On the Choice of Coordinates for Control Problems on SO(3)

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Abstract

Many alternative coordinate choices exist for the description of the rotational motion of a rigid body. Not all the choices are equivalent with respect to their domain of validity for accurate attitude representation or the ease they offer in the control design process. In this paper we present two choices of coordinates which seem to be appropriate for two important control problems: the stabilization of a rigid body with reduced control actuation and stabilization with limited sensor information. Both coordinate sets have the common characteristic of introducing a passive map from the angular velocities to the attitude coordinates.

1. Introduction

In most control problems the choice of coordinates has a profound effect on the design process and the final properties of the proposed control law. This is true even for linear systems where an insightful choice of coordinates is often the key for successful control law design [6, 11]. For nonlinear systems (which generically live on manifolds) the choice of coordinates is even more crucial. This is clearly evident from the recent advances in nonlinear control design methodologies using geometric methods (e.g., feedback linearization) where the search for a diffeomorphic transformation (i.e., coordinate change) is perhaps the most difficult step the control engineer has to face during the control design process [10]. Coordinate neighborhoods (charts) are then used to translate operations on manifolds to operations on the euclidean space $\mathbb{R}^n$.

In this paper we address the coordinate choice problem for a particular nonlinear system which often appears in control applications. Namely, the rotational motion of a rigid body. Although there is a plethora of valid attitude representations commonly used by both scientists and engineers alike, their effect on analysis, stabilization and control problems varies a great deal. General statements about the relative merit of each coordinate choice cannot be made easily, since the ramifications of each choice depends on the particular problem at hand. We provide a preliminary discussion on certain coordinates for the attitude kinematics which can be very useful for control applications. In particular, we investigate two coordinate sets with the following attractive properties: the associated singularity of the coordinate set is moved as far from the equilibrium as possible; in addition, the coordinates introduce a passive map from the angular velocity vector in body frame to the kinematic variables.

A number of reasons motivates our close examination of the equations of the motion of a rotating rigid body. First, it is probably the most widely encountered nonlinear system in applications. Second, it has enough geometric structure which allows for the derivation of certain elegant results. And third, it illustrates the close connection between dynamical system analysis and nonlinear control.

2. The Configuration Space SO(3)

The orientation of a rigid body which rotates freely in space can be uniquely described by an orthogonal matrix R with positive determinant. Therefore, the configuration manifold of the motion is the special orthogonal group $SO(3)$. The time evolution of the matrix R on $SO(3)$ gives rise to the *attitude kinematics* of the motion

$$\dot{R} = S(\omega)R \tag{1}$$

where $S(\cdot)$ is the skew-symmetric matrix

$$S(\omega) := \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{2}$$

and $\omega := (\omega_1, \omega_2, \omega_3)^T$ is the angular velocity vector in body-axes.

There are many parameterizations of the attitude kinematics. Loosely speaking, one can think of different parameterizations as different choices of coordinates on the rotation group $SO(3)$, that is, as different parameterizations of the matrix R. Standard coordinates on $SO(3)$ are the Eulerian angles, the Euler parameters (quaternions), the Cayley-Rodrigues parameters, the Cayley-Klein parameters, the axis-azimuth parameters, etc. [12, 17]. A major classification of these parameterizations is according to their dimension. Since $SO(3)$ is a three-dimensional compact manifold [2], a minimal parameterization is necessarily of third order.

3. Effect of Singularities

The Eulerian angles, the Cayley-Rodrigues parameters and the axis-azimuth parameters are examples of three dimensional parameterizations of $SO(3)$. Three dimensional parameterizations introduce necessarily a *singularity*, as it is not possible to find a globally diffeomorphic transformation between $SO(3)$ (which is compact) and the euclidean space $\mathbb{R}^3$ (which is not). In order to appreciate the effect of the coordinate singularity let us consider the problem of spin-axis stabilization of a rotating, symmetric body (e.g., a spinning top).

Typically, a 3-1-3 Eulerian angle set (ϕ, θ, ψ) is chosen for the description of the attitude kinematics for this problem [8, 15, 23]. This Eulerian angle choice has the advantage that it requires only one angle (the second angle θ) in order to describe the departure of the local spin-axis from its initial (e.g.,

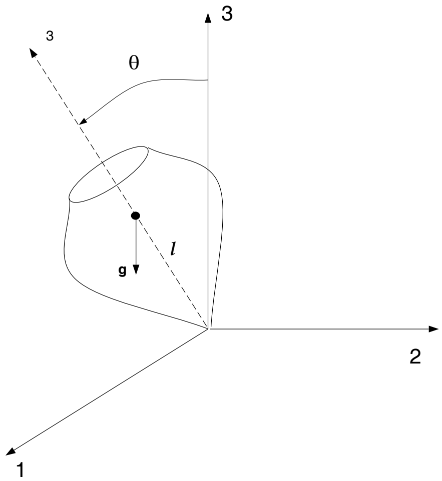

Figure 1: Spinning top and tilt angle θ.

inertial) orientation (cf. Fig 1). The kinematic equations in terms of ϕ, θ and ψ are given by [23, 15, 8]

$$\dot{\phi} = (\omega_2 \cos\psi + \omega_1 \sin\psi)/\sin\theta \quad (3a)$$

$$\dot{\theta} = \omega_1 \cos\psi - \omega_2 \sin\psi \quad (3b)$$

$$\dot{\psi} = \omega_3 - (\omega_2 \cos\psi + \omega_1 \sin\psi)/\tan\theta \quad (3c)$$

The equilibrium point is $\phi = \theta = \psi = 0$. Thus, the objective of the control law is to make $\theta = 0$. Notice however, that this 3-1-3 Eulerian angle set has a singularity (i.e., the kinematic differential equations are not defined) at $\theta = 0$. It is therefore clear that, although all minimal parameterizations will introduce a singularity in the description of the kinematics, the *effect* of this singularity for a particular problem can vary a great deal. The appropriate choice of coordinates for such systems (or for general nonlinear systems, for that matter) cannot be overemphasized. For the previous example, the poor choice of kinematic parameters introduced the singularity exactly at the equilibrium point! The previous discussion reveals that (at least for stabilization problems) it is desirable to move the singularity associated with three-dimensional attitude parameterizations as far from the equilibrium as possible. In the sequel we introduce two nonstandard parameterizations which have exactly this property. The derivation of both these representations is based on the idea of stereographic projection [1, 19].

4. Two Nonstandard Coordinate Choices

4.1. The (w, z) coordinates

The method of stereographic projection can be used to eliminate the unity constraint associated with the elements of any column (or row) of the rotation matrix R. This procedure introduces a complex coordinate (w) in order to describe the relative location of a given axis in the body frame. Going back to the example of the spinning top, one can define the unit vector $\gamma := (\gamma_1, \gamma_2, \gamma_3)^T$ in the negative gravity direction expressed in body coordinates. In other words, $\gamma_1, \gamma_2, \gamma_3$ are the direction cosines of the inertial 3-axis with respect to the local body-fixed axes. The kinematic equations in terms of γ are given by

$$\dot{\gamma} = S(\omega)\gamma \quad (4)$$

Because of the constraint $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ the vector γ lies on the unit sphere S^2 in $\mathbb{R}^3$. Let us consider the stereographic projection $S^2 \to \mathbb{C}_\infty$ of the unit sphere S^2 onto the extended complex plane $\mathbb{C}_\infty \triangleq \mathbb{C} \cup \{\infty\}$, defined by

$$\mathrm{w} = \mathrm{w}_1 + i\,\mathrm{w}_2 := \frac{\gamma_2 - i\,\gamma_1}{1 + \gamma_3} \quad (5)$$

where $i := \sqrt{-1}$. The kinematic equation for this coordinate satisfies a complex Riccati equation and is related to the work of Darboux in classical differential geometry [3]. Using Eqs. (4) and (5) one obtains [21, 22]

$$\dot{\mathrm{w}} = -i\,\omega_3 \mathrm{w} + \frac{\omega_c}{2} + \frac{\bar{\omega}_c}{2}\mathrm{w}^2 \quad (6)$$

where the bar denotes complex conjugate and where $\omega_c := \omega_1 + i\,\omega_2$. This stereographic projection establishes a one-to-one correspondence between the unit sphere and the extended complex plane. Note that by choosing the pole of the stereographic projection to be the point $(0, 0, -1)$ we have moved – in effect – the singularity of this coordinate choice as far from the equilibrium point $(0, 0, 1)$ as possible.

A complete coordinate set for $SO(3)$ requires the introduction of an additional coordinate to complement (w). It can be shown [22, 20] that this coordinate (z) satisfies the following differential equation

$$\dot{z} = \omega_3 + \frac{i}{2}(\bar{\omega}_c \mathrm{w} - \omega_c \bar{\mathrm{w}}) \quad (7)$$

4.2. The (σ) Coordinates

Higher-dimensional parameterizations, such as the Euler parameters, are singularity-free and therefore provide a global description; such parameterizations are, however, redundant. Since the Euler parameter vector $q := (q_0, q_1, q_2, q_3)^T$ obeys the constraint $q^T q = 1$ it evolves on the four-dimensional unit sphere S^3 in $\mathbb{R}^4$. One can therefore apply the stereographic projection $S^3 \to \mathbb{R}^3 \cup \{\infty\}$ on the Euler parameters and introduce a set of coordinates (referred to in the literature as the Modified Rodrigues parameters [17, 14, 18, 16]) by

$$\sigma_i := \frac{q_i}{1 + q_0}, \quad i = 1, 2, 3 \quad (8)$$

These attitude coordinates allow for eigenaxis rotations of up to 360 deg whereas the classical Rodrigues parameters are limited to eigenaxis rotations of only up to 180 deg. Therefore, the Rodrigues parameters necessarily introduce an infinite number of singular orientations, whereas the Modified Rodrigues parameters only introduce a single singular configuration on $SO(3)$; the Modified Rodrigues parameters have a larger domain of validity when compared to other three dimensional parameterizations.

The kinematics in terms of the Modified Rodrigues parameter vector $\sigma := (\sigma_1, \sigma_2, \sigma_3)^T$ is given by

$$\dot{\sigma} = G(\sigma)\omega, \qquad \sigma(0) = \sigma_0 \quad (9)$$

where

$$G(\sigma) := \tfrac{1}{2}\left(I_3 - S(\sigma) + \sigma\sigma^T - \frac{1 + \sigma^T\sigma}{2} I_3\right). \quad (10)$$

and I_3 is the 3×3 identity matrix. Direct calculation shows that the matrix $G(\sigma)$ in Eq. (9) satisfies the following two identities

$$\sigma^T G(\sigma)\omega = \left(\frac{1 + \sigma^T\sigma}{4}\right)\sigma^T\omega \quad (11)$$

and

$$G^T(\sigma)G(\sigma) = \left(\frac{1+\sigma^T\sigma}{4}\right)^2 I_3 \tag{12}$$

for all $\omega, \sigma \in \mathbb{R}^3$.

5. Two Control Problems

A new family of small spacecraft has been recently proposed, which intends to reduce the overall operational cost of today's ultra-expensive satellites [4, 7]. These proposed satellites will be task-specific, thus reducing the time and effort required for their design, construction and operation. Moreover, if the cost of these spacecraft is to be kept at a minimum, sensor and/or actuator redundancy must be avoided.

It is therefore of prime interest the problem of stabilization without complete actuation or without complete sensor information. In this section we show how the two sets of coordinates previously introduced can be used to solve these two problems. We emphasize the useful property of *passivity* for these choices of coordinates; that is, for these kinematic parameters the map from the angular velocities to the orientation coordinates is passive. This property – along with the well-known passivity from the torques to the angular velocities – allows for *linear* globally asymptotically stabilizing control laws and for global stabilization with no angular velocity feedback.

Let $\mathcal{H}$ and $\mathcal{X}$ denote two Hilbert spaces and let $<\cdot,\cdot>_{\mathcal{H}}$ denote the inner product in $\mathcal{H}$.

Definition 5.1 A system with input $u \in \mathcal{H}$ and output $y \in \mathcal{H}$ is called *passive* (with storage function V) if there exists a positive definite function $V : \mathcal{X} \to \mathbb{R}_+$ such that

$$\int_0^T < y(t), u(t) >_{\mathcal{H}} \, dt \geq V(x(T)) - V(x(0)) \tag{13}$$

where $x \in \mathcal{X}$ is the state of the system.

Definition 5.2 A system with input $u \in \mathcal{H}$ and output $y \in \mathcal{H}$ is called *strictly passive* (with storage function V and dissipation rate χ) if there exist positive definite functions $V : \mathcal{X} \to \mathbb{R}_+$ and $\chi : \mathcal{X} \to \mathbb{R}_+$ such that

$$\int_0^T < y(t), u(t) >_{\mathcal{H}} \, dt \geq V(x(T)) - V(x(0)) + \int_0^T \chi(x(t))\, dt \tag{14}$$

Passivity is a very desirable property for a system. First, because it implies some very good robustness properties. Second, because, under mild assumptions, feedback interconnection of a passive and a strictly passive system is globally asymptotically stable [5].

5.1. The under-actuated body

Consider an axially-symmetric rigid body with I_a and I_r being the axial and the radial moments of inertia. If there is no available torque along the symmetry axis (say, the 3-axis) then the angular velocity component along this axis is constant. That is, $\omega_3(0) = \omega_{30}$. The equations of motion can then be written in the form

$$\dot{\omega}_c = -i\,\alpha\omega_{30}\omega_c + u_c, \qquad \omega_c(0) = \omega_{c0} \tag{15a}$$

$$\dot{\mathrm{w}} = -i\,\omega_{30}\mathrm{w} + \frac{\omega_c}{2} + \frac{\bar{\omega}_c}{2}\mathrm{w}^2, \qquad \mathrm{w}(0) = \mathrm{w}_0 \tag{15b}$$

where $\alpha = (I_r - I_a)/I_r$ and $u_c := u_1 + i\,u_2$ is the acting torque. Equation (15a) will be referred to as the *dynamics* and Eq. (15b) will be referred to as the *kinematics* of the motion.

In the sequel $|\cdot|$ denotes absolute value of a complex number and $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^n$. Moreover, the inner products in $\mathbb{C}$ and $\mathbb{R}^n$ are defined, as usual, by $< x_c, y_c >_{\mathbb{C}} = Re(\bar{x}_c y_c)$ and $< x, y >_{\mathbb{R}^n} = x^T y$ for $x_c, y_c \in \mathbb{C}$ and $x, y \in \mathbb{R}^n$, respectively.

The system in Eqs. (15) has some nice passivity characteristics.

Proposition 5.1 *(i) Consider the system (15a) with input u_c and output ω_c. This system is passive with storage function*

$$V_1(\omega_c) = \tfrac{1}{2}|\omega_c|^2 \tag{16}$$

(ii) Consider the system (15b) with input ω_c and output w*. This system is passive with storage function*

$$V_2(\mathrm{w}) = \ln(1 + |\mathrm{w}|^2) \tag{17}$$

Proof. (i) In order to show that the dynamics subsystem in Eq. (15a) is passive notice that the derivative of V_1 in Eq. (16) along the trajectories of (15a) is

$$\frac{dV_1}{dt} = Re(\bar{\omega}_c u_c) \tag{18}$$

Integrating both sides of the previous equation from 0 to T, we arrive at Eq. (13).

(ii) In order to show that the kinematics subsystem in Eq. (15b) is passive notice that the derivative of V_2 in Eq. (17) along the trajectories of (15b) is

$$\frac{dV_2}{dt} = Re(\bar{\mathrm{w}}\omega_c) \tag{19}$$

Integrating both sides we arrive at Eq. (13). ∎

This proposition shows that the system in Eqs. (15) is a cascade interconnection of two passive systems. We now show that the cascade interconnection of the two passive systems in Eqs. (15a) and (15b) can be globally asymptotically stabilized using *linear* feedback in terms of the subsystem outputs. This result is a consequence of the particular choice of coordinates which make the kinematics subsystem passive. Hence the following lemma.

Lemma 5.1 *The control law*

$$u_c = -k_1\omega_c + \nu_c \tag{20}$$

with $k_1 > 0$ renders the subsystem (15a) strictly passive from ν_c to ω_c with storage function V_1 and dissipation rate $\chi(\omega_c) = k_1|\omega_c|^2$.

Proof. Letting V_1 as in Eq. (16) and using Eqs. (18) and (20) we get that

$$\frac{dV_1}{dt} = -k_1|\omega_c|^2 + Re(\bar{\omega}_c\nu_c) \tag{21}$$

Integrating both sides of the previous equation one obtains

$$\int_0^T Re(\bar{\omega}_c\nu_c)\, dt = V_1(\omega_c(T)) - V_1(\omega_c(0)) + k_1\int_0^T |\omega_c|^2\, dt \tag{22}$$

which, according to Eq. (14) implies that the system from ν_c to ω_c is strictly passive. ∎

Choosing now a negative feedback $\nu_c = -k_2 \mathrm{w}$ one obtains a feedback interconnection of a strictly passive system and a passive system which, using a certain observability condition can be shown to be globally asymptotically stable [5, 9].

Theorem 5.1 *Consider the cascade interconnection (15a)-(15b). The linear control*

$$u_c = -k_1 \omega_c - k_2 \mathrm{w} \tag{23}$$

where $k_1, k_2 > 0$ globally asymptotically stabilizes this system at the origin.

Proof. Consider the positive definite, radially unbounded function

$$V(\omega_c, \mathrm{w}) = V_1(\omega_c) + k_2 V_2(\mathrm{w}) = \tfrac{1}{2}|\omega_c|^2 + k_2 \ln(1 + |\mathrm{w}|^2) \tag{24}$$

Taking the derivative of V along the trajectories of Eqs. (15)-(23) one obtains

$$\begin{aligned} \dot{V} &= Re(\bar{\omega}_c \dot{\omega}_c) + \frac{2k_2}{1+|\mathrm{w}|^2} Re(\bar{\mathrm{w}}\dot{\mathrm{w}}) \\ &= -k_1 |\omega_c|^2 - k_2 Re(\bar{\omega}_c \mathrm{w}) + \frac{k_2}{1+|\mathrm{w}|^2} Re(\bar{\mathrm{w}}\omega_c + \bar{\mathrm{w}}\bar{\omega}_c \mathrm{w}^2) \\ &= -k_1 |\omega_c|^2 \end{aligned} \tag{25}$$

and the system is stable. Asymptotic stability follows using a standard LaSalle-type argument. ∎

5.2. The under-sensed body

In this section we address the problem of stabilization on $SO(3)$ with incomplete state information. In particular, we investigate the case of no angular velocity feedback. Control laws which do not require angular velocity feedback can be utilized in control strategies for small satellites for the reasons stated earlier. In addition, even for traditional spacecraft, angular velocity information is usually provided by rate gyros which are prone to failure; thus, implementation of feedback control schemes without angular velocity information is clearly desirable.

Consider again the axi-symmetric body described by Eqs. (15) where now it is assumed that the sensors provide only attitude orientation signals as measured by the coordinate (w). The passivity properties of this system can be utilized for velocity-free stabilizing control laws. Lemma 5.1 shows that the feedback control law $u_c = -k_1 \omega_c + \nu_c$ makes the system (15a) strictly passive from ν_c to ω_c. Consider now instead the control law

$$u_c = -k_2 \mathrm{w} + v_c \tag{26}$$

Lemma 5.2 *Let the system (15) and the control law in Eq. (26). This system with input v_c and output ω_c is passive.*

Proof. Let the function $V(\omega_c, \mathrm{w}) = V_1(\omega_c) + k_2 V_2(\mathrm{w})$ where V_1 and V_2 as in Eqs. (16) and (17), respectively. Differentiation along the trajectories of (15) yields that $\dot{V}(\omega_c, \mathrm{w}) = Re(\bar{\omega}_c v_c) + k_2 Re(\bar{\mathrm{w}}\omega_c)$. Using Eq. (26) we get that $\dot{V}(\omega_c, \mathrm{w}) = Re(\bar{\omega}_c v_c)$. Integrating both sides we arrive at Eq. (13). ∎

It should be clear from the previous proof that the system in Eqs. (15) with the control law in Eq. (26) is passive with storage function $V(\omega_c, \mathrm{w}) = \frac{1}{2}|\omega_c|^2 + k_2 \ln(1 + |\mathrm{w}|^2)$.

For the axi-symmetric body with no control along the symmetry axis we have shown that $\omega_3(t) = \omega_{30}$ is constant. Let us further assume that $\omega_{30} = 0$ (e.g., as in a rest-to-rest maneuver).

Proposition 5.2 *Consider the system (15) with $\omega_{30} = 0$ and the control law in Eq. (26). This system with input*

$$y_c = \frac{2}{1 - |\mathrm{w}|^4}(v_c - \mathrm{w}^2 \bar{v}_c) \tag{27}$$

and output

$$w_c = \frac{\omega_c}{2} + \frac{\bar{\omega}_c}{2}\mathrm{w}^2 = \dot{\mathrm{w}} \tag{28}$$

is passive.

Proof. For input y_c and output w_c, one obtains

$$\begin{aligned} \int_0^T Re(\bar{w}_c y_c)\, dt &= \frac{1}{1-|\mathrm{w}|^4} \int_0^T Re(\bar{\omega}_c v_c - \omega_c \bar{v}_c |\mathrm{w}|^4)\, dt \\ &= \int_0^T Re(\bar{\omega}_c v_c)\, dt \end{aligned} \tag{29}$$

and the result follows from lemma 5.2. ∎

Notice that if y_c is the new input as defined by proposition 5.2 then v_c is given by

$$v_c = \frac{y_c}{2} + \frac{\bar{y}_c}{2}\mathrm{w}^2 \tag{30}$$

Since the map from y_c to w_c is passive we can explore the possibility of a feedback interconnection between w_c and y_c with a strictly passive system. This motivates the control law in the following theorem.

Theorem 5.2 *Consider the system in Eqs. (15) and let the control law*

$$u_c = -k_2 \mathrm{w} - \frac{k_1}{2}(y_c + \bar{y}_c \mathrm{w}^2) \tag{31}$$

with $k_1 > 0, k_2 > 0$, and where y_c is the output of the linear, time-invariant system

$$\dot{x}_c = -a\, x_c + \mathrm{w} \tag{32a}$$

$$y_c = -a\, x_c + \mathrm{w} \tag{32b}$$

where $a > 0$. Then $\lim_{t\to\infty}(\omega_c(t), \mathrm{w}(t)) = 0$, for all initial conditions $(\omega_{c0}, \mathrm{w}_0) \in \mathbb{C} \times \mathbb{C}$).

Proof. Consider the function

$$V(\omega_c, w, \dot{x}_c) = \tfrac{1}{2}|\omega_c|^2 + k_2 \ln(1 + |\mathrm{w}|^2) + \frac{k_1}{2}|\dot{x}_c|^2 \tag{33}$$

Taking the derivative of V along the closed-loop trajectories, one obtains

$$\begin{aligned} \dot{V} &= Re(u_c \bar{\omega}_c) + \frac{2k_2}{1+|\mathrm{w}|^2} Re(\dot{\mathrm{w}}\bar{\mathrm{w}}) + k_1 Re(\ddot{x}_c \dot{\bar{x}}_c) \\ &= -\frac{k_1}{2}(y_c \bar{\omega}_c + \bar{y}_c \mathrm{w}^2 \bar{\omega}_c) + k_1 Re(\ddot{x}_c \dot{\bar{x}}_c) \\ &= -\frac{k_1}{2}(\dot{x}_c \bar{\omega}_c + \dot{\bar{x}}_c \mathrm{w}^2 \bar{\omega}_c) + k_1 Re(\dot{\bar{x}}_c a \dot{x}_c) \\ &\quad + \frac{k_1}{2} Re(\dot{\bar{x}}_c \omega + \dot{\bar{x}}_c \bar{\omega} \mathrm{w}^2) \\ &= -k_1 a |\dot{x}_c|^2 \end{aligned} \tag{34}$$

Since $\dot{V} \leq 0$ and V is radially unbounded, all solutions are bounded. Consider the set $\mathcal{E}_c = \{(\omega_c, \mathrm{w}, x_c) : \dot{V} = 0\}$. Notice that $\dot{V} \equiv 0$ if and only if $\dot{x}_c \equiv 0$, which implies that $y_c = 0$. Moreover, $\ddot{x}_c \equiv 0$ implies that $\dot{\mathrm{w}} = 0$ and from Eq. (15b) that $\omega_c = 0$. Equation (15a) then implies that $u_c = 0$ and therefore from Eq. (31) that $\mathrm{w} = 0$. In short, we have shown that $\dot{V} \equiv 0$ if and only if $\omega_c = \mathrm{w} = 0$. That is, the largest invariant set in $\mathcal{E}_c$ is the set $\mathcal{M}_c = \{(\omega_c, \mathrm{w}, x_c) \in \mathcal{E}_c : \omega_c = \mathrm{w} = 0\}$. By LaSalle's Invariance Principle [13], all trajectories of the closed-loop system asymptotically approach $\mathcal{M}_c$, thus $\lim_{t\to\infty}(\omega_c(t), \mathrm{w}(t)) = 0$. ■

Remark 5.1 The transfer function from $\dot{\mathrm{w}}$ to y_c is strictly positive real. The system in Eqs. (32) is (non-strictly) proper, and (non-strictly) positive real. It is, in essence, a lead filter of the orientation parameter (w) which provides derivative information to be used in the control law.

The case of a general, non-symmetric case can be treated similarly. First, one can show that passivity properties similar to the ones of the (w) coordinate also hold for the (σ) coordinates. To this end, recall that the equations for a general rigid body are

$$J\dot{\omega} = -S(\omega)J\omega + u, \qquad \omega(0) = \omega_0 \tag{35a}$$
$$\dot{\sigma} = G(\sigma)\omega, \qquad \sigma(0) = \sigma_0 \tag{35b}$$

where J is the inertia matrix, $u := (u_1, u_2, u_3)^T$ is the input torque in body-axes and $G(\sigma)$ as in Eq. (9).

Proposition 5.3 *(i) The system (35a) with input u and output ω is passive.*

(ii) The system (35b) with input ω and output σ is passive.

Proof. (i) Let the function $V_1(\omega) = \frac{1}{2}\omega^T J\omega$. Differentiation along the trajectories of Eq. (35a) yields that $\dot{V}_1(\omega) = \omega^T u$, therefore

$$\int_0^T \omega^T u \, dt = V_1(\omega(T)) - V_1(\omega_0) \tag{36}$$

(ii) Let the function $V_2(\sigma) = 2\ln(1+\sigma^T\sigma)$. Differentiation along the trajectories of Eq. (35b) and use of Eq. (12) yields that $\dot{V}_2(\sigma) = \sigma^T\omega$, therefore

$$\int_0^T \sigma^T \omega \, dt = V_2(\sigma(T)) - V_2(\sigma_0) \tag{37}$$

■

Consider now the more general control law

$$u = -k_2\sigma + v \tag{38}$$

with $k_1 > 0$, where v is the new input. The following lemma shows that the passivity between the new input v and the output ω is preserved for the system in Eqs. (35).

Lemma 5.3 *Let the system (35) and the control law (38). This system with input v and output ω is passive.*

Proof. Let the function $V(\omega,\sigma) = V_1(\omega) + k_2V_2(\sigma)$ where V_1 and V_2 as in proposition 5.3. Differentiation along the trajectories of Eq. (35a) yields that $\dot{V}(\omega,\sigma) = \omega^T u + k_2\sigma^T\omega$. Using Eq. (38) we get that $\dot{V}(\omega,\sigma) = \omega^T v$. The rest of the proof follows as in Proposition 5.3. ■

Property (12) implies an "orthogonality" condition for the matrix $G(\sigma)$; in particular, the matrix $G(\sigma)$ times its transpose yields the identity matrix times a non-vanishing, time-varying function. Similarly to proposition 5.2 one can use this result to establish "orthogonal" input/output transformations for Eqs. (35)-(38) which preserve passivity.

Proposition 5.4 *The system in Eqs. (35) with input $y = \left(\frac{4}{1+\sigma^T\sigma}\right)^2 G(\sigma)v$ and output $w = G(\sigma)\omega = \dot{\sigma}$ is passive.*

Proof. Using Eq. (12) we have that

$$\int_0^T w^T y \, dt = \int_0^T \left(\frac{4}{1+\sigma^T\sigma}\right)^2 \omega^T G^T(\sigma)G(\sigma)v \, dt = \int_0^T \omega^T v \, dt \tag{39}$$

Using now lemma 5.3 we establish the desired result. ■

Notice that if y is the new input as defined by proposition 5.4 then v is given by

$$v = G^T(\sigma)y \tag{40}$$

Since the map from y to w is passive, one may explore the possibility of globally asymptotically stabilizing the system by choosing a feedback such that the map from w to y is strictly passive [9]).

Let A be any stability matrix, B any full column rank matrix, with the pair (A,B) controllable, and Q any positive definite matrix. Let also the matrix P be the solution of the Lyapunov equation

$$A^TP + PA = -Q \tag{41}$$

Clearly then P is positive definite. We are now ready to state the main result for asymptotic stabilization in the large of the general rigid body in Eqs. (35) without angular velocity feedback.

Theorem 5.3 *Consider the system (35) and let the control law*

$$u = -k_2\sigma - k_1G^T(\sigma)y \tag{42}$$

with $k_1 > 0$, $k_2 > 0$, and where y is the output of the linear, time-invariant system

$$\dot{x} = Ax + B\sigma \tag{43a}$$
$$y = B^TPAx + B^TPB\sigma \tag{43b}$$

Then $\lim_{t\to\infty}(\omega(t),\sigma(t)) = 0$, for all initial conditions $(\omega_0,\sigma_0) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Proof. Consider the function

$$V(\omega,\sigma,\dot{x}) = \tfrac{1}{2}\omega^TJ\omega + 2k_2\ln(1+\sigma^T\sigma) + \frac{k_1}{2}\dot{x}^TP\dot{x} \tag{44}$$

The time derivative of V along the trajectories of the closed-loop system is then

$$\begin{aligned}\dot{V} &= \omega^TJ\dot{\omega} + k_2\left(\frac{4}{1+\sigma^T\sigma}\right)\sigma^TG(\sigma)\omega \\ &\quad + k_1\dot{x}^TP\ddot{x} \\ &= \omega^T(-k_2\sigma - k_1G^T(\sigma)y) \\ &\quad + k_2\sigma^T\omega + k_2\dot{x}^TPA\dot{x} + k_1\dot{x}^TPBG(\sigma)\omega \\ &= \frac{k_2}{2}\dot{x}^T(PA + A^TP)\dot{x} = -\frac{k_2}{2}\dot{x}^TQ\dot{x} \le 0\end{aligned} \tag{45}$$

First observe that since V is radially unbounded, all solutions are bounded. Consider now the set $\mathcal{E} = \{(\omega,\sigma,x) : \dot{V} = 0\}$. Trajectories in $\mathcal{E}$ then satisfy $\dot{x} = 0$ and hence $x(t) = x_0$ for all $t \ge 0$ and from (43a) also $\sigma(t) = \sigma_0$ for all $t \ge 0$. Then $\dot{\sigma} = 0$ and from (9) also $\omega(t) = 0$ for all

$t \geq 0$. Since $y = B^T P \dot{x}$ one has also that $y = 0$, and using (35a) and (42) we have that $\omega = \dot{\omega} = 0$ and $y = 0$ implies that $\sigma = 0$. The largest invariant set in $\mathcal{E}$ is therefore the set $\mathcal{M} = \{(\omega, \sigma, x) \in \mathcal{E} : \omega = 0, \sigma = 0, x = x_0\}$. By LaSalle's Invariance Principle [13] all trajectories of the system asymptotically approach $\mathcal{M}$, thus $\lim_{t \to \infty}(\omega(t), \sigma(t)) = 0$, as claimed. ■

Remark 5.2 Similarly, to the results of section 5.1, and using again the passivity characteristics of the σ coordinates, it should not be very difficult for the reader to verify that in case of angular velocity feedback, the *linear* control law

$$u = -k_1 \omega - k_2 \sigma \tag{46}$$

where $k_1 > 0, k_2 > 0$, globally asymptotically stabilizes the system in Eqs. (35); see also [18].

6. Conclusions

We have addressed the issue of "good" coordinate choices for control problems on $SO(3)$. We have shown that for three-dimensional parameterizations the associated singularities have a great impact on the stabilization problem. In fact, it is always advisable to choose coordinates such that the singularity is as far from the equilibrium as possible. Moreover, parameterizations having certain passivity properties can be used for global asymptotic stabilization using linear control laws and for stabilization without angular velocity feedback. We have presented two special, nonstandard parameterizations which appear to be useful for axially-symmetric and non-symmetric rigid bodies, respectively. We hope that the results of this work will contribute to our current understanding on the impact of appropriate coordinate choices for problems on $SO(3)$.

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