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Suboptimal Control of Rigid Body Motion with a Quadratic Cost

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Abstract. In this paper we consider the problem of controlling the rotational motion of a rigid body using three independent control torques. Given a quadratic cost we seek stabilizing state feedback controllers which are suboptimal in the sense that, given a scalar γ, they guarantee that all motions starting within a specified bounded set result in a cost bounded by γ. For a special class of cost functions, we present explicit expressions for suboptimal stabilizing controllers yielding a cost arbitrarily close to the infimal cost. For the general case, we present sufficient conditions which guarantee the existence of linear, suboptimal, stabilizing controllers.

Keywords: Rigid body, quadratic cost, Lyapunov function, linear matrix inequalities, stabilization.

1. Introduction

In this paper we consider the problem of controlling the rotational motion of a rigid body using three independent control torques. The minimal requirement on the controller is to stabilize the body about a specified orientation. In addition to this, we require the controller to guarantee that a specified quadratic cost, or performance index, is bounded for all initial states lying in a given set. Ideally, we would like to minimize the cost, but since this is, in general, a difficult task we are contented with obtaining an upper bound for the cost. By minimizing this upper bound, it is hoped that one can achieve acceptable performance close to the optimal one.

The problem addressed in this paper is of significant importance in aerospace engineering since it corresponds to the control of the orientation of a spacecraft. This problem has received a great deal of attention in the literature; the main thrust of previous research, however, has been directed towards the time or fuel-optimal control problem; see, for example, [1, 4, 7, 11] and the recent survey paper [16].

As far as the optimal regulation of the angular velocity or the angular momentum vector is concerned, the earliest results seem to be the ones reported in [6, 14] and [18]. More

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recently Dabbous and Ahmed [5] and Bourdache-Siguerdidjane [2] have also considered the optimal regulation of the angular momentum (dynamic) equations. In this work we are interested however with the more complicated problem of optimal control for the *complete* attitude equations, i.e. dynamics and kinematics.

The equations describing the rotational motion of a rigid body are nonlinear. Thus, in general, to obtain optimal feedback controllers for nonlinear systems one has to solve the associated Hamilton-Jacobi equation (HJE). This is a partial differential equation and except for very special cases, one cannot obtain solutions. As a result of the difficulty in obtaining optimal controllers, in this paper we look for suboptimal stabilizing feedback controllers.

We consider the following problem: Given a subset of the state space, find a feedback controller which results in an asymptotically stable closed loop system and which guarantees bounded cost for all initial states in the given set. For a special class of cost functions, we present an explicit expression for suboptimal controllers which achieve a cost which is arbitrarily close to the infimal cost. The main result of the paper is a sufficient condition which guarantees the existence of *linear* suboptimal controllers for a *general* quadratic cost. This condition involves the solution of a matrix inequality. The motivation for considering linear controllers stems from the fact that, with the kinematic description used here, the equations describing the rotational motion of a rigid body have the nice property that, although nonlinear, they admit linear controllers which yield *global* asymptotic stability and finite cost for any initial state.

The paper is organized as follows. In section 2 we present the equations of motion of a rotating rigid body and we state the optimal control problem; we call it the Quadratic Regulation Problem (QRP). The description of the dynamics is standard, whereas for the kinematics we choose the Cayley-Rodrigues kinematic parameters. Section 3 presents simple linear controllers which render the systems under consideration globally asymptotically stable. The results here motivate our approach to later results. A special class of cost functions without penalty on the control input is considered in section 4. We first focus on the *kinematics only* with the angular velocity as the control input. We explicitly exhibit the optimal controller for this system and the optimal cost function. This is achieved by solving the Hamilton-Jacobi equation for the associated optimization problem. We then consider the full system (i.e., dynamics plus kinematics) and present controllers which yield cost values arbitrarily close to the cost obtained by considering the kinematics only. The main results are contained in section 5. For the general quadratic cost, we present sufficient conditions which, if satisfied, guarantee the existence of a linear, suboptimal, stabilizing controller. These conditions take the form of a convex optimization problem plus a line search. We provide an algorithm—based on Linear Matrix Inequalities—for the numerical solution of this problem.

1.1. Notation

A' is the transpose of matrix A.

$A > B$ ($A \geq B$), where A and B are real symmetric matrices, means $A - B$ has positive (nonnegative) eigenvalues.

I_n the identity matrix of size n.

$\|x\| = \sqrt{x'x}$.

$\mathcal{B}_2(d) = \{x \in \mathbb{R}^s \mid \|x\| \leq d\}$.

$\mathcal{B}_\infty(d) = \{x \in \mathbb{R}^s \mid |x_i| \leq d \text{ for } i = 1, \ldots, s\}$.

$V : \mathbb{R}^n \rightarrow \mathbb{R}$ is positive definite if $V(x) \geq 0$ for all $x \in \mathbb{R}^n$, $V(x) = 0$ only if $x = 0$, and $\lim_{x \to \infty} V(x) = \infty$.

$V_x(x) = \frac{\partial V(x)}{\partial x}$ ($1 \times n$ row matrix).

$\mathbf{Co}\{\mathcal{S}\}$ = convex hull of the set $\mathcal{S}$.

2. Problem Formulation

We consider the rotational motion of a rigid body subject to three independent scalar control torques; these torques are applied about axes which are fixed in the body and aligned with the body principal axes. The rotational motion of a rigid body can be described by a system of six first order differential equations. Three of these equations govern the angular velocity (dynamic equations) while the other three describe the evolution of the body orientation (kinematic equations).

Choosing a body-fixed coordinate system aligned with the principal axes, the dynamic equations can be written in the form

$$\dot{\omega} = F(\omega)\omega + J^{-1}u, \quad \omega(0) = \omega_0, \tag{1}$$

where $\omega = [\omega_1 \ \omega_2 \ \omega_3]'$ and ω_1, ω_2, and ω_3 are the components of the angular velocity vector. The matrix $F(\omega)$ is given by

$$F(\omega) = \begin{bmatrix} 0 & -J_3\omega_3/J_1 & J_2\omega_2/J_1 \\ J_3\omega_3/J_2 & 0 & -J_1\omega_1/J_2 \\ -J_2\omega_2/J_3 & J_1\omega_1/J_3 & 0 \end{bmatrix} \tag{2}$$

where the positive scalars J_1, J_2, and J_3 are the principal moments of inertia of the rigid body at the mass center. The matrix J is the diagonal matrix

$$J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}.$$

If $S(\omega)$ is the skew-symmetric matrix defined by

$$S(\omega) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{3}$$

then $F(\omega)$ can be written as

$$F(\omega) = J^{-1}S(J\omega). \tag{4}$$

To describe the orientation of the rigid body in space, kinematic equations are necessary. One possible choice of kinematic parameters is given by the so-called Cayley-Rodrigues parameters ρ_1, ρ_2, and ρ_3 [12]. These parameters lead to a minimal three-dimensional representation of the rotation group. The corresponding kinematic equations are

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0 \tag{5}$$

where $\rho = [\rho_1 \; \rho_2 \; \rho_3]'$ is the kinematic vector,

$$G(\rho) := \frac{1}{2}(I_3 + S(\rho) + \rho\rho'), \tag{6}$$

and $S(\rho)$ is the skew-symmetric matrix defined in (3).

A useful property of the Cayley-Rodrigues representation is that, for any $\rho \in \mathbb{R}^3$, we have

$$\rho' G(\rho) = \frac{1}{2}(1 + \|\rho\|^2)\rho'. \tag{7}$$

This property will be used when computing Lyapunov derivatives associated with the nonlinear system given by (1) and (5).

When the Cayley-Rodrigues parameters are zero, $\rho_1 = \rho_2 = \rho_3 = 0$, the rotation matrix is the identity matrix and the body and inertial coordinate systems coincide. *This will be the equilibrium (rest) or desired orientation in this paper.*

We associate with the system given by (1) and (5) a *performance output*

$$z = C \begin{bmatrix} \rho \\ \omega \end{bmatrix} + Du, \tag{8}$$

where C and D are given real matrices. For each initial state $[\rho_0' \; \omega_0']' \in \mathbb{R}^6$, and control input $u(\cdot)$, the performance index or cost associated with this output is given by

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) := \int_0^\infty \|z(t)\|^2 \, dt. \tag{9}$$

The objective of this paper is to solve the following problem.

Quadratic Regulation Problem (QRP). Consider the nonlinear system

$$\dot{\rho} = G(\rho)\omega \tag{10a}$$
$$\dot{\omega} = F(\omega)\omega + J^{-1}u \tag{10b}$$

where $F(\cdot)$ is defined in (2), $G(\cdot)$ is defined in (6), and the performance index is given by (8) and (9). Given any bounded set $\mathcal{C} \subset \mathbb{R}^6$ containing the origin and any positive scalar γ, obtain a memoryless state-feedback controller $u = k(\rho, \omega)$ such that:

(i) the closed loop system is asymptotically stable about zero with $\mathcal{C}$ contained in the region of attraction;

(ii) for each initial state $[\rho_0' \; \omega_0']' \in \mathcal{C}$ the performance index satisfies the bound

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) \leq \gamma. \tag{11}$$

3. Globally Asymptotically Stabilizing Linear Controllers

In this section we present simple linear controllers which render the nonlinear system (10) *globally* asymptotically stable. The result given here provides motivation for later developments.

LEMMA 1 *The linear controller*

$$u = -\kappa_1 \omega - \kappa_2 \rho, \tag{12}$$

where κ_1 and κ_2 are any positive scalars, globally asymptotically stabilizes the system (10) Moreover, given any initial state $(\rho_0 , \omega_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, we have

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) < \infty.$$

Proof. Define the positive definite function

$$V(\rho, \omega) := \frac{1}{2}\omega' J \omega + \kappa_2 \ln(1 + \|\rho\|^2) \tag{13}$$

where $\ln(\cdot)$ denotes the natural logarithm. We show that this is a Lyapunov function for the closed-loop system. Differentiating (13) along the trajectories of the closed-loop system obtained by applying (12) to (10), and using (7), we obtain

$$\dot{V} = -\kappa_1 \omega' J \omega \leq 0. \tag{14}$$

Since V is radially unbounded, it now follows that all trajectories are bounded. Since $\dot{V} \equiv 0$ implies that $\omega \equiv 0$, which gives $\dot{\omega} \equiv 0$ and $\rho \equiv 0$, it follows from LaSalle's theorem that the closed-loop system is globally asymptotically stable about zero.

In order to show that the cost (9) is bounded, notice first that the control law (12) is *locally* exponential stabilizing, or equivalently, the linearization of the closed-loop system is asymptotically stable [13]. The linearization of the closed-loop system about the origin is given by

$$\begin{bmatrix} \dot{\rho} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & I_3/2 \\ -J^{-1}\kappa_2 & -J^{-1}\kappa_1 \end{bmatrix} \begin{bmatrix} \rho \\ \omega \end{bmatrix} \tag{15}$$

Consider the positive definite function

$$V(\rho, \omega) = \kappa_2 \rho' \rho + \frac{1}{2}\omega' J \omega. \tag{16}$$

Differentiating (16) along the trajectories of the linearized closed loop system (15) gives

$$\dot{V} = -\kappa_1 \omega' \omega \leq 0.$$

Notice also that $\dot{V} \equiv 0$ implies that $\omega \equiv 0$, which gives $\dot{\omega} \equiv 0$ and $\rho \equiv 0$.Therefore, the system (15) is asymptotically stable for all $\kappa_1 > 0$ and $\kappa_2 > 0$ and, consequently, the system (10) with control (12) is (locally) exponentially stable.

Since the closed-loop system is locally exponentially stable, there exist positive scalars δ, η_1 and η_2 such that $\|x(t_0)\| \leq \delta$ implies

$$\|x(t)\| \leq \eta_1 \|x(t_0)\| e^{-\eta_2(t-t_0)}, \quad \forall t \geq t_0 \tag{17}$$

where $x(t)$ is the trajectory starting from $x(t_0)$ at $t = t_0$. Moreover, due to global asymptotic stability of the closed-loop system, for every initial state $x_0 = [\rho_0', \omega_0'] \in \mathbb{R}^6$ there exists a time t^* such that $\|x(t)\| \leq \delta$ for all $t \geq t^*$.

In order now to show that the cost (11) is bounded, consider a trajectory starting from any initial state x_0. Then

$$\begin{aligned}\mathcal{J}(\rho_0, \omega_0, u(\cdot)) &= \lim_{T\to\infty} \int_0^T \|z(t)\|^2 \, dt \\ &= \int_0^{t^*} \|z(t)\|^2 \, dt + \lim_{T\to\infty} \int_{t^*}^T \|z(t)\|^2 \, dt \end{aligned} \tag{18}$$

For the part of the trajectory starting from $x(t^*)$ at $t = t^*$ we have, using (17), that

$$\begin{aligned}\lim_{T\to\infty} \int_{t^*}^T \|x(t)\|^2 \, dt &= \lim_{T\to\infty} \int_0^T \|x(\tau + t^*)\|^2 \, d\tau \\ &\leq \lim_{T\to\infty} \eta_1 ||x(t^*)|| \int_0^T e^{-\eta_2 \tau} \, d\tau < \infty \end{aligned}$$

Using (12) and the definition of z we have finally that the second integral in (17) is bounded and thus

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) < \infty.$$

as claimed. □

This lemma provides the main motivation for the methodology used in the paper. According to this lemma, the system (10) has the–rather unusual for a nonlinear system–property that it admits linear *globally* asymptotically stabilizing control laws. In addition, the linear control law (12) provides a bounded value for the cost (9). It is natural then to search over the class of linear controllers to find the one yielding the minimum value of the cost.

4. Special Cost Functions

In this section we consider the special class of QRP problems with performance output z of the form

$$z = \begin{bmatrix} r_1 \rho \\ r_2 \omega \end{bmatrix}$$

where r_1, r_2 are positive scalars. This performance output is a special case of the general performance output (8); the corresponding cost is given by

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) = \int_0^\infty \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} \, dt. \tag{19}$$

Note that (10) is a system in cascade form; i.e., ρ does not enter the right-hand-side of (10a) and u does not enter (10b). In essence, ω acts as a "control" for the subsystem (10b). Therefore, when u does not enter in the cost function, it is natural to consider first the suboptimal control problem for the kinematics only with ω treated as a control-like variable. Such problems are simpler than suboptimal control problems for both (10b) and (10a). Actually, for the class of performance outputs considered here, we can obtain *optimal* controllers. Optimal control problems with ω as the control provide lower bounds on the optimal performance that can be achieved when u is the control variable.

LEMMA 2 *Consider the nonlinear system*

$$\dot{\rho} = G(\rho)\omega, \quad \rho(0) = \rho_0 \tag{20}$$

with ω as the control input. Let r_1 and r_2 denote two positive scalars and define $r = r_1/r_2$. The controller

$$\omega_{\text{opt}}(\rho) = -r\rho \tag{21}$$

has the following properties:

(i) The corresponding closed-loop system is globally exponentially stable about zero.

(ii) For every initial state ρ_0, the controller (21) minimizes the performance index

$$\mathcal{H}(\rho_0, \omega) := \int_0^\infty \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} \, dt \tag{22}$$

over the set of control inputs $\omega(\cdot)$ which result in $\lim_{t\to\infty} \rho(t) = 0$, and the minimum of the performance index is

$$\mathcal{H}_{\text{opt}}(\rho_0) = 2r_1 r_2 \ln(1 + \|\rho_0\|^2) \,. \tag{23}$$

Proof. To demonstrate global exponential stability of the closed loop system

$$\dot{\rho} = -rG(\rho)\rho$$

introduce the Lyapunov function candidate

$$W(\rho) = \rho'\rho.$$

From (7) it follows that the derivative of W along any solution of the closed loop system satisfies

$$\begin{aligned} \dot{W} &= -r(1 + \|\rho\|^2)\|\rho\|^2 \\ &\leq -rW. \end{aligned}$$

This guarantees global exponential stability about zero with rate of convergence $r/2$.

To demonstrate the optimality properties of controller (21), consider the positive definite function

$$V(\rho) := 2r_1 r_2 \ln(1 + \|\rho\|^2).$$

Take now any initial state ρ_0 and any control input $\omega(\cdot)$ which results in $\lim_{t\to\infty} \rho(t) = 0$. The derivative of V along the corresponding solution of system (20) satisfies (this computation makes use of (7))

$$\begin{aligned} \dot{V} &= 4r_1 r_2 (1 + \|\rho\|^2)^{-1} \rho' G(\rho)\omega \\ &= 2r_1 r_2 \rho' \omega \\ &= -r_1^2 \|\rho\|^2 - r_2^2 \|\omega\|^2 + \|r_1\rho + r_2\omega\|^2. \end{aligned}$$

Considering any time $T \geq 0$ and integrating this last equality over the interval $[0,\, T]$ yields

$$\int_0^T \left\{ r_1^2 \|\rho(t)\|^2 + r_2^2 \|\omega(t)\|^2 \right\} \, dt = V(\rho_0) - V(\rho(T)) + \int_0^T \|r_1\rho(t) + r_2\omega(t)\|^2 \, dt.$$

Since $\lim_{T\to\infty} \rho(T) = 0$, we have $\lim_{T\to\infty} V(\rho(T)) = 0$ and

$$\mathcal{H}(\rho_0, \omega) = V(\rho_0) + \int_0^\infty \left\{ \|r_1\rho(t) + r_2\omega(t)\|^2 \right\} \, dt. \tag{24}$$

The optimality properties of controller (21) now follow from (24). □

The next result shows that, for the complete system (10), one can asymptotically recover, on compact sets of initial states $[\rho_0' \ \omega_0']$, the optimal cost $\mathcal{H}_{\rm opt}(\rho_0) = 2r_1 r_2 \ln(1 + ||\rho_0||^2)$ (of the kinematics) through the original control inputs u, if the control inputs are not penalized and the feedback controller is permitted to be nonlinear.

THEOREM 1 *Consider the nonlinear system (10) with performance index $\mathcal{J}(\rho_0, \omega_0, u(\cdot))$ given by (19), where r_1 and r_2 are positive scalars. Then, given any positive number κ, the controller*

$$u = -JF(\omega)\omega - rJG(\rho)\omega - \kappa J(\omega + r\rho) \tag{25}$$

with $r = r_1/r_2$ has the following properties:

(i) The corresponding closed-loop system is globally asymptotically stable about zero.

(ii) For every initial state $(\rho_0\,, \omega_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, we have

$$\mathcal{J}(\rho_0, \omega_0, u(\cdot)) = 2r_1 r_2 \ln(1 + \|\rho_0\|^2) + \frac{1}{2\kappa} \|r_1\rho_0 + r_2\omega_0\|^2.$$

Proof. The closed-loop system, with the control law (25), is

$$\dot{\rho} = G(\rho)\omega \quad (26a)$$
$$\dot{\omega} = -rG(\rho)\omega - \kappa(\omega + r\rho) \quad (26b)$$

Consider the Lyapunov function candidate

$$V(\rho, \omega) = \frac{1}{2\kappa}\|r_1\rho + r_2\omega\|^2 + 2r_1r_2\ln(1 + \|\rho\|^2) \quad (27)$$

Taking the derivative of (27) along a trajectory of (26), and using (7), one obtains

$$\begin{aligned} \dot{V} &= -\|r_2\omega + r_1\rho\|^2 + 2r_1r_2\rho'\omega \\ &= -r_2^2\|\omega\|^2 - r_1^2\|\rho\|^2 \end{aligned} \quad (28)$$

and the closed-loop system is globally asymptotically stable about zero. This proves property (i).

To show (ii), use (27), and integrate (27) from 0 to T and take the limit as $T \to \infty$ to get

$$\begin{aligned} \mathcal{J}(\rho_0, \omega_0, u(\cdot)) &= \lim_{T\to\infty} \int_0^T \{r_1^2\|\rho(t)\| + r_2^2\|\omega(t)\|\}\, dt \\ &= V(\rho_0, \omega_0) - \lim_{T\to\infty} V(\rho(T), \omega(T)) \\ &= V(\rho_0, \omega_0). \end{aligned}$$

□

5. Main Results

We now consider a control problem for the nonlinear system (10) with a more general performance index than the one in Theorem 1. In particular, we now include a term with the control input u. Unfortunately, when the performance index is arbitrary, we cannot solve the optimal control problem. Instead, we give sufficient conditions for the solvability of the QRP problem introduced in section 2.

In section 4 we have shown that for some special cases of performance outputs, Lyapunov functions which include a logarithmic term in the kinematic parameters give rise to linear controllers and, in addition, give rise to a finite quadratic cost. We therefore anticipate that, for more general quadratic cost functionals of the form (8)-(9), the use of Lyapunov functions including a logarithmic term in the kinematic parameters, will be beneficial for the computation of the cost. Specifically, in this section we consider positive definite functions of the form

$$V(x) = \lambda \ln(1 + \|\rho\|^2) + x'Px$$

for some positive definite symmetric matrix $P \in \mathbb{R}^{6\times 6}$ and some nonnegative scalar λ, as Lyapunov function candidates for the computation of general quadratic costs of the form (8)-(9).

In order to state our main result, we need to compute a few preliminary quantities. First, note that we can write the nonlinear system (10) in the form

$$\dot{x} = A(x)x + Bu, \qquad x(0) = x_0 \tag{29a}$$
$$z = Cx + Du, \tag{29b}$$

where $x := [\rho' \; \omega']'$ and

$$A(x) := \begin{bmatrix} 0 & G(\rho) \\ 0 & F(\omega) \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix}. \tag{30}$$

Using (30) it can be shown that $A(x)$ can also be written as

$$A(x) = A_0 + \sum_{i=1}^{6} x_i A_i + B_0 x x' C_0, \tag{31}$$

where $A_0, A_1, \ldots, A_6, B_0, C_0$ are real matrices in $\mathbb{R}^{6\times 6}$, which are uniquely determined by $G(\cdot)$ and $F(\cdot)$. These matrices are fairly easy to compute and they are given in the appendix. Equation (31) shows that the matrix $A(x)$ is the sum of two parts; the first part is affine in the state x and the second is quadratic in the state x.

Let $\mathcal{B}_\infty(d)$ denote the hypercube of radius d in $\mathbb{R}^6$; i.e,

$$\mathcal{B}_\infty(d) = \left\{ x \in \mathbb{R}^6 \;\middle|\; |x_i| \le d, i = 1, 2, \ldots, 6 \right\}.$$

Compute real matrices $A_1^{\#}, \ldots, A_p^{\#}$ such that

$$\left\{ A_0 + \sum_{i=1}^{6} x_i A_i \;\middle|\; x \in \mathcal{B}_\infty(d) \right\} = \mathbf{Co}\{A_1^{\#}, \ldots, A_p^{\#}\}. \tag{32}$$

The matrices $A_1^{\#}, \ldots, A_p^{\#}$ exist because the set in the left hand side of (32) is a polytope; these matrices are given in the appendix.

The next result yields a solution to the suboptimal quadratic regulation problem for the nonlinear system (29).

THEOREM 2 *Consider the nonlinear system (29) together with the cost function*

$$\mathcal{J}(x_0, u(\cdot)) = \int_0^\infty \|Cx(t) + Du(t)\|^2 \, dt. \tag{33}$$

Suppose that $D'[C \;\; D] = [0 \;\; I_3]$. Let d denote a positive constant and let the matrices $A_1^{\#}, \ldots A_p^{\#}$, B_0, and C_0 be defined by (32) and (31). Suppose there exists a positive definite symmetric matrix $P \in \mathbb{R}^{6\times 6}$, positive scalars $\sigma_1, \ldots, \sigma_p$, and $\lambda \ge 0$ such that, for each $i = 1, \ldots, p$, we have

$$\begin{aligned} A_i^{\#\prime} P + P A_i^{\#} &+ 3d^2(\sigma_i P B_0 + \sigma_i^{-1} C_0')(\sigma_i P B_0 + \sigma_i^{-1} C_0')' \\ &- PBB'P + C'C + \lambda \Pi < 0, \end{aligned} \tag{34}$$

where

$$\Pi := \frac{1}{2}\begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix}.$$

Define the positive definite function

$$V(x) := \lambda \ln(1 + ||\rho||^2) + x' P x, \tag{35}$$

where ρ denotes the first 3 components of $x \in \mathrm{IR}^6$, and define the set

$$\Omega(\gamma) := \left\{ x \in \mathrm{IR}^6 \mid V(x) \le \gamma \right\}, \tag{36}$$

where γ is a given positive number. If $\Omega(\gamma) \subset \mathcal{B}_\infty(d)$, then the linear state-feedback control law

$$u = -B' P x$$

is such that, given any initial state $x_0 \in \Omega(\gamma)$, the resulting closed loop trajectory converges to zero, and the closed loop cost satisfies the bound

$$\begin{aligned} \mathcal{J}(x_0, -B'Px) &= \int_0^\infty ||(C - DB'P)x(t)||^2\, dt \\ &\le \lambda \ln(1 + ||\rho_0||^2) + x_0' P x_0 \le \gamma. \end{aligned} \tag{37}$$

The intuition behind this theorem is as follows. If the matrix inequalities in (33) hold, one can show, using the Lyapunov function (35), that the control $u = -B'Px$ asymptotically stabilizes the nonlinear system (29) whenever the initial state belongs to an invariant set of closed loop trajectories contained in $\mathcal{B}_\infty(d)$. The set $\Omega(\gamma)$ is one such invariant set because $x(t) \in \Omega(\gamma)$ implies $x(t) \in \mathcal{B}_\infty(d)$; hence, $\dot{V}(x(t)) \le 0$. Moreover, the same Lyapunov function can be used to show the performance bound in (36) by a simple "completion of squares" argument. To formalize this intuitive argument we shall make use of the following analysis result, which holds for arbitrary nonlinear systems. For the sake of continuity, this result is proven in the appendix.

LEMMA 3 *Consider the autonomous nonlinear system*

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{38a}$$

$$z = h(x) \tag{38b}$$

where $x \in \mathrm{IR}^n$ is the state vector and $z \in \mathrm{IR}^p$ is output vector. Suppose that $f(\cdot)$ and $h(\cdot)$ are continuous vector fields. Suppose also that there exists a positive scalar γ and a positive definite, continuously differentiable, function $V(\cdot)$ defined on IR^n, such that $\zeta \in \Omega(\gamma) = \{\zeta \in \mathrm{IR}^n \,|\, V(\zeta) \le \gamma\}$ and $\zeta \ne 0$ imply

$$V_x(\zeta) f(\zeta) + h'(\zeta) h(\zeta) < 0. \tag{39}$$

Then, $f(0) = 0$ and $x_0 \in \Omega(\gamma)$ implies that (38) has a solution with $x(0) = x_0$ and every solution with this initial state converges to zero asymptotically and satisfies

$$\int_0^\infty \|z(t)\|^2 \, dt \leq V(x_0) \leq \gamma.$$

Proof of Theorem 2. Define the state-feedback gain $K = -B'P$. Then, inequality (33) is equivalent to

$$(A_i^\# + BK)'P + P(A_i^\# + BK) + 3d^2(\sigma_i PB_0 + \sigma_i^{-1}C_0')(\sigma_i PB_0 + \sigma_i^{-1}C_0')' \\ +(C + DK)'(C + DK) + \lambda\Pi < 0. \tag{40}$$

We will show that the satisfaction of (40), for $i = 1, \ldots, p$, implies that for any vector $\zeta \in \mathcal{B}_\infty(d)$ the following matrix inequality holds

$$(A(\zeta) + BK)'P + P(A(\zeta) + BK) + (C + DK)'(C + DK) + \lambda\Pi < 0, \tag{41}$$

where $A(\cdot)$ is the matrix function defined in (30). To see this note that the set of symmetric matrices

$$\mathbf{Co}\Big\{(A_i^\# + BK)'P + P(A_i^\# + BK) \\ +(C + DK)'(C + DK) + \lambda\Pi \;\Big|\; i = 1, \ldots, p\Big\} \tag{42}$$

is equal to the set

$$\Big\{\left(A_0 + \textstyle\sum_{i=1}^6 \zeta_i A_i + BK\right)' P + P\left(A_0 + \textstyle\sum_{i=1}^6 \zeta_i A_i + BK\right) \\ +(C + DK)'(C + DK) + \lambda\Pi \;\Big|\; \zeta \in \mathcal{B}_\infty(d)\Big\}.$$

Hence, from Lemma 5 in the appendix, we get that (40) holds, for some $\sigma_1, \ldots, \sigma_p > 0$, if and only if

$$\Big(A_0 + \textstyle\sum_{i=1}^6 \zeta_i A_i + B_0 qq' C_0 + BK\Big)' P + P\Big(A_0 + \textstyle\sum_{i=1}^6 \zeta_i A_i + B_0 qq' C_0 + BK\Big) \\ +(C + DK)'(C + DK) + \lambda\Pi < 0$$

for each $\zeta \in \mathcal{B}_\infty(d)$ and $q \in \mathcal{B}_2(\sqrt{6}d)$. Since $\mathcal{B}_\infty(d) \subset \mathcal{B}_2(\sqrt{6}d)$, we conclude that (41) holds for all $\zeta \in \mathcal{B}_\infty(d)$.

To complete the proof, take $\zeta \in \mathcal{B}_\infty(d)$ and consider the function $V(\cdot)$ defined in (35). Using property (7), a simple calculation yields

$$V_x(\zeta)(A(\zeta) + BK)\zeta = 2\zeta' P(A(\zeta) + BK)\zeta + \lambda\zeta'\Pi\zeta.$$

From this equation and (41) we get that, given any nonzero $\zeta \in \mathcal{B}_\infty(d)$,

$$V_x(\zeta)(A(\zeta) + BK)\zeta + \|(C + DK)\zeta\|^2 < 0.$$

Since the level set $\Omega(\gamma)$ is contained in $\mathcal{B}_\infty(d)$, we conclude from Lemma 3 that the assertion of the theorem holds. □

6. Numerical Solution of the QRP

Let $\mathcal{C}$ denote the bounded set of initial states where the QRP should be solved. Then, the best suboptimal controller that results from Theorem 2 is obtained by solving

$$\boxed{\begin{aligned}\gamma_{\rm opt} \;=\; & \inf_{(\gamma,\lambda,\sigma_1,\ldots,\sigma_p,P)}\ \gamma \\ & \text{subject to } \lambda \geq 0,\ \sigma_1 > 0,\ldots,\sigma_p > 0, \\ & \qquad P = P' > 0,\ \text{and (33)} \\ & \qquad \mathcal{C} \subset \Omega(\gamma) \subset \mathcal{B}_\infty(d)\end{aligned}} \tag{43}$$

where $\Omega(\gamma)$ is defined in (36). Indeed, the state-feedback matrix $K = -B'P_{\rm opt}$, where $P_{\rm opt}$ denotes a solution to (43), stabilizes the origin, with $\mathcal{C}$ in the region of attraction, and guarantees that the quadratic performance index (33) is bounded by $\gamma_{\rm opt}$ for all initial states in $\mathcal{C}$.

It turns out that (43) does not exhibit any convexity properties that can be exploited to compute a global solution. To see this, suppose that all optimization variables except P are fixed. Then the matrix inequalities (33) cannot be made convex in P due to the presence of an *indefinite* quadratic term in P; similarly, if we write (33) in terms of P^{-1}, the presence of an *indefinite* quadratic term in P^{-1} shows that (33) is not convex in P^{-1} either. Below we will give an iterative method for finding suboptimal solutions to (43). This method can be implemented by solving a sequence of Linear Matrix Inequalities (LMIs); each LMI problem can be solved efficiently [3, 9].

Notice first that, with $K = -B'P$, we can write each matrix inequality in (33) as

$$\begin{aligned}(A_i^{\#} + BK)'P + P(A_i^{\#} + BK) + 3d^2(\sigma_i PB_0 + \sigma_i^{-1}C_0')(\sigma_i PB_0 + \sigma_i^{-1}C_0')' \\ +(C+DK)'(C+DK) + \lambda\Pi < 0.\end{aligned} \tag{44}$$

Fix $\gamma > 0$ and K. It follows that there exist $\lambda \geq 0$, positive numbers $\sigma_1,\ldots,\sigma_p$, and $P = P' > 0$, such that (44) holds and

$$\mathcal{C} \subset \Omega(\gamma) \subset \mathcal{B}_\infty(d) \tag{45}$$

if and only if there exist $\beta_0 \geq 0$, positive numbers $\beta_1,\ldots,\beta_p$, and $X = X' > 0$ such that

$$\begin{aligned}(A_i^{\#} + BK)'X + X(A_i^{\#} + BK) + 3d^2\beta_i^{-1}(XB_0 + \beta_i C_0')(XB_0 + \beta_i C_0')' \\ +\gamma^{-1}(C+DK)'(C+DK) + \beta_0\Pi < 0\end{aligned} \tag{46}$$

and

$$\mathcal{C} \subset \Phi \subset \mathcal{B}_\infty(d) \tag{47}$$

where

$$\Phi := \left\{ x \in \mathbb{R}^6 \;\middle|\; \beta_0 \ln(1 + \|\rho\|^2) + x'Xx \leq 1 \right\}. \tag{48}$$

To show this equivalence, introduce the change of variable $P = \gamma X$, $\beta_0 = \lambda/\gamma$, and $\beta_i = 1/(\sigma_i^2\gamma)$.

Introducing the change of variables $\alpha = \gamma^{-1}$, and using the Schur complement formula, (46) is equivalent to

$$\boxed{\begin{bmatrix} (A_i^{\#} + BK)'X + X(A_i^{\#} + BK) + \\ \alpha(C+DK)'(C+DK) + \beta_0 \Pi & XB_0 + \beta_i C_0' \\ B_0'X + \beta_i C_0 & -\frac{1}{3d^2}\beta_i I \end{bmatrix} < 0.} \tag{49}$$

Finally, from the equivalence between the pair of conditions (44)-(45) and the pair of conditions (46)-(47), we get that the optimization problem (43) is equivalent to

$$\begin{aligned} \gamma_{\mathrm{opt}}^{-1} \;=\; & \sup_{(\alpha,\beta_0,\beta_1,\ldots,\beta_p,X,K)} \alpha \\ & \text{subject to } \beta_0 \geq 0, X = X' > 0, \beta_i > 0 \\ & \qquad (49) \text{ holds for } i = 1,\ldots,p, \\ & \qquad \text{the set inclusion (47) holds,} \\ & \qquad \text{and } K = -\frac{1}{\alpha}B'X. \end{aligned} \tag{50}$$

We will now show how to compute a suboptimal solution to (49) by solving a sequence of LMI problems when the set $\mathcal{C}$ in (47) is the polytope given by

$$\mathcal{C} = \mathbf{Co}\{v_1, \ldots, v_r\} \tag{51}$$

for some vectors $v_1, \ldots, v_r$ in $\mathbb{R}^6$.

First we show that, if certain LMIs in the variables β_0 and X hold, the set inclusion in (47) holds. Consider first the inclusion $\mathcal{C} \subset \Phi$. Suppose that, given $\beta_0 \geq 0$, $X > 0$ and $h = 1, \ldots, r$, we have

$$\boxed{\beta_0 \; \|[I_3 \; 0]v_h\|^2 + v_h' X v_h \leq 1.} \tag{52}$$

Then, since $\mathcal{C}$ is a polytope and

$$\Phi_- := \left\{ x \;\middle|\; \beta_0 \; \|[I_3 \; 0]x\|^2 + x'Xx \leq 1 \right\}$$

is convex, we get $\mathcal{C} \subset \Phi_-$. Since $\ln(1+\theta^2) \leq \theta^2$ for all $\theta \in \mathbb{R}$, we obtain $\Phi_- \subset \Phi$. Hence, if (52) holds, $\mathcal{C} \subset \Phi$. The important point is that (52) is affine in β_0 and X.

Now, to enforce $\Phi \subset \mathcal{B}_\infty(d)$ we use

$$\boxed{d^2 X - e_s e_s' \geq 0,} \tag{53}$$

for $s = 1, \ldots, 6$, where e_s denotes the vector in $\mathbb{R}^6$ with zeros everywhere except for the component s, which is equal to one. To show that this condition implies $\Phi \subset \mathcal{B}_\infty(d)$, take $x \in \Phi$. Since $\beta_0 \geq 0$ we get $x'Xx \leq 1$. If Φ_+ denotes the ellipsoid

$$\Phi_+ := \{x \mid x'Xx \leq 1\}$$

then clearly, $\Phi \subset \Phi_+$; hence, it suffices to show $\Phi_+ \subset \mathcal{B}_\infty(d)$. If (53) holds and $x \in \Phi_+$, then

$$d^2 - |e_s' x|^2 \geq 0.$$

This gives $|e_s' x| \leq d$; since s is arbitrary, it follows that $x \in \mathcal{B}_\infty(d)$, thus $\Phi_+ \subset \mathcal{B}_\infty(d)$.

From these (conservative) characterizations of the set inclusion (47), we get

$$\boxed{\begin{array}{lll} \gamma_{\mathrm{opt}}^{-1} & \geq & \sup_{(\alpha,\beta_0,\beta_1,\ldots,\beta_p,X,K)} \quad \alpha \\ & & \text{subject to } \beta_0 \geq 0, X = X' > 0, \beta_i > 0 \\ & & \qquad\qquad \text{(49) holds for } i = 1,\ldots,p, \\ & & \qquad\qquad \text{(52) holds for } h = 1,\ldots,r, \\ & & \qquad\qquad \text{(53) holds for } s = 1,\ldots,6, \\ & & \qquad\qquad \text{and } K = -\frac{1}{\alpha}B'X. \end{array}} \tag{54}$$

For fixed K, (54) is a convex LMI problem in the variables $(\alpha, \beta_0, \beta_1, \ldots, \beta_p, X)$. Further, for fixed $(\alpha, \beta_0, \beta_1, \ldots, \beta_p, X)$, satisfying all the constraints in (54) but the last one, a new gain can be generated according to the formula

$$K_{\mathrm{new}} = -\frac{1}{\alpha}B'X \tag{55}$$

such that $(\alpha, \beta_0, \beta_1, \ldots, \beta_p, X, K_{\mathrm{new}})$ is a feasible solution for (54). To see this, substitute (55) (for K) in (49) and note that (52) and (53) are independent of K. Hence, suboptimal controller gains for (54) can be obtained by iteratively computing (α, X), together with $\beta_0, \beta_1, \ldots, \beta_p$, and then computing K from (55). This is summarized in the following algorithm.

The $(\alpha, X) - K$ iteration

1. Choose d with $\mathcal{C} \subset \mathcal{B}_\infty(d)$ and compute the data necessary to write down the LMIs (49), (52), and (53).
2. Compute K_0 the solution of the LQR problem corresponding to the linearized (about $x = 0$) system. (A unique K_0 exists.) Set the iteration index $\ell = 0$ and goto 3.
3. Fix $K = K_\ell$ in (54) and solve it (without the last equality constraint) to obtain (α_ℓ, X_ℓ).
4. Compute $K_{\ell+1} = -\alpha_\ell^{-1}B'X_\ell$. Stop if $\|K_{\ell+1} - K_\ell\|_m$ is less than a specified tolerance; otherwise, set $\ell = \ell + 1$ and goto 3. (Here $\|\cdot\|_m$ denotes a matrix norm.)

6.1. A Quadratic Lyapunov Function Approach

When $\lambda = 0$ in (33), and when the set $\mathcal{C}$ is a polytope, the optimization problem (43) is equivalent to a *single* convex programming problem. In this special case, there is no need for iterations and, an optimal controller gain $K = -B'P_{\rm opt}$, where $P_{\rm opt}$ is a solution to (43) can be derived "in one shot." Notice that the case $\lambda = 0$ corresponds to bounding the cost (33) using Lyapunov functions of the form

$$V(x) = x'Px.$$

This motivates us to designate this approach as a "quadratic Lyapunov function approach."

Obviously, when $\lambda = 0$ in (43) the resulting upper bound $\gamma_{\rm opt}$ for the quadratic cost will be, in general, larger than the optimal upper bound obtained without this constraint. This is because $\lambda = 0$ need not be optimal for (43). This may suggest that solving the problem with $\lambda = 0$ makes no sense. Nevertheless, the fact that

- in general we cannot solve (43) exactly, and
- if $\lambda = 0$, (43) is equivalent to a convex program, which can be solved exactly,

justify the quadratic Lyapunov function approach. In fact, there could be problems for which the results with $\lambda = 0$ could be better than without such constraint in the sense that, with $\lambda = 0$, one may obtain controller gains yielding a smaller quadratic cost over the same set of initial states. This is, of course, problem dependent.

Enforcing $\lambda = 0$ in (43) yields the following optimization problem

$$\boxed{\begin{aligned} \gamma_{\rm opt} \;=\; & \inf_{(\gamma,\sigma_1,\ldots,\sigma_p,P)} \;\gamma \\ & \text{subject to } \lambda = 0,\ \sigma_1 > 0, \ldots, \sigma_p > 0 \\ & \qquad P = P' > 0, \text{ and (33)} \\ & \qquad \mathcal{C} \subset \left\{ x \in \mathbb{R}^6 \mid x'Px \le \gamma \right\} \subset \mathcal{B}_\infty(d). \end{aligned}} \tag{56}$$

As before, the state-feedback gain $K = -B'P_{\rm opt}$, where $P_{\rm opt}$ denotes a solution to (56), stabilizes the nonlinear system, with the set $\mathcal{C}$ in the region of attraction, and guarantees that the quadratic performance index is bounded by $\gamma_{\rm opt}$ for all initial states in $\mathcal{C}$.

To get a convex program, equivalent to (56), we proceed as follows. Given any $\gamma > 0$, simple algebra and the Schur complement formula yield that, the positive numbers $\sigma_1, \ldots, \sigma_p$, and $P = P' > 0$, satisfy (33) if and only if the positive numbers $\beta_1 = \gamma\sigma_1^2, \ldots, \beta_p = \gamma\sigma_p^2$, and the positive definite matrix $X = \gamma P^{-1}$, satisfy

$$\boxed{\begin{bmatrix} A_i^{\#}X + XA_i^{\#'} - \gamma BB' & \beta_i B_0 + XC_0' & XC' \\ \beta_i B_0' + C_0 X & -\frac{1}{3d^2}\beta_i I & 0 \\ CX & 0 & -\gamma I \end{bmatrix} < 0} \tag{57}$$

for $i = 1, \ldots, p$. Therefore, we may write

$$\begin{aligned} \gamma_{\rm opt} \;=\; & \inf_{(\gamma,\sigma_1,\ldots,\sigma_p,X)} \;\gamma \\ & \text{subject to } X = X' > 0, \text{ and (57)} \\ & \qquad \mathcal{C} \subset \left\{ x \in \mathbb{R}^6 \mid x'X^{-1}x \le 1 \right\} \subset \mathcal{B}_\infty(d). \end{aligned}$$

If the set $\mathcal{C}$ is given by the polytope (51) the constraint $\mathcal{C} \subset \left\{x \in \mathbb{R}^6 \mid x'X^{-1}x \leq 1\right\}$ holds if and only if, for $h = 1, \ldots, r$ we have $v_h \in \left\{x \in \mathbb{R}^6 \mid x'X^{-1}x \leq 1\right\}$. Since $X = X' > 0$, this last condition is equivalent to

$$\boxed{v_h v_h' - X \leq 0.} \tag{58}$$

Similarly, the constraint $\left\{x \in \mathbb{R}^6 \mid x'X^{-1}x \leq 1\right\} \subset \mathcal{B}_\infty(d)$ holds if and only if, for $s = 1, \ldots, 6$, we have

$$\boxed{e_s' X e_s - d^2 \leq 0.} \tag{59}$$

These results lead to the following *exact* formula for the optimization problem (56):

$$\boxed{\begin{aligned} \gamma_{\rm opt} \;=\; & \inf_{(\gamma, \beta_1, \ldots, \beta_p, X)} \; \gamma \\ & \text{subject to } X = X' > 0, \beta_i > 0 \\ & \qquad\qquad \text{(57) holds for } i = 1, \ldots, p, \\ & \qquad\qquad \text{(58) holds for } h = 1, \ldots, r, \\ & \qquad\qquad \text{(59) holds for } s = 1, \ldots, 6. \end{aligned}} \tag{60}$$

Notice that (60) is a convex LMI problem in the variables $(\gamma, \beta_1, \ldots, \beta_p, X)$ and therefore it can be solved very efficiently. Once $\gamma_{\rm opt}$ and $X_{\rm opt}$ are known, an optimal controller gain can be computed from

$$K = -\gamma_{\rm opt} B' X_{\rm opt}^{-1}. \tag{61}$$

7. Numerical Example

In this section we provide a numerical example which illustrates the previous theoretical results. The results of this paper are most beneficial for systems exhibiting large angular motions and large angular velocities; i.e., rigid bodies subject to nonlinear behavior. A new class of spacecraft, recently promoted by NASA, exhibits this behavior. This new "smaller, cheaper" series of satellites is a radical departure from the traditional philosophy; they are mission-specific, designed to perform only a handful of scientific experiments at a time [10]. These new spacecraft are indeed small; while conventional satellites have moments of inertia in the order of $10^4 - 10^5\, kg \cdot m^2$, this new family of satellites has moments of inertia in the order of $10\, kg \cdot m^2$. Because of their size, these small satellites can exhibit large angular motions and large angular velocities in response to external disturbances. The attitude control system has to address the possibility of the satellite entering the nonlinear region.

To demonstrate the efficacy of the design method in section 6 we consider a rigid-body with principal moments of inertia given by

$$J_1 = 15\, kg \cdot m^2, \qquad J_2 = 22\, kg \cdot m^2, \qquad J_3 = 17\, kg \cdot m^2 \tag{62}$$

These values approximately correspond to the SAMPEX satellite. The SAMPEX (Solar, Anomalous and Magnetospheric Particle Explorer) satellite is the first of the Small Explorer (SMEX) series of spacecraft. The actual SAMPEX control system uses three magnetic torquer bars and a pitch axis momentum wheel as control actuators. For the purposes of illustrating the theory, however, we will assume that control thrusters are used instead.

The output matrices C and D where chosen as

$$C = \begin{bmatrix} 2.3I_3 & 0 \\ 0 & 4I_3 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ I_3 \end{bmatrix}.$$

In this example, the set in which the QRP is to be solved is taken to be $\mathcal{C} = \mathcal{B}_\infty(v)$; i.e., a hypercube of length v in $\mathbb{R}^6$, where the positive number v should be as large as possible in order to increase the domain of validity of the controller. For the hypercube $\mathcal{B}_\infty(d)$ (see step 1 in the (α, X)–K iteration) we choose $d = 1$. The largest value of v, such that the problem was found to be feasible, is $v = 0.08$. As shown in the simulations below, the actual domain of validity may be larger than the one predicted by the analysis.

Two controllers were constructed and compared. The first controller is the LQR controller derived using the linearized equations. The second controller was derived using the (α, X)-K iteration algorithm of section 6. We denote this controller QRP, which stands for the problem (Quadratic Regulator Problem) it solves.

The solution to the minimization problem (54), required in step 3 of the (α, X)–K iteration, was obtained using the LMI solver LMITOOL [8]. This software provides an interface between the semi-definite optimization package SP developed by Vandenberghe and Boyd [17] with MATLAB.

The resulting controllers for the two cases are

$$K_{\text{lqr}} = \begin{bmatrix} -2.2918 & 0 & 0 & -7.0977 & 0 & 0 \\ 0 & -2.2918 & 0 & 0 & -8.1499 & 0 \\ 0 & 0 & -2.2918 & 0 & 0 & -7.4136 \end{bmatrix} \tag{63}$$

$$K_{\text{qrp}} = \begin{bmatrix} -3.9816 & 0 & 0 & -35.1367 & 0 & 0 \\ 0 & -4.2430 & 0 & 0 & -37.2046 & 0 \\ 0 & 0 & -4.0505 & 0 & 0 & -35.6869 \end{bmatrix} \tag{64}$$

The smallest value for γ for the QRP controller we achieved is

$$\gamma_{\text{qrp}} = 18.6957. \tag{65}$$

This is an upper bound on the largest quadratic cost, over all initial states in $\mathcal{C} = \mathcal{B}_\infty(0.08)$ that the controller (64) delivers.

Simulations were carried out for several initial states in the box $\mathcal{C} = \mathcal{B}_\infty(0.08)$, as well as initial states outside $\mathcal{C}$ but inside $\mathcal{B}_\infty(1)$. For initial states in the box $\mathcal{C} = \mathcal{B}_\infty(0.08)$, the upper bound (65) for the cost was satisfied. Below only simulations with initial states outside $\mathcal{C} = \mathcal{B}_\infty(0.08)$ are shown. We consider two cases: small initial states and large initial state.

7.1. Small initial states.

Figures 1 through 4 depict simulation results for the initial states

$$\rho_i = 0.1 \qquad \omega_i = 0.1\, rad/sec \qquad (i = 1, 2, 3)$$

In Figure 1 we show the running cost (the quadratic cost from time zero to t) for the two controllers. This figure shows that the LQR cost is the smallest; hence, for this initial state one may conjecture that the system is almost in the linear domain. Figures 2 and 3 show the trajectories for the LQR and the QRP controllers. In those figures we have plotted the time history of the kinematic parameter vector ρ and the angular velocity vector ω. Only the first component is shown here since the other components exhibit similar behavior. Notice that with the QRP controller the dynamics and kinematics show less overshoot than the with LQR controller. Figure 4 shows the corresponding control histories. bsequent ones, correspond to the following initial states:

$$\rho_i = 1 \qquad \omega_i = 0.75\, rad/sec \qquad (i = 1, 2, 3)$$

The trajectories are shown in Figures 6 and 7. The control history is shown in Figure 8. The superior behavior of the kinematic and dynamic variables, and the larger control torques, of the QRP controller are evident from these plots.

These numerical results indicate that the controller $K_{\rm qrp}$ is rather conservative, in the sense that the actual value of the cost is, in general, much lower than the one predicted by the algorithm. Moreover, although the algorithm guarantees that the actual cost will be smaller than $\gamma_{\rm qrp}$ for all initial states inside the hypercube $\mathcal{C} = \mathcal{B}_\infty(0.08)$ the simulations showed that the controller performed well for initial states outside this set.

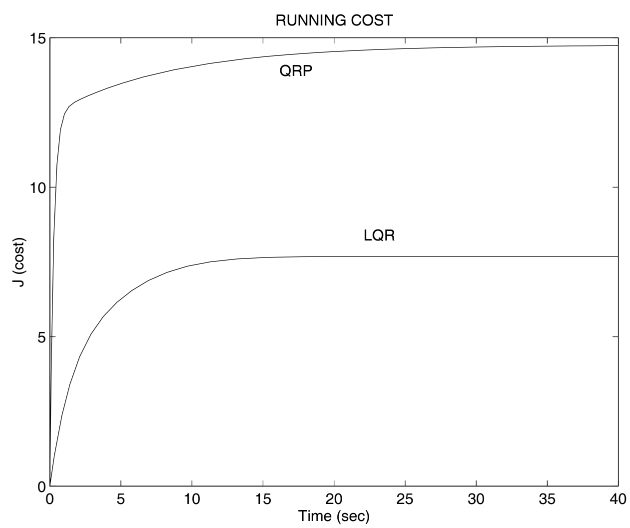

Figure 1. Running cost for the two controllers (small initial states).

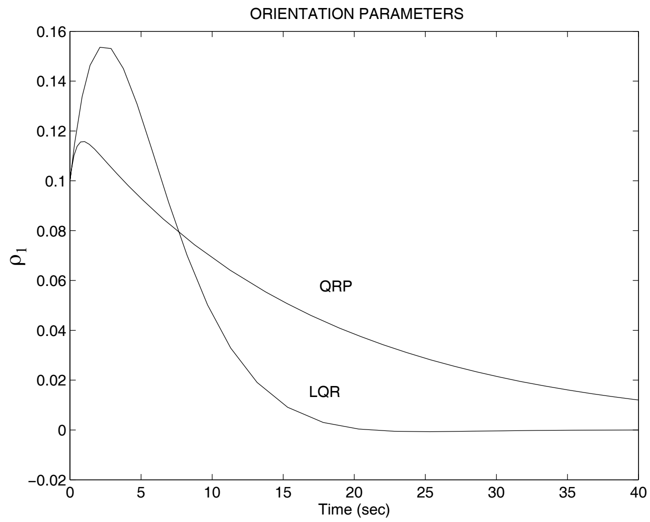

Figure 2. Kinematic parameters (small initial states).

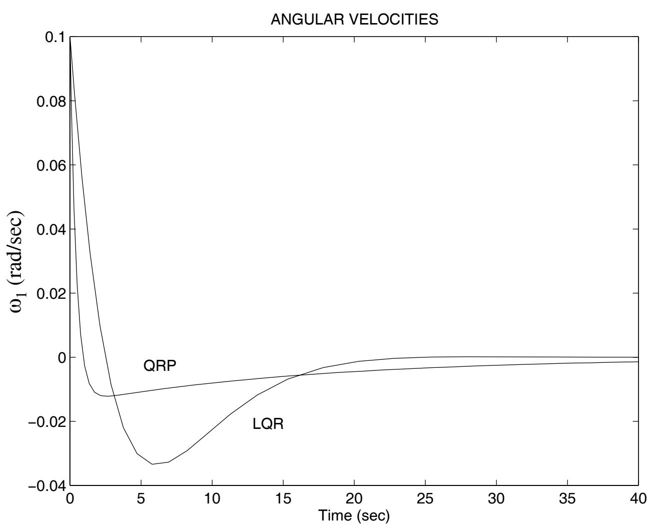

Figure 3. Angular velocities (small initial states).

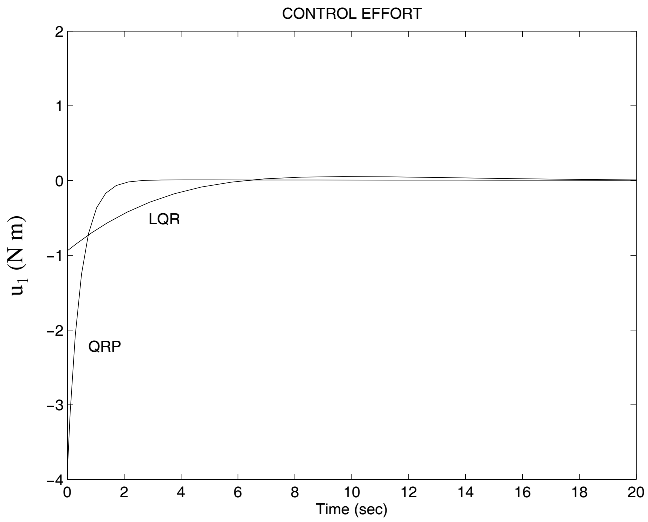

Figure 4. Control histories (small initial states).

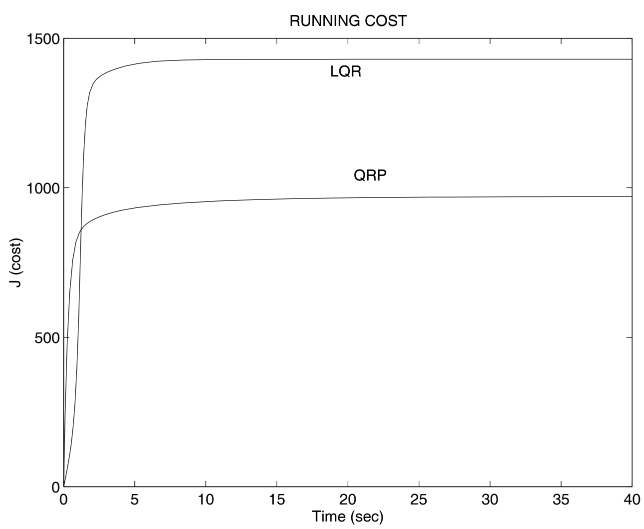

Figure 5. Running cost for the two controllers (large initial states).

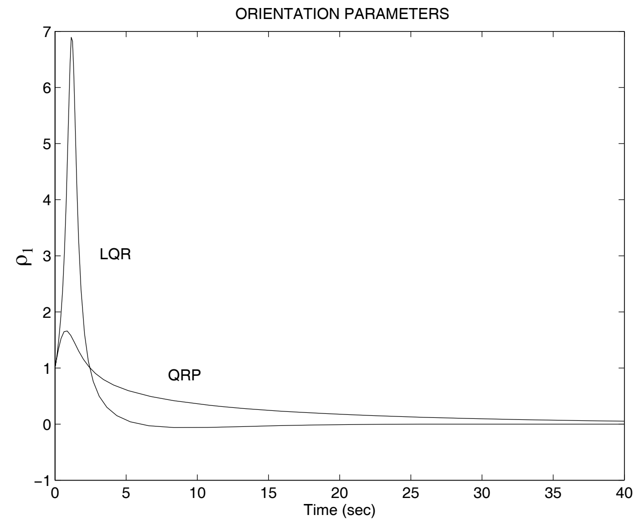

Figure 6. Kinematic parameters (large initial states).

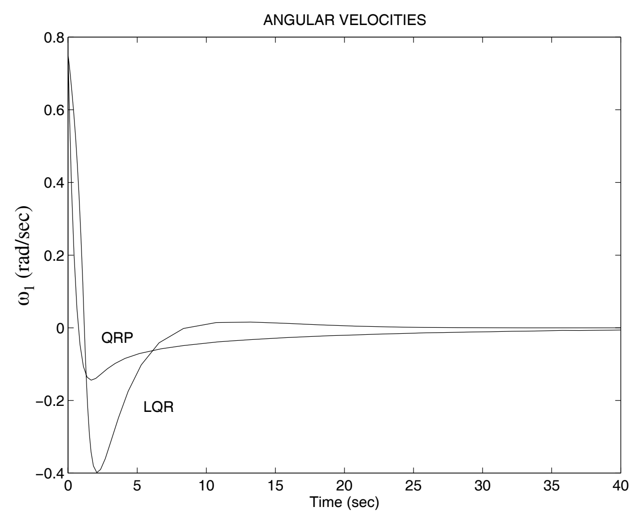

Figure 7. Angular velocities (large initial states).

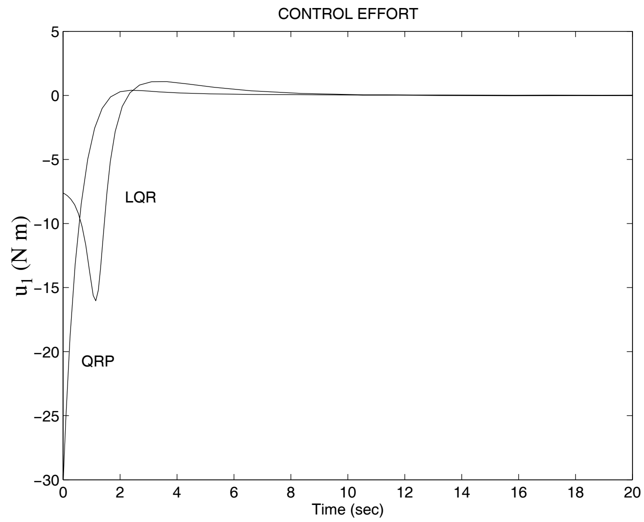

Figure 8. Control histories (large initial states).

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Appendix

The matrices in equation (31) are given by

$$A_0 := \frac{1}{2}\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

$$A_1 := \frac{1}{2}\begin{bmatrix} 0 & \Sigma_1 \\ 0 & 0 \end{bmatrix}, \quad A_2 := \frac{1}{2}\begin{bmatrix} 0 & \Sigma_2 \\ 0 & 0 \end{bmatrix}$$

$$A_3 := \frac{1}{2}\begin{bmatrix} 0 & \Sigma_3 \\ 0 & 0 \end{bmatrix}, \quad A_4 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_1 J_1 \end{bmatrix},$$

$$A_5 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_2 J_2 \end{bmatrix}, \quad A_6 := \begin{bmatrix} 0 & 0 \\ 0 & J^{-1}\Sigma_3 J_3 \end{bmatrix}$$

where

$$\Sigma_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Sigma_2 := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$\Sigma_3 := \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$B_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad C_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

The matrices $A_1^{\#}, \ldots, A_p^{\#}$ required in equation (32) are given by

$$A_k^{\#} := A_0 + \sum_{i=1}^{6} \hat{d}_{k,i} A_i$$

where $\hat{d}_{k,i}$ denotes the i-th component of the k-th $(1 \leq k \leq p = 2^6)$ vertex of the 6-dimensional cube $\mathcal{B}_\infty(d)$.

Proof of Lemma 3 The continuity of $f(\cdot)$ implies that, given any initial state x_0, there exists $t^* > 0$ such that a solution exists in the interval $[0, t^*]$. Using condition (39) and a standard Lyapunov argument we may now conclude that any solution $x(\cdot)$, starting from $x_0 \in \Omega(\gamma)$, remains in the set $\Omega(\gamma)$. Since this set is bounded, we get that solutions starting in this set are bounded; hence, any solution starting in $\Omega(\gamma)$ exists for all $t \geq 0$ and remains in $\Omega(\gamma)$.

Next, we show that any solution starting from $x_0 \in \Omega(\gamma)$ converges to zero. First note that, under the assumptions, a standard Lyapunov stability argument (e.g., Theorem 3.1, p. 101 in [13]) proves asymptotic stability of the zero state $x = 0$. Moreover, since condition (39) holds for all nonzero $\zeta \in \Omega(\gamma)$ we may conclude that $\Omega(\gamma)$ is an invariant set of trajectories contained in the region of attraction. This shows that any solution starting from $x_0 \in \Omega(\gamma)$ converges to zero asymptotically.

Note that, given any $t \geq 0$ and any initial state $x_0 \in \Omega(\gamma)$, inequality (39) and the invariance of $\Omega(\gamma)$ yield

$$\dot{V}(x(t)) = V_x(x(t)) f(x(t)) \leq -h'(x(t)) h(x(t)),$$

where $x(\cdot)$ denotes a trajectory starting at x_0. Integrating this last inequality, one obtains

$$V(x(T)) - V(x_0) \leq -\int_0^T \|z(t)\|^2 \, dt$$

Taking the limit as $T \to \infty$ yields

$$\int_0^\infty \|z(t)\|^2 \, dt \leq V(x_0)$$

which completes the proof. □

LEMMA 4 *Given any $v, z \in \mathbb{R}^n$ and $\delta > 0$, we have the following identity:*

$$\max_{||q|| \leq \sqrt{\delta}} v'qq'z = \frac{\delta}{2}(z'v + ||z|| \, ||v||) \tag{A.1}$$

Proof. Notice first that

$$\max_{||q|| \leq \sqrt{\delta}} v'qq'z = \max_{||q|| \leq \sqrt{\delta}} q'vz'q = \max\left\{0, \frac{\delta}{2}\lambda_{\max}(vz' + zv')\right\} \tag{A.2}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue. An easy calculation shows that, if $n > 1$, $\lambda_{\max}(vz' + zv') = z'v + ||z|| \, ||v|| \geq 0$. Thus, using (A.2)

$$\max_{||q|| \leq \sqrt{\delta}} v'qq'z = \frac{\delta}{2}(z'v + ||z|| \, ||v||). \tag{A.3}$$

The maximum is attained for $q = \sqrt{\delta}\mathrm{v}/||\mathrm{v}||$, where v is an eigenvector corresponding to the maximum eigenvalue of the matrix $vz' + zv'$. If $n = 1$, (A.3) also holds because with v and z scalars we have $zv + |z||v| = \max\{0, 2zv\}$. □

LEMMA 5 *Let $S_1, \ldots, S_h$ denote real symmetric matrices in $\mathbb{R}^{n \times n}$. Let M_1 and M_2 denote real matrices in $\mathbb{R}^{m \times n}$. The following statements are equivalent:*

1. *For all $S \in \mathbf{Co}\,\{S_1, \ldots, S_h\}$ and $q \in \mathcal{B}_2(\sqrt{\delta})$, we have*

$$S + M_1'qq'M_2 + M_2'qq'M_1 < 0.$$

2. *There exist positive numbers $\sigma_1, \ldots, \sigma_h$ such that for $i = 1, \ldots, h$ we have*

$$S_i + \frac{\delta}{2}(\sigma_i M_1 + \sigma_i^{-1} M_2)'(\sigma_i M_1 + \sigma_i^{-1} M_2) < 0.$$

Proof. The first condition in the lemma holds if and only if, given any $i = 1, \ldots, h$, $q \in \mathcal{B}_2(\sqrt{\delta})$, and a real vector $x \neq 0$, we have

$$x'S_i x + 2x'M_1'qq'M_2 x < 0. \tag{A.4}$$

Condition (A.4) is equivalent to the satisfaction of

$$x'S_i x + 2 \max_{\|q\| \leq \sqrt{\delta}} (x'M_1'qq'M_2 x) < 0, \tag{A.5}$$

for $i = 1, \ldots, h$ and any real vector $x \neq 0$. Hence, from Lemma 4, we conclude that the first condition in the lemma holds if and only if

$$x'S_i x + \delta(x'M_1'M_2 x + \|M_1 x\|\,\|M_2 x\|) < 0, \tag{A.6}$$

for $i = 1, \ldots, h$ and any real vector $x \neq 0$. It now follows from [15] that condition (A.6) is equivalent to the existence of positive numbers $\sigma_1, \ldots, \sigma_h$ such that for any $x \neq 0$ and $i = 1, \ldots, h$, we have

$$x'S_i x + \delta(x'M_1'M_2 x + \frac{\sigma_i^2}{2}\|M_1 x\|^2 + \frac{1}{2\sigma_i^2}\|M_2 x\|^2) < 0.$$

A simple completion of squares shows that this last condition is equivalent to the second condition in the lemma. □

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