tracking controller without angular velocity feedback.**

I. INTRODUCTION

Dual quaternions provide a compact way to represent not only the attitude but also the position of a rigid body. A list of successful applications of dual quaternions is given in [1]. They have been argued to be the most compact and efficient way to express the translation and rotation of robotic kinematic chains [2]. Moreover, combined position and attitude control laws based on dual quaternions automatically take into account the natural coupling between the rotational and translational motion [3], [4]. In addition, dual quaternions make it possible to write a single control law for both position and attitude. However, the most useful property of dual quaternions is the that the combined translational and rotational kinematic and dynamic equations of motion written in terms of dual quaternions have the same for as the rotational-only kinematic and dynamic equations of motion written in terms of quaternions [1]. In this paper, we take advantage of this property to extend an existing attitude-only tracking controller without angular velocity feedback [5] into a combined attitude and position tracking controller without linear and angular velocity feedback. This paper also extends the combined attitude and position setpoint controller without linear and angular velocity feedback presented in [1] to the tracking case.

Compared with techniques based on the Special Euclidean group $\mathrm{SE}(3)$ [6], [7], [8], where rotations are represented directly by rotation matrices, our technique does not require the definition of two separate error functions for position and attitude. Instead, we use a single error function, the *error dual quaternion* (defined by analogy to the classical rotation error quaternion) to represent both errors. Moreover, we prove the *almost* asymptotical stability of our combined position and attitude controller in one step by using a Lyapunov function with the same form as the Lyapunov function used to prove the *almost* asymptotical stability of the attitude-only controller. Hence, we do not need to divide the position and attitude control problem in two separate subproblems, as in [6]. Furthermore, whereas controllers based on quaternions produce two closed-loop equilibrium points [9] (both representing the identity rotation matrix), controllers based on rotation matrices produce a minimum of four closed-loop equilibrium points [7], [6], only one of which is the identity rotation matrix.

II. MATHEMATICAL PRELIMINARIES

A. Quaternions

A quaternion can be represented as an ordered pair $q = (\bar{q}, q_4)$, where $\bar{q} = [q_1\ q_2\ q_3]^\mathsf{T} \in \mathbb{R}^3$ is the *vector part* of the quaternion and $q_4 \in \mathbb{R}$ is the *scalar part*. Henceforth, quaternions with zero scalar part and with zero vector part will be referred to as *vector quaternions* and *scalar quaternions*, respectively. The set of quaternions, vector quaternions, and scalar quaternions will be denoted by $\mathbb{H} = \{q :\ q = (\bar{q}, q_4),\ \bar{q} \in \mathbb{R}^3,\ q_4 \in \mathbb{R}\}$, $\mathbb{H}^v = \{q \in \mathbb{H} :\ q_4 = 0\}$, and $\mathbb{H}^s = \{q \in \mathbb{H} :\ \bar{q} = \bar{0}\}$, respectively. The basic quaternion operations are given below:

Addition: $a + b = (\bar{a} + \bar{b}, a_4 + b_4)$,
Multiplication by a scalar: $\lambda a = (\lambda \bar{a}, \lambda a_4)$,
Multiplication: $ab = (a_4\bar{b} + b_4\bar{a} + \bar{a} \times \bar{b}, a_4 b_4 - \bar{a} \cdot \bar{b})$,
Conjugation: $a^* = (-\bar{a}, a_4)$,
Dot product: $a \cdot b = (\bar{0}, a_4 b_4 + \bar{a} \cdot \bar{b})$,
Cross product: $a \times b = (b_4\bar{a} + a_4\bar{b} + \bar{a} \times \bar{b}, 0)$,
Norm: $\|a\|^2 = aa^* = a^*a = a \cdot a = (\bar{0}, a_4^2 + \bar{a} \cdot \bar{a})$,
Scalar part: $\mathrm{sc}(a) = (\bar{0}, a_4) \in \mathbb{H}^s$,
Vector part: $\mathrm{vec}(a) = (\bar{a}, 0) \in \mathbb{H}^v$,

where $a, b \in \mathbb{H}$, $\lambda \in \mathbb{R}$, and $\bar{0} = [0\ 0\ 0]^\mathsf{T}$. Under the natural isomorphism between $\mathbb{H}^s$ and $\mathbb{R}$, we will often identify, with a slight abuse of notation, $(\bar{0}, q_4)$ with q_4. The multiplication of a 4-by-4 matrix with a quaternion will be defined as $M * q = (M_{11}\bar{q} + M_{12}q_4, M_{21}\bar{q} + M_{22}q_4) \in \mathbb{H}$, where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{4\times 4},$$

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$q \in \mathbb{H}$, $M_{11} \in \mathbb{R}^{3\times3}$, $M_{12} \in \mathbb{R}^{3\times1}$, $M_{21} \in \mathbb{R}^{1\times3}$, and $M_{22} \in \mathbb{R}$. It can also be easily shown that the following properties follow from the previous definitions: $a\cdot(bc) = b\cdot(ac^*) = c\cdot(b^*a)$, $\|ab\| = \|a\|\|b\|$, and $(M*a)\cdot b = a\cdot(M^\intercal * b)$, where $a, b, c \in \mathbb{H}$ and $M \in \mathbb{R}^{4\times4}$.

The relative orientation between the body frame and the inertial frame can be represented by the *unit* quaternion $q_{\text{B/I}} \in \mathbb{H}^u = \{q \in \mathbb{H} : \ q\cdot q = 1\}$. Then, the body coordinates of a vector, $\bar{v}^{\text{B}}$, can be calculated from its inertial coordinates, $\bar{v}^{\text{I}}$, and vice-versa, through $v^{\text{B}} = q_{\text{B/I}}^* v^{\text{I}} q_{\text{B/I}}$ and $v^{\text{I}} = q_{\text{B/I}} v^{\text{B}} q_{\text{B/I}}^*$, where $v^{\text{B}} = (\bar{v}^{\text{B}}, 0)$ and $v^{\text{I}} = (\bar{v}^{\text{I}}, 0)$. The rotational kinematic equations of the body frame and a desired frame, both represented with respect to the inertial frame by the unit quaternions $q_{\text{B/I}}$ and $q_{\text{D/I}}$, respectively, take the form: $\dot{q}_{\text{B/I}} = \frac{1}{2} q_{\text{B/I}} \omega_{\text{B/I}}^{\text{B}} = \frac{1}{2}\omega_{\text{B/I}}^{\text{I}} q_{\text{B/I}}$ and $\dot{q}_{\text{D/I}} = \frac{1}{2} q_{\text{D/I}} \omega_{\text{D/I}}^{\text{D}} = \frac{1}{2}\omega_{\text{D/I}}^{\text{I}} q_{\text{D/I}}$, where $\omega_{\text{Y/Z}}^{\text{X}} = (\bar{\omega}_{\text{Y/Z}}^{\text{X}}, 0)$, and $\bar{\omega}_{\text{Y/Z}}^{\text{X}}$ is the angular velocity of the Y-frame with respect to the Z-frame expressed in the X-frame. The error quaternion between $q_{\text{B/I}}$ and $q_{\text{D/I}}$ is the unit quaternion that describes the orientation of the body frame relative to the desired frame and is given by $q_{\text{B/D}} = q_{\text{D/I}}^* q_{\text{B/I}}$. By differentiating $q_{\text{B/D}}$, the error quaternion kinematic equation turns out to be

$$\dot{q}_{\text{B/D}} = \tfrac{1}{2} q_{\text{B/D}} \omega_{\text{B/D}}^{\text{B}} = \tfrac{1}{2}\omega_{\text{B/D}}^{\text{D}} q_{\text{B/D}}, \tag{1}$$

where $\omega_{\text{B/D}}^{\text{B}} = \omega_{\text{B/I}}^{\text{B}} - \omega_{\text{D/I}}^{\text{B}}$ (and $\omega_{\text{B/D}}^{\text{D}} = \omega_{\text{B/I}}^{\text{D}} - \omega_{\text{D/I}}^{\text{D}}$).

B. Dual Quaternions

A dual quaternion is defined as $\hat{q} = q_r + \epsilon q_d$, where $q_r, q_d \in \mathbb{H}$ are the *real* and *dual part* of the dual quaternion, respectively, and ϵ is the *dual unit* defined as $\epsilon^2 = 0$ and $\epsilon \neq 0$. Hereafter, dual quaternions with $q_r, q_d \in \mathbb{H}^v$ and with $q_r, q_d \in \mathbb{H}^s$ will be referred to as *dual vector quaternions* and *dual scalar quaternions*, respectively. The set of dual quaternions, dual scalar quaternions, and dual vector quaternions will be denoted by $\mathbb{H}_d = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d, \ q_r, q_d \in \mathbb{H}\}$, $\mathbb{H}_d^s = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d, \ q_r, q_d \in \mathbb{H}^s\}$, and $\mathbb{H}_d^v = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d, \ q_r, q_d \in \mathbb{H}^v\}$, respectively. Moreover, we will also denote the set of dual scalar quaternions with zero dual part as $\mathbb{H}_d^r = \{\hat{q} : \ \hat{q} = q_r + \epsilon(\bar{0}, 0), \ q_r \in \mathbb{H}^s\}$. The elementary operations on dual quaternions are given by [4], [10]:

Addition: $\hat{a} + \hat{b} = (a_r + b_r) + \epsilon(a_d + b_d)$,

Multiplication by a scalar: $\lambda\hat{a} = (\lambda a_r) + \epsilon(\lambda a_d)$,

Multiplication: $\hat{a}\hat{b} = (a_r b_r) + \epsilon(a_r b_d + a_d b_r)$,

Conjugation: $\hat{a}^* = a_r^* + \epsilon a_d^*$,

Swap: $\hat{a}^{\mathsf{s}} = a_d + \epsilon a_r$,

Dot product: $\hat{a}\cdot\hat{b} = a_r\cdot b_r + \epsilon(a_d\cdot b_r + a_r\cdot b_d) \in \mathbb{H}_d^s$,

Cross product: $\hat{a}\times\hat{b} = a_r\times b_r + \epsilon(a_d\times b_r + a_r\times b_d) \in \mathbb{H}_d^v$,

Dual norm: $\|\hat{a}\|_d^2 = (a_r\cdot a_r) + \epsilon(2a_r\cdot a_d) \in \mathbb{H}_d^s$,

Scalar part: $\text{sc}(\hat{a}) = \text{sc}(a_r) + \epsilon\,\text{sc}(a_d) \in \mathbb{H}_d^s$,

Vector part: $\text{vec}(\hat{a}) = \text{vec}(a_r) + \epsilon\,\text{vec}(a_d) \in \mathbb{H}_d^v$,

where $\hat{a}, \hat{b} \in \mathbb{H}_d$ and $\lambda \in \mathbb{R}$. Note that $\hat{a}\hat{b} \neq \hat{b}\hat{a}$, in general. Hereafter, we will use the following *dual quaternion norm* [10]: $\|\hat{a}\|^2 = \hat{a}\circ\hat{a}$, where $\circ$ is defined as the *dual quaternion circle product*, given by $\hat{a}\circ\hat{b} = a_r\cdot b_r + a_d\cdot b_d$, for $\hat{a}, \hat{b} \in \mathbb{H}_d$. Under the natural isomorphism between $\mathbb{H}_d^r$ and $\mathbb{R}$, we will often identify, with a slight abuse of notation, $(\bar{0}, q_4) + \epsilon(\bar{0}, 0)$ with q_4. The multiplication of a 8-by-8 matrix with a dual quaternion will be defined as $M \star \hat{q} = (M_{11} * q_r + M_{12} * q_d) + \epsilon(M_{21} * q_r + M_{22} * q_d)$, where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad M_{11}, M_{12}, M_{21}, M_{22} \in \mathbb{R}^{4\times4}.$$

The next properties follow from the above definitions:

$$\hat{a}\circ(\hat{b}\hat{c}) = \hat{b}^{\mathsf{s}}\circ(\hat{a}^{\mathsf{s}}\hat{c}^*) = \hat{c}^{\mathsf{s}}\circ(\hat{b}^*\hat{a}^{\mathsf{s}}) \in \mathbb{R}, \quad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d, \tag{2}$$

$$\hat{a}\circ(\hat{b}\times\hat{c}) = \hat{b}^{\mathsf{s}}\circ(\hat{c}\times\hat{a}^{\mathsf{s}}) = \hat{c}^{\mathsf{s}}\circ(\hat{a}^{\mathsf{s}}\times\hat{b}), \quad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d^v, \tag{3}$$

$$\hat{a}\times\hat{a} = 0, \quad \hat{a} \in \mathbb{H}_d^v, \tag{4}$$

$$\hat{a}\times\hat{b} = -\hat{b}\times\hat{a}, \quad \hat{a}, \hat{b} \in \mathbb{H}_d^v, \tag{5}$$

$$\hat{a}^{\mathsf{s}}\circ\hat{b}^{\mathsf{s}} = \hat{a}\circ\hat{b}, \quad \hat{a}, \hat{b} \in \mathbb{H}_d, \tag{6}$$

$$\|\hat{a}^{\mathsf{s}}\| = \|\hat{a}\|, \quad \hat{a} \in \mathbb{H}_d, \tag{7}$$

$$\|\hat{a}^*\| = \|\hat{a}\|, \quad \hat{a} \in \mathbb{H}_d, \tag{8}$$

$$(M\star\hat{a})\circ\hat{b} = \hat{a}\circ(M^\intercal\star\hat{b}), \ \hat{a}, \hat{b} \in \mathbb{H}_d, \ M \in \mathbb{R}^{8\times8} \tag{9}$$

$$|\hat{a}\circ\hat{b}| \leq \|\hat{a}\|\|\hat{b}\|, \quad \hat{a}, \hat{b} \in \mathbb{H}_d. \tag{10}$$

Lemma 1. *For any $\hat{a}, \hat{b} \in \mathbb{H}_d$, the following inequality holds:*

$$\|\hat{a}\hat{b}\| \leq \sqrt{3/2}\,\|\hat{a}\|\|\hat{b}\|. \tag{11}$$

Proof. By definition, $\|\hat{a}\hat{b}\|^2 = \|(a_r b_r) + \epsilon(a_r b_d + a_d b_r)\|^2 = \|a_r b_r\|^2 + \|a_r b_d + a_d b_r\|^2 \leq \|a_r b_r\|^2 + (\|a_r b_d\| + \|a_d b_r\|)^2 = \|a_r b_r\|^2 + \|a_r b_d\|^2 + \|a_d b_r\|^2 + 2\|a_r b_d\|\|a_d b_r\| = \|a_r\|^2\|b_r\|^2 + \|a_r\|^2\|b_d\|^2 + \|a_d\|^2\|b_r\|^2 + 2\|a_r\|\|b_d\|\|a_d\|\|b_r\| = \|a_r\|^2(\|b_r\|^2 + \|b_d\|^2) + \|a_d\|^2\|b_r\|^2 + 2\|a_r\|\|b_d\|\|a_d\|\|b_r\|$. Using $\|a_r\|\|a_d\| \leq \frac{1}{2}(\|a_r\|^2 + \|a_d\|^2) = \frac{1}{2}\|\hat{a}\|^2$ and $\|b_r\|\|b_d\| \leq \frac{1}{2}(\|b_r\|^2 + \|b_d\|^2) = \frac{1}{2}\|\hat{b}\|^2$, we have that $\|a_r\|\|b_d\|\|a_d\|\|b_r\| \leq \frac{1}{4}\|\hat{a}\|^2\|\hat{b}\|^2$. It follows that $\|\hat{a}\hat{b}\|^2 \leq \|a_r\|^2\|\hat{b}\|^2 + \|a_d\|^2\|\hat{b}\|^2 + \frac{1}{2}\|\hat{a}\|^2\|\hat{b}\|^2 = (\|a_r\|^2 + \|a_d\|^2)\|\hat{b}\|^2 + \frac{1}{2}\|\hat{a}\|^2\|\hat{b}\|^2 = \|\hat{a}\|^2\|\hat{b}\|^2 + \frac{1}{2}\|\hat{a}\|^2\|\hat{b}\|^2 \leq \frac{3}{2}\|\hat{a}\|^2\|\hat{b}\|^2$. The result follows immediately by taking the square root of both sides of the last inequality. □

A compact way to represent the relationship between the body frame and the inertial frame when they are related by a rotation quaternion $q_{\text{B/I}}$ and a translation vector $\bar{r}_{\text{B/I}}$ is to use the dual quaternion [11] $\hat{q}_{\text{B/I}} = q_{\text{B/I}} + \epsilon\frac{1}{2} r_{\text{B/I}}^{\text{I}} q_{\text{B/I}} = q_{\text{B/I}} + \epsilon\frac{1}{2} q_{\text{B/I}} r_{\text{B/I}}^{\text{B}}$, where $r_{\text{Y/Z}}^{\text{X}} = (\bar{r}_{\text{Y/Z}}^{\text{X}}, 0)$ and $\bar{r}_{\text{Y/Z}}^{\text{X}}$ is the translation vector from the origin of the Z-frame to the origin of the Y-frame expressed in the X-frame. Note that $\hat{q}_{\text{B/I}}$ is a *unit* dual quaternion [1], i.e., it belongs to the set $\mathbb{H}_d^u = \{\hat{q} \in \mathbb{H}_d : \ \hat{q}\cdot\hat{q} = \hat{q}\hat{q}^* = \hat{q}^*\hat{q} = \|\hat{q}\|_d = 1\}$.

Assume that the desired orientation and position of the body with respect to the inertial frame are given by the unit dual quaternion $\hat{q}_{\text{D/I}} = q_{\text{D/I}} + \epsilon\frac{1}{2} r_{\text{D/I}}^{\text{I}} q_{\text{D/I}} = q_{\text{D/I}} + \epsilon\frac{1}{2} q_{\text{D/I}} r_{\text{D/I}}^{\text{D}}$. Then, by direct analogy to the quaternion case, the *dual error quaternion* [10], [3] is defined as $\hat{q}_{\text{B/D}} = \hat{q}_{\text{D/I}}^* \hat{q}_{\text{B/I}} = q_{\text{B/D}} + \epsilon\frac{1}{2} q_{\text{B/D}} r_{\text{B/D}}^{\text{B}}$. Note that $\hat{q}_{\text{B/D}}$ is also a unit dual quaternion [1]. Hence, the dual error quaternion $\hat{q}_{\text{B/D}}$ represents the rotation ($q_{\text{B/D}}$) and the translation ($r_{\text{B/D}}^{\text{B}}$) necessary to align the desired and the body frames.

The combined translational and rotational kinematic equations of the body and desired frames expressed in terms of dual quaternions are [11] $\dot{\hat{q}}_{\text{B/I}}=\frac{1}{2}\hat{\omega}^{\text{I}}_{\text{B/I}}\hat{q}_{\text{B/I}}=\frac{1}{2}\hat{q}_{\text{B/I}}\hat{\omega}^{\text{B}}_{\text{B/I}}$ and $\dot{\hat{q}}_{\text{D/I}}=\frac{1}{2}\hat{\omega}^{\text{I}}_{\text{D/I}}\hat{q}_{\text{D/I}}=\frac{1}{2}\hat{q}_{\text{D/I}}\hat{\omega}^{\text{D}}_{\text{D/I}}$, where $\hat{\omega}^{\text{X}}_{\text{Y/Z}}$ is the *dual velocity* of the Y-frame with respect to the Z-frame expressed in the X-frame, so that $\hat{\omega}^{\text{X}}_{\text{Y/Z}} = \omega^{\text{X}}_{\text{Y/Z}}+\epsilon(v^{\text{X}}_{\text{Y/Z}}+\omega^{\text{X}}_{\text{Y/Z}}\times r^{\text{X}}_{\text{X/Y}})$, $v^{\text{X}}_{\text{Y/Z}} = (\bar{v}^{\text{X}}_{\text{Y/Z}}, 0)$, $\bar{v}^{\text{X}}_{\text{Y/Z}}$ is the linear velocity of the Y-frame with respect to the Z-frame expressed in the X-frame.

By differentiating $\hat{q}_{\text{B/D}}$, the dual error quaternion kinematic equations become [10]

$$\dot{\hat{q}}_{\text{B/D}} = \tfrac{1}{2}\hat{q}_{\text{B/D}}\hat{\omega}^{\text{B}}_{\text{B/D}} = \tfrac{1}{2}\hat{\omega}^{\text{D}}_{\text{B/D}}\hat{q}_{\text{B/D}}, \tag{12}$$

where $\hat{\omega}^{\text{B}}_{\text{B/D}} = \hat{\omega}^{\text{B}}_{\text{B/I}} - \hat{\omega}^{\text{B}}_{\text{D/I}}$ is the *dual relative velocity* between the body frame and the desired frame expressed in the body frame. Note that $\hat{\omega}^{\text{B}}_{\text{D/I}} = \hat{q}^*_{\text{B/D}}\hat{\omega}^{\text{D}}_{\text{D/I}}\hat{q}_{\text{B/D}}$ and $\hat{\omega}^{\text{D}}_{\text{B/I}} = \hat{q}_{\text{B/D}}\hat{\omega}^{\text{B}}_{\text{B/I}}\hat{q}^*_{\text{B/D}}$. Note also that (12) has the same form as (1).

III. RIGID BODY RELATIVE DYNAMICS IN TERMS OF DUAL QUATERNIONS

We will use the following proposition in the derivation of the rigid body relative dynamic equations. A proof based on infinitesimal displacements is given in [12]. Here, we give an alternative proof based on dual quaternion algebra.

Proposition 1. *Given a dual vector quaternion expressed in the desired frame, $\hat{v}^{\text{D}}$, and the dual error quaternion $\hat{q}_{\text{B/D}}$ describing the relationship between the desired frame and the body frame, such that $\hat{v}^{\text{B}} = \hat{q}^*_{\text{B/D}}\hat{v}^{\text{D}}\hat{q}_{\text{B/D}}$, then the time derivative of $\hat{v}^{\text{B}}$ can be written as $\dot{\hat{v}}^{\text{B}} = \hat{q}^*_{\text{B/D}}(\dot{\hat{v}}^{\text{D}} + \hat{\omega}^{\text{D}}_{\text{D/B}} \times \hat{v}^{\text{D}})\hat{q}_{\text{B/D}}$.*

Proof. Calculate the time of $\hat{v}^{\text{B}}$: $\dot{\hat{v}}^{\text{B}} = \frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}^*_{\text{B/D}}\hat{v}^{\text{D}}\hat{q}_{\text{B/D}}) = \dot{\hat{q}}^*_{\text{B/D}}\hat{v}^{\text{D}}\hat{q}_{\text{B/D}} + \hat{q}^*_{\text{B/D}}\dot{\hat{v}}^{\text{D}}\hat{q}_{\text{B/D}} + \hat{q}^*_{\text{B/D}}\hat{v}^{\text{D}}\dot{\hat{q}}_{\text{B/D}}$. Replacing $\dot{\hat{q}}_{\text{B/D}}$ by (12) yields $\dot{\hat{v}}^{\text{B}} = \frac{1}{2}\hat{q}^*_{\text{B/D}}(\hat{\omega}^{\text{D}}_{\text{B/D}})^*\hat{v}^{\text{D}}\hat{q}_{\text{B/D}} + \hat{q}^*_{\text{B/D}}\dot{\hat{v}}^{\text{D}}\hat{q}_{\text{B/D}} + \hat{q}^*_{\text{B/D}}\hat{v}^{\text{D}}\frac{1}{2}\hat{\omega}^{\text{D}}_{\text{B/D}}\hat{q}_{\text{B/D}} = \hat{q}^*_{\text{B/D}}(\frac{1}{2}(\hat{\omega}^{\text{D}}_{\text{B/D}})^*\hat{v}^{\text{D}} + \dot{\hat{v}}^{\text{D}} + \hat{v}^{\text{D}}\frac{1}{2}\hat{\omega}^{\text{D}}_{\text{B/D}})\hat{q}_{\text{B/D}}$. Finally, and since $\hat{v}^{\text{D}}$ is a dual vector quaternion, we have that $(\hat{v}^{\text{D}})^* = -\hat{v}^{\text{D}}$, and we can write $\dot{\hat{v}}^{\text{B}} = \hat{q}^*_{\text{B/D}}(\dot{\hat{v}}^{\text{D}} + \frac{1}{2}\hat{v}^{\text{D}}\hat{\omega}^{\text{D}}_{\text{B/D}} - \frac{1}{2}(\hat{\omega}^{\text{D}}_{\text{B/D}})^*(\hat{v}^{\text{D}})^*)\hat{q}_{\text{B/D}} = \hat{q}^*_{\text{B/D}}(\dot{\hat{v}}^{\text{D}}+\hat{v}^{\text{D}}\times\hat{\omega}^{\text{D}}_{\text{B/D}})\hat{q}_{\text{B/D}} = \hat{q}^*_{\text{B/D}}(\dot{\hat{v}}^{\text{D}}+\hat{\omega}^{\text{D}}_{\text{D/B}}\times\hat{v}^{\text{D}})\hat{q}_{\text{B/D}}$. □

Note that Proposition 1 is the dual quaternion counterpart to the classical *transport theorem*.

The following proposition gives the rigid body relative dynamic equations between the body frame and the desired frame in terms of dual quaternions. Equivalent equations have been used in [10].

Proposition 2. *The rigid body relative dynamic equations in terms of dual quaternions are given by*

$$\begin{aligned}(\dot{\hat{\omega}}^{\text{B}}_{\text{B/D}})^{\mathsf{s}} =& (M^{\text{B}})^{-1}\star\Big(\hat{f}^{\text{B}} - (\hat{\omega}^{\text{B}}_{\text{B/D}}+\hat{\omega}^{\text{B}}_{\text{D/I}})\times\big(M^{\text{B}}\star((\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\text{B}}_{\text{D/I}})^{\mathsf{s}})\big)\\ &-M^{\text{B}}\star(\hat{q}^*_{\text{B/D}}\dot{\hat{\omega}}^{\text{D}}_{\text{D/I}}\hat{q}_{\text{B/D}})^{\mathsf{s}} - M^{\text{B}}\star(\hat{\omega}^{\text{B}}_{\text{D/I}}\times\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}}\Big),\end{aligned} \tag{13}$$

where $\hat{f}^{\text{B}} = f^{\text{B}} + \epsilon\tau^{\text{B}}$ is the total external dual force applied to the body about its center of mass expressed in the body frame, $f^{\text{B}} = (\bar{f}^{\text{B}}, 0)$, $\bar{f}^{\text{B}}$ is the total external force vector applied to the body, $\tau^{\text{B}} = (\bar{\tau}^{\text{B}}, 0)$, and $\bar{\tau}^{\text{B}}$ is the total external moment vector applied to the body about its center of mass. Finally, $M^{\text{B}} \in \mathbb{R}^{8\times 8}$ is the dual inertia matrix defined as

$$M^{\text{B}}=\begin{bmatrix} mI_3 & 0_{3\times1} & 0_{3\times3} & 0_{3\times1}\\ 0_{1\times3} & 1 & 0_{1\times3} & 0\\ 0_{3\times3} & 0_{3\times1} & \bar{I}^{\text{B}} & 0_{3\times1}\\ 0_{1\times3} & 0 & 0_{1\times3} & 1\end{bmatrix},\quad I^{\text{B}}=\begin{bmatrix}\bar{I}^{\text{B}} & 0_{3\times1}\\ 0_{1\times3} & 1\end{bmatrix},$$

$\bar{I}^{\text{B}} \in \mathbb{R}^{3\times3}$ is the mass moment of inertia of the body about its center of mass written in the body frame, and m is the mass of the body.

Proof. Differentiating the dual relative velocity and noting that the swap of the addition is equal to the addition of the swaps yields $(\dot{\hat{\omega}}^{\text{B}}_{\text{B/D}})^{\mathsf{s}} = (\dot{\hat{\omega}}^{\text{B}}_{\text{B/I}})^{\mathsf{s}} - (\dot{\hat{\omega}}^{\text{B}}_{\text{D/I}})^{\mathsf{s}}$. It has been shown in [1] that $(\dot{\hat{\omega}}^{\text{B}}_{\text{B/I}})^{\mathsf{s}} = (M^{\text{B}})^{-1}\star(\hat{f}^{\text{B}} - \hat{\omega}^{\text{B}}_{\text{B/I}}\times(M^{\text{B}}\star(\hat{\omega}^{\text{B}}_{\text{B/I}})^{\mathsf{s}}))$. Moreover, using Proposition 1, we can write $(\dot{\hat{\omega}}^{\text{B}}_{\text{D/I}})^{\mathsf{s}} = (\hat{q}^*_{\text{B/D}}(\dot{\hat{\omega}}^{\text{D}}_{\text{D/I}} + \hat{\omega}^{\text{D}}_{\text{D/B}}\times\hat{\omega}^{\text{D}}_{\text{D/I}})\hat{q}_{\text{B/D}})^{\mathsf{s}} = (\hat{q}^*_{\text{B/D}}\dot{\hat{\omega}}^{\text{D}}_{\text{D/I}}\hat{q}_{\text{B/D}})^{\mathsf{s}} + (\hat{\omega}^{\text{B}}_{\text{D/I}}\times\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}}$. Finally, subtracting $(\dot{\hat{\omega}}^{\text{B}}_{\text{D/I}})^{\mathsf{s}}$ from $(\dot{\hat{\omega}}^{\text{B}}_{\text{B/I}})^{\mathsf{s}}$ yields (13). □

IV. POSITION AND ATTITUDE TRACKING WITH DUAL RELATIVE VELOCITY FEEDBACK

In [3], a position and attitude tracking law is suggested based on the feedback of the dual relative velocity and the logarithm of the dual error quaternion. One drawback of this control law is that it is not written in terms of the dual force ($\hat{f}^{\text{B}}$), but in terms of a dual quaternion defined component-wise in terms of its real and dual parts as a function of $\bar{f}$ and $\bar{\tau}$. This is solved in [10], where an adaptive Terminal Sliding Mode (TSM) tracking law for the relative position and attitude of a leader-follower spacecraft formation is written in terms of the dual force. The tracking law in [10] is based on the special operator $\frac{\mathrm{d}}{\mathrm{d}\epsilon}$. Below, we propose an alternative tracking law in terms of the dual force that does not involve the special operator $\frac{\mathrm{d}}{\mathrm{d}\epsilon}$ and can be readily extended to a tracking law that does not need dual relative velocity feedback, thus extending the results of [5] and [1] for the case of combined position and attitude tracking.

First, we define the $\mathcal{L}_\infty$-norm of a function $\hat{u} : [0,\infty) \to \mathbb{H}_d$ as $\|\hat{u}\|_\infty = \sup_{t\geq0}\|\hat{u}(t)\|$. The dual quaternion $\hat{u} \in \mathcal{L}_\infty$, if and only if $\|\hat{u}\|_\infty < \infty$.

Theorem 1. *Consider the rigid body relative kinematic and dynamic equations* (12) *and* (13)*. Let the input dual force be defined by the feedback control law*

$$\begin{aligned}\hat{f}^{\text{B}} =& -k_p\mathrm{vec}\big(\hat{q}^*_{\text{B/D}}(\hat{q}^{\mathsf{s}}_{\text{B/D}} - \epsilon)\big) - k_d(\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}} + M^{\text{B}}\star(\hat{q}^*_{\text{B/D}}\dot{\hat{\omega}}^{\text{D}}_{\text{D/I}}\hat{q}_{\text{B/D}})^{\mathsf{s}}\\ &+\hat{\omega}^{\text{B}}_{\text{D/I}}\times(M^{\text{B}}\star(\hat{\omega}^{\text{B}}_{\text{D/I}})^{\mathsf{s}}),\quad k_p, k_d > 0,\end{aligned} \tag{14}$$

and assume that $\dot{\hat{\omega}}^{\text{D}}_{\text{D/I}}, \hat{\omega}^{\text{D}}_{\text{D/I}} \in \mathcal{L}_\infty$. Then, $\hat{q}_{\text{B/D}} \to \pm1$ (i.e., $q_{\text{B/D}} \to \pm1$ and $r^{\text{B}}_{\text{B/D}} \to 0$) and $\hat{\omega}^{\text{B}}_{\text{B/D}} \to 0$ (i.e., $\omega^{\text{B}}_{\text{B/D}} \to 0$ and $v^{\text{B}}_{\text{B/D}} \to 0$) as $t \to +\infty$ for all initial conditions.

Proof. First, note that $\hat{q}_{\text{B/D}} = \pm1$ and $\hat{\omega}^{\text{B}}_{\text{B/D}} = 0$ are, in fact, the equilibrium conditions for the closed-loop system formed by (13), (12), and (14). Consider now the following candidate Lyapunov function for the equilibrium point $\hat{q}_{\text{B/D}} = +1$ and $\hat{\omega}^{\text{B}}_{\text{B/D}} = 0$ (equivalently, $(\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}} = 0$): $V(\hat{q}_{\text{B/D}}, \hat{\omega}^{\text{B}}_{\text{B/D}}) = k_p(\hat{q}_{\text{B/D}} - 1)\circ(\hat{q}_{\text{B/D}} - 1) + \frac{1}{2}(\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}}\circ(M^{\text{B}}\star(\hat{\omega}^{\text{B}}_{\text{B/D}})^{\mathsf{s}})$. Note that V is a valid candidate Lyapunov function since $V(\hat{q}_{\text{B/D}} = 1, \hat{\omega}^{\text{B}}_{\text{B/D}} = 0) = 0$ and $V(\hat{q}_{\text{B/D}}, \hat{\omega}^{\text{B}}_{\text{B/D}}) > 0$ for all $(\hat{q}_{\text{B/D}}, \hat{\omega}^{\text{B}}_{\text{B/D}}) \in \mathbb{H}^u_d \times \mathbb{H}^v_d\backslash\{1, 0\}$.

The time derivative of V is equal to $\dot{V} = 2k_p(\hat{q}_{\mathrm{B/D}}-1)\circ\dot{\hat{q}}_{\mathrm{B/D}} + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(M^{\mathrm{B}}\star(\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})$. Then, by plugging in (13) and (12), and using (3), it follows that $\dot{V}=(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(k_p\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)+\hat{f}^{\mathrm{B}}-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})\times(M^{\mathrm{B}}\star((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}}-M^{\mathrm{B}}\star(\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}}-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})$. Introducing the feedback control law (14), we get $\dot{V} = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-k_d(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(k_p\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)-k_p\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)\big)) + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})\times(M^{\mathrm{B}}\star((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))$. Note that the second term is zero because it is the circle product of a dual vector quaternion with a dual scalar quaternion. Moreover, the third term can be shown to be equal to zero as follows: $(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})\times(M^{\mathrm{B}}\star((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})) = ((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}-(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})\circ(-\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}-\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})) = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(-\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))-(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(-\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})) = -(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}))-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})+(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}))+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})-(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))$. Note that the first and last terms are zero due to (3) and (4). Moreover, using (9) and (6), we can rewrite the second and fifth terms as $-(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})+(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}))+(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})$. Finally, applying (3) and (5) to the first and last terms of the previous expression yields $-(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}))+(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}))+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}))-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})) = 0$. Therefore, $\dot{V}$ is equal to $\dot{V} = -k_d(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} \leq 0$, for all $(\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}) \in \mathbb{H}^u_d\times\mathbb{H}^v_d\backslash\{1,0\}$. Hence, $\hat{q}_{\mathrm{B/D}}$ and $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}$ are uniformly bounded, i.e., $\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$. Since $V \geq 0$ and $\dot{V} \leq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. By integrating both sides of $\dot{V} = -k_d(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} \leq 0$, one obtains $\lim_{t\to\infty}\int_0^t \dot{V}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} V(t) - V(0) \leq -\lim_{t\to\infty}\int_0^t k_d(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(\tau))^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(\tau))^{\mathsf{s}}\mathrm{d}\tau$ or

$$\lim_{t\to\infty}\int_0^t k_d(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(\tau))^{\mathsf{s}}\circ(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(\tau))^{\mathsf{s}}\,\mathrm{d}\tau \leq V(0). \tag{15}$$

Since $\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$ and $\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}},\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$ by assumption, from (14) it follows that $\hat{f}^{\mathrm{B}} \in \mathcal{L}_\infty$ as well. From (13) then it also follows that $\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$. Along with (15), this yields $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(t) \to 0$ as $t \to \infty$.

We will now also prove that $\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ as $t\to\infty$. First, note that $\lim_{t\to\infty}\int_0^t \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty}\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(t) - \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(0) = -\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(0)$ exits and is finite. Furthermore, since $\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}},\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$ and $\hat{q}_{\mathrm{B/D}} \in \mathcal{L}_\infty$, it follows that $\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}},\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$ as well, and since $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} = \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}$ it also follows that $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \in \mathcal{L}_\infty$. Now note that $\ddot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$ since $\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{D/I}},\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}},\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}},\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/I}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}},\hat{q}_{\mathrm{B/D}},\dot{\hat{q}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ as $t\to\infty$.

Finally, calculating the limit as $t\to\infty$ of both sides of equation (13) yields $\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)\big) \to 0$ as $t\to\infty$, which, as shown in [1], is equivalent to $\hat{q}_{\mathrm{B/D}} \to \pm 1$. □

V. POSITION AND ATTITUDE TRACKING WITHOUT DUAL RELATIVE VELOCITY FEEDBACK

The feedback law given in Section IV for relative position and attitude tracking assumes that the dual error quaternion ($\hat{q}_{\mathrm{B/D}}$) and the dual relative velocity ($\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}$) are known. Theorem 2 below shows that relative position and attitude tracking can also be performed without relative linear and angular velocity measurements.

Theorem 2. *Consider the rigid body relative kinematic and dynamic equations* (12) *and* (13). *Let the input dual force be defined by the feedback control law*

$$\hat{f}^{\mathrm{B}} = -k_p\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)\big)-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big)+M^{\mathrm{B}}\star(\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} + \hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}), \quad k_p > 0, \tag{16}$$

where $\hat{z}$ is the output of the following LTI system: $\dot{\hat{x}}_{\mathrm{p}} = A\star\hat{x}_{\mathrm{p}} + B\star\hat{q}_{\mathrm{B/D}}$ and $\hat{z} = (CA)\star\hat{x}_{\mathrm{p}} + (CB)\star\hat{q}_{\mathrm{B/D}}$, where (A,B,C) is a minimal realization of a strictly positive real transfer matrix $C_{\mathrm{sp}}(s)$ with B a full rank matrix, and assume that $\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}},\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$. Then, $\hat{q}_{\mathrm{B/D}} \to \pm 1$, $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$, and $\hat{x}_{\mathrm{sp}} = \dot{\hat{x}}_{\mathrm{p}} \to 0$ as $t\to+\infty$ for any initial condition.

Proof. First, rewrite the LTI system as follows:

$$\dot{\hat{x}}_{\mathrm{sp}} = A\star\hat{x}_{\mathrm{sp}} + B\star\dot{\hat{q}}_{\mathrm{B/D}},\;\; \hat{z} = C\star\hat{x}_{\mathrm{sp}}. \tag{17}$$

Note that $\hat{q}_{\mathrm{B/D}} = \pm 1$, $\hat{\omega}_{\mathrm{B}} = 0$, and $\hat{x}_{\mathrm{sp}} = 0$ is the equilibrium condition for the closed-loop system formed by (13), (12), (17), and (16). Consider the candidate Lyapunov function $V(\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}},\hat{x}_{\mathrm{sp}}) = k_p(\hat{q}_{\mathrm{B/D}}-1)\circ(\hat{q}_{\mathrm{B/D}}-1) + \frac{1}{2}(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + 2\hat{x}_{\mathrm{sp}}\circ(P\star\hat{x}_{\mathrm{sp}})$, for the equilibrium point $\hat{q}_{\mathrm{B/D}} = 1$, $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$, and $\hat{x}_{\mathrm{sp}} = 0$, where $P > 0$ satisfies $A^{\mathsf{T}}P + PA = -Q$, $PB = C^{\mathsf{T}}$, and $Q > 0$. By the Kalman-Yakubovich-Popov conditions [13], there always exist matrices P and Q satisfying these conditions. Note that V is a valid candidate Lyapunov function since $V(\hat{q}_{\mathrm{B/D}} = 1, \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0, \hat{x}_{\mathrm{sp}} = 0) = 0$ and $V(\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}},\hat{x}_{\mathrm{sp}}) > 0$ for all $(\hat{q}_{\mathrm{B/D}},\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}},\hat{x}_{\mathrm{sp}}) \in \mathbb{H}^u_d\times\mathbb{H}^v_d\times\mathbb{H}_d\backslash\{1,0,0\}$. The time derivative of V is equal to $\dot{V} = 2k_p(\hat{q}_{\mathrm{B/D}}-1)\circ\dot{\hat{q}}_{\mathrm{B/D}} + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(M^{\mathrm{B}}\star(\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + 4\dot{\hat{x}}_{\mathrm{sp}}\circ(P\star\hat{x}_{\mathrm{sp}})$. By plugging in (12) and (13) into the previous equation and applying (2) and the KYP conditions, it follows that $\dot{V} = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(k_p\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)+\hat{f}^{\mathrm{B}}-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})\times(M^{\mathrm{B}}\star((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}}-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})+4(A\star\hat{x}_{\mathrm{sp}}+B\star\dot{\hat{q}}_{\mathrm{B/D}})\circ(P\star\hat{x}_{\mathrm{sp}})$. Introducing the feedback control law (16), we get $\dot{V} = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big)) + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(k_p\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)-k_p\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}}-\epsilon)\big)) + (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-(\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})\times(M^{\mathrm{B}}\star((\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}))^{\mathsf{s}})-M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}+\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}}\times(M^{\mathrm{B}}\star(\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})) + 4(A\star\hat{x}_{\mathrm{sp}}+B\star\dot{\hat{q}}_{\mathrm{B/D}})\circ(P\star\hat{x}_{\mathrm{sp}})$. Again, note that the second term is zero because it is the circle product of a dual vector quaternion with a dual scalar quaternion. Moreover, the third term has been shown to be equal to zero in the proof of Theorem 1. As for the fourth term, it can be simplified as follows: $\dot{V} = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big))+4(A\star\hat{x}_{\mathrm{sp}})\circ(P\star\hat{x}_{\mathrm{sp}})+4(B\star\dot{\hat{q}}_{\mathrm{B/D}})\circ(P\star\hat{x}_{\mathrm{sp}}) = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big)) + 2((A^{\mathsf{T}}P+PA)\star\hat{x}_{\mathrm{sp}})\circ\hat{x}_{\mathrm{sp}} + 4\dot{\hat{q}}_{\mathrm{B/D}}\circ((B^TP)\star\hat{x}_{\mathrm{sp}}) = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big))-2\hat{x}_{\mathrm{sp}}\circ(Q\star\hat{x}_{\mathrm{sp}})+2(\hat{q}_{\mathrm{B/D}}\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})\circ(C\star\hat{x}_{\mathrm{sp}}) = (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\circ(2\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}-2\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}\hat{z}^{\mathsf{s}}\big)) - 2\hat{x}_{\mathrm{sp}}\circ(Q\star\hat{x}_{\mathrm{sp}}) =$

$-2\hat{x}_{\rm sp} \circ (Q \star \hat{x}_{\rm sp}) \leq 0$, for all $(\hat{q}_{\rm B/D}, \hat{\omega}^{\rm B}_{\rm B/D}, \hat{x}_{\rm sp}) \in \mathbb{H}^u_d \times \mathbb{H}^v_d \times \mathbb{H}_d \backslash \{1,0,0\}$. Hence, $\hat{q}_{\rm B/D}$, $\hat{\omega}^{\rm B}_{\rm B/D}$, and $\hat{x}_{\rm sp}$ are uniformly bounded, i.e., $\hat{q}_{\rm B/D}, \hat{\omega}^{\rm B}_{\rm B/D}, \hat{x}_{\rm sp} \in \mathcal{L}_\infty$.

We will now prove that $\hat{x}_{\rm sp} \to 0$ as $t \to \infty$. Since $V \geq 0$ and $\dot{V} \leq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. By integrating both sides of $\dot{V} = -2\hat{x}_{\rm sp} \circ (Q \star \hat{x}_{\rm sp}) \leq 0$, one obtains $\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} V(t) - V(0) \leq -\lim_{t\to\infty} \int_0^t 2\hat{x}_{\rm sp}(\tau) \circ (Q \star \hat{x}_{\rm sp}(\tau))\mathrm{d}\tau$ or

$$\lim_{t\to\infty} \int_0^t 2\hat{x}_{\rm sp}(\tau) \circ (Q \star \hat{x}_{\rm sp}(\tau))\,\mathrm{d}\tau \leq V(0). \tag{18}$$

Since $\hat{x}_{\rm sp}, \dot{\hat{q}}_{\rm B/D} \in \mathcal{L}_\infty$, it follows that $\dot{\hat{x}}_{\rm sp} \in \mathcal{L}_\infty$. Along with (18), this yields $\hat{x}_{\rm sp} \to 0$ as $t \to \infty$. This, in turn, implies that $\hat{z} \to 0$ as $t \to \infty$ from (17).

We will now also prove that $\dot{\hat{x}}_{\rm sp} \to 0$ as $t \to \infty$. First, note that $\lim_{t\to\infty} \int_0^t \dot{\hat{x}}_{\rm sp}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} \hat{x}_{\rm sp}(t) - \hat{x}_{\rm sp}(0) = -\hat{x}_{\rm sp}(0)$ exits and is finite. Note that $\ddot{\hat{x}}_{\rm sp} = A \star \dot{\hat{x}}_{\rm sp} + B \star \ddot{\hat{q}}_{\rm B/D}$ from which it follows that $\ddot{\hat{x}}_{\rm sp} \in \mathcal{L}_\infty$ since $\ddot{\hat{q}}_{\rm B/D}, \dot{\hat{\omega}}^{\rm B}_{\rm B/D}, \hat{z}, \hat{q}_{\rm B/D}, \dot{\hat{\omega}}^{\rm D}_{\rm D/I}, \hat{\omega}^{\rm D}_{\rm D/I}, \hat{\omega}^{\rm B}_{\rm B/I} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\dot{\hat{x}}_{\rm sp} \to 0$ as $t \to \infty$.

Thus, calculating the limit as $t \to \infty$ of both sides of equation (17) yields $\dot{\hat{q}}_{\rm B/D} \to 0$ as $t \to \infty$, since B is assumed to be full rank. Given that (12) can be rewritten as $\hat{\omega}^{\rm B}_{\rm B/D} = 2\hat{q}^*_{\rm B/D}\dot{\hat{q}}_{\rm B/D}$, this also implies that $\hat{\omega}^{\rm B}_{\rm B/D} \to 0$ as $t \to \infty$.

Similarly to Theorem 1, we can prove that $\dot{\hat{\omega}}^{\rm B}_{\rm B/D} \to 0$ as $t \to \infty$. First, note that $\lim_{t\to\infty} \int_0^t \dot{\hat{\omega}}^{\rm B}_{\rm B/D}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} \hat{\omega}^{\rm B}_{\rm B/D}(t) - \hat{\omega}^{\rm B}_{\rm B/D}(0) = -\hat{\omega}^{\rm B}_{\rm B/D}(0)$ exits and is finite. Also note that $(\ddot{\hat{\omega}}^{\rm B}_{\rm B/D})^{\rm s} = (M^{\rm B})^{-1} \star (-k_p \mathrm{vec}(\dot{\hat{q}}^*_{\rm B/D}(\hat{q}^{\rm s}_{\rm B/D} - \epsilon)) - k_p \mathrm{vec}(\hat{q}^*_{\rm B/D}(\dot{\hat{q}}^{\rm s}_{\rm B/D})) - 2\mathrm{vec}(\dot{\hat{q}}^*_{\rm B/D}\hat{z}^{\rm s}) - 2\mathrm{vec}(\hat{q}^*_{\rm B/D}(\dot{\hat{z}})^{\rm s}) + \dot{\hat{\omega}}^{\rm B}_{\rm D/I} \times (M^{\rm B} \star (\hat{\omega}^{\rm B}_{\rm D/I})^{\rm s}) + \hat{\omega}^{\rm B}_{\rm D/I} \times (M^{\rm B} \star (\dot{\hat{\omega}}^{\rm B}_{\rm D/I})^{\rm s}) - \dot{\hat{\omega}}^{\rm B}_{\rm B/I} \times (M^{\rm B} \star (\hat{\omega}^{\rm B}_{\rm B/I})^{\rm s}) - \hat{\omega}^{\rm B}_{\rm B/I} \times (M^{\rm B} \star (\dot{\hat{\omega}}^{\rm B}_{\rm B/I})^{\rm s}) - M^{\rm B} \star (\dot{\hat{\omega}}^{\rm B}_{\rm D/I} \times \hat{\omega}^{\rm B}_{\rm B/D})^{\rm s} - M^{\rm B} \star (\hat{\omega}^{\rm B}_{\rm D/I} \times \dot{\hat{\omega}}^{\rm B}_{\rm B/D})^{\rm s})$ and, hence, $\ddot{\hat{\omega}}^{\rm B}_{\rm B/D} \in \mathcal{L}_\infty$ since $\dot{\hat{\omega}}^{\rm B}_{\rm B/D}, \hat{\omega}^{\rm B}_{\rm B/D}, \dot{\hat{\omega}}^{\rm B}_{\rm B/I}, \hat{\omega}^{\rm B}_{\rm B/I}, \dot{\hat{q}}_{\rm B/D}, \hat{q}_{\rm B/D}, \hat{z}, \dot{\hat{z}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\dot{\hat{\omega}}^{\rm B}_{\rm B/D} \to 0$ as $t \to \infty$.

Finally, calculating the limit as $t \to \infty$ of both sides of equation (13) yields $\mathrm{vec}(\hat{q}^*_{\rm B/D}(\hat{q}^{\rm s}_{\rm B/D} - \epsilon)) \to 0$ as $t \to \infty$, which, as shown in [1], is equivalent to $\hat{q}_{\rm B/D} \to \pm 1$. □

Remark 1. According to the proofs of Theorems 1 and 2, $\hat{q}_{\rm B/D}$ converges to either $+1$ or -1. In fact, all solutions converge to $\hat{q}_{\rm B/D} = +1$ except for the solution starting at $\hat{q}_{\rm B/D} = -1$, in which case the system remains in $\hat{q}_{\rm B/D} = -1$. Note, however, that $\hat{q}_{\rm B/D} = +1$ and $\hat{q}_{\rm B/D} = -1$ represent the same physical relative position and attitude between frames, so either equilibrium is acceptable. This creates the annoyance however that for initial conditions close to $\hat{q}_{\rm B/D} = -1$, a large rotation (larger than 180 degrees) will be performed, despite the fact that a shorter rotation (less than 180 degrees) to the equilibrium exists. This is a well-known issue of quaternions and can be easily solved by switching the gains in (14) and (16) in order to follow the shortest path. For details, see [4], [14].

Remark 2. If $\hat{\omega}^{\rm D}_{\rm D/I} = 0$, the model-dependent tracking controllers (14) and (16) simplify into the model-independent setpoint controllers given in [1].

VI. SIMULATION RESULTS

To compare the performance of control laws (14) and (16) (with and without dual relative velocity feedback, respectively), a simple example is considered here.

A rigid body with mass moment of inertia

$$\bar{I}^{\rm B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.63 & 0 \\ 0 & 0 & 0.85 \end{bmatrix} \text{ Kg.m}^2,$$

and mass $m = 1$ kg is assumed. The center of mass of the rigid body is positioned relatively to the origin of the desired frame at $\bar{r}^{\rm B}_{\rm B/D} = [x^{\rm B}_{\rm B/D}\ y^{\rm B}_{\rm B/D}\ z^{\rm B}_{\rm B/D}]^{\sf T} = [20,\ 20,\ 10]^{\sf T}$ m. Moreover, the initial error quaternion and relative linear and angular velocities of the body frame with respect to the desired frame are set to $q_{\rm B/D} = [q_{\rm B/D1}\ q_{\rm B/D2}\ q_{\rm B/D3}\ q_{\rm B/D4}]^{\sf T} = [0.4618,\ 0.1917,\ 0.7999,\ 0.3320]^{\sf T}$, $\bar{v}^{\rm B}_{\rm B/D} = [u^{\rm B}_{\rm B/D}\ v^{\rm B}_{\rm B/D}\ w^{\rm B}_{\rm B/D}]^{\sf T} = [0.1,\ -0.2,\ 0.3]^{\sf T}$ m/s, and $\bar{\omega}^{\rm B}_{\rm B/D} = [p^{\rm B}_{\rm B/D}\ q^{\rm B}_{\rm B/D}\ r^{\rm B}_{\rm B/D}]^{\sf T} = [-0.1,\ 0.2,\ -0.3]^{\sf T}$ rad/s, respectively.

The linear and angular velocity of the desired frame with respect to the inertial frame, expressed in the desired frame, are defined as $\bar{v}^{\rm D}_{\rm D/I} = [u^{\rm D}_{\rm D/I}\ v^{\rm D}_{\rm D/I}\ w^{\rm D}_{\rm D/I}]^{\sf T} = -[0.1,\ 0.2,\ 0.3]^{\sf T}\cos(2\pi[10^{-1},\ 10^{-1},\ 10^{-1}]^{\sf T} + \frac{\pi}{180}[30,\ 60,\ 90]^{\sf T})$ m/s and $\bar{\omega}^{\rm D}_{\rm D/I} = [p^{\rm D}_{\rm D/I}\ q^{\rm D}_{\rm D/I}\ r^{\rm D}_{\rm D/I}]^{\sf T} = [0.1,\ 0.2,\ 0.3]^{\sf T}\cos(2\pi[10^{-1},\ 10^{-1},\ 10^{-1}]^{\sf T} + \frac{\pi}{180}[0,\ 45,\ 90]^{\sf T})$ rad/s, respectively. They are illustrated in Figure 1. The control gains are set to $k_p = 0.2$ (both in (14) and (16)) and $k_d = 0.4$ (in (14)). To simplify the calculations, A and B are chosen as $-k_f I_8$ and $k_f I_8$, respectively. By defining $Q = -k_d(B^{-\sf T}A + A^{\sf T}B^{-\sf T})$ as in [15], the KYP conditions yield $P = k_d B^{-\sf T}$ and $C = k_d I_8$.

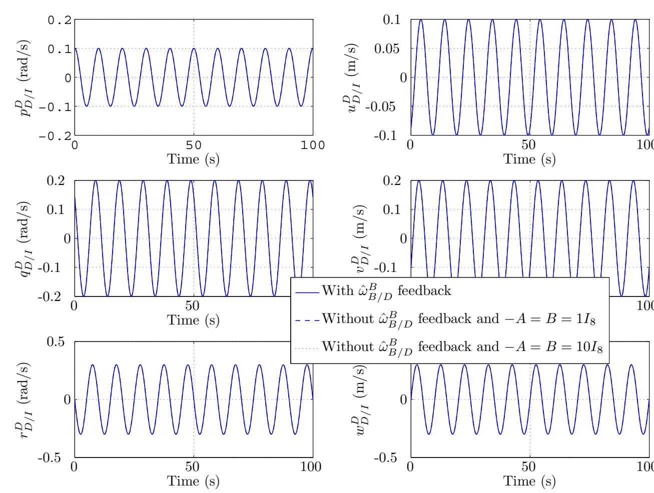

Fig. 1. Desired linear and angular velocity expressed in the desired frame.

The relative position and attitude of the body frame with respect to the desired frame with controller (14) (with dual relative velocity feedback) and with controller (16) (without dual relative velocity feedback) with $k_f = 1$ and $k_f = 10$ are compared in Figure 2. In all three cases, $q_{\rm B/D} \to 1$ and $\bar{r}^{\rm B}_{\rm B/D} \to 0$ as $t \to \infty$, as expected. Figure 3 shows the relative linear and angular velocity of the body frame with respect to

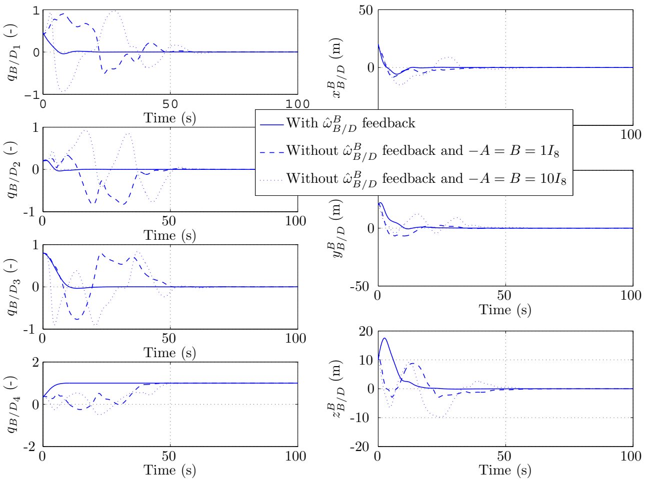

Fig. 2. Relative attitude and position.

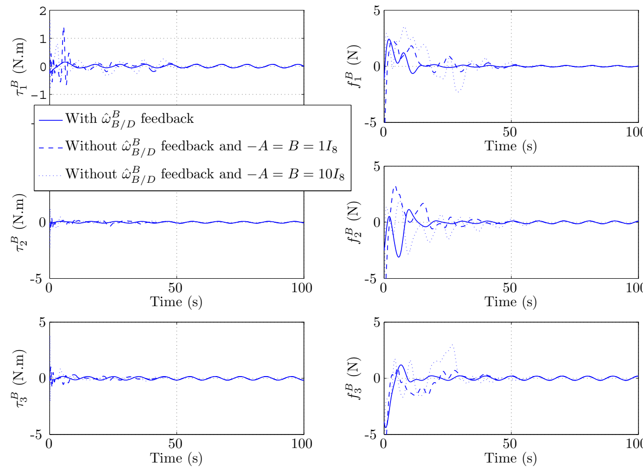

Fig. 4. Control force and torque.

the desired frame for the same three cases studied in Figure 2. As expected, $\bar{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ and $\bar{v}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ as $t \to \infty$. For

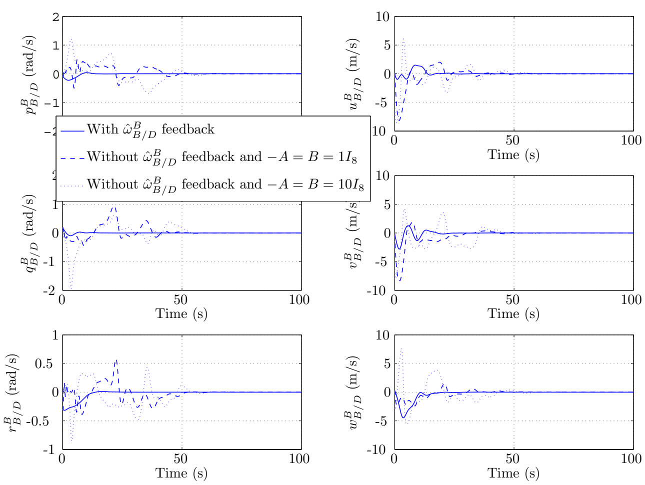

Fig. 3. Relative linear and angular velocity expressed in the body frame.

completeness, Figure 4 shows the control force and torque applied to the body for the same three cases.

VII. CONCLUSION

A velocity-free tracking controller for the relative position and attitude of a rigid body with respect to some desired frame is presented in this paper. It can be used when no relative linear and angular velocity information is available. Also, and more importantly, this paper, together with [1], shows how it can be relatively straightforward to extend attitude controllers based on quaternions into combined position and attitude controllers based on dual quaternions. Future work includes redesigning the proposed controllers such that no model information (e.g., inertia, mass) is required.

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