Toward Less Conservative Stability Analysis of Time-Delay Systems

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Abstract

The stability of linear time-delay systems is investigated via the robustness analysis of several uncertain delay-free comparison systems. Several new delay-dependent stability criteria, which are formulated as linear matrix inequalities (LMIs), are then derived. Finally, an example problem demonstrates that these new stability criteria may be significantly less conservative than those existing in the literature. **Keywords.** Time-delay systems; stability; linear matrix inequalities.

1 Introduction

The analysis of time-delay systems has attracted much interest over a half century, especially in the last decade. The recent book [6] contains an extensive collection of papers dealing with both delay-dependent and delay-independent stability. Many of the stability analyses have been formulated in the time domain based on Lyapunov's Second Method using Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions [9, 11, 12, 13, 16]. Frequency domain techniques for analysis of time-delay systems have also been developed [12], such as polynomial criteria [8], matrix pencils [3], integral quadratic constraints [7], the singular value test [18], and $\mu-$based criteria [1, 4, 14], etc. Recently, the authors introduced a comparison system-based approach [19] for analysis of time-delay linear systems, and demonstrated that several existing Lyapunov-based results are, in fact, equivalent to robust stability analysis of a comparison system via the scaled small-gain lemma. An examination of the manner in which delay elements are covered with norm-bounded uncertainty sets in this approach, directly indicates a potential source of significant conservatism and a possible remedy. In this paper, we use this insight to develop several new, less conservative conditions for delay-dependent stability. These conditions can be formulated in terms of LMIs. A numerical example indicates that these new criteria may be significantly less conservative than previous ones. **Notation**. Let $\Re_e := \Re \cup \{-\infty, \infty\}$, $\Re_e^+ := [0, \infty) \cup \{\infty\}$, and I_n be $n \times n$ identity matrix. A minimum realization (A, B, C, D) of a transfer function matrix $G(s)$ is denoted by $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. $\mathbf{D}(q; r) := \{z \in \mathbf{C} | \, |z - q| \leq r\}$ represents the closed disk in the complex plane with center q and radius r, and the closed unit disk in the complex plane is denoted by $\mathbf{B} := \mathbf{D}(0; 1)$.

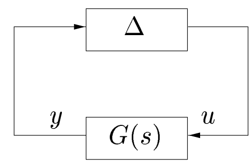

Figure 1: A feedback system.

2 A Comparison System

Consider the linear time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau) \quad (1)$$

where the delay $\tau \in [0, \bar{\tau}]$ is constant but unknown. Equation (1) can be expressed in the frequency domain as[1]

$$sX(s) \quad = \quad AX(s) + A_d e^{-\tau s} X(s). \quad (2)$$

For the stability analysis tests we develop, we need the following definition and preliminary results.

Definition 1 *Consider a linear, time-invariant (finite-dimensional) system $G(s)$ interconnected with an uncertain block $\Delta \in \underline{\Delta}$, as shown in Figure 1. Then the system is said to be* ***robustly stable*** *if $G(s)$ is internally stable, the interconnection is well-posed and it remains internally stable for all $\Delta \in \underline{\Delta}$.*

The following lemma introduces a frequency-dependent covering of the delay elements that will be used later in the robust stability analysis of the comparison system.

[1] While the notation using Laplace transforms is used for convenience and is somewhat abusive, the results of this paper can be proven by using characteristic equations.

Lemma 1 *Define the following set of complex-valued functions*

$$\mathbf{E}_\Omega := \{\delta(s) |\, \|\delta\|_\infty < \infty, \text{ and } \delta(j\omega) \in \Omega(j\omega), \ \forall \omega \in \Re_e\}$$

Given the ***delay elements***

$$\phi_1(s,\tau) = \begin{cases} \frac{e^{-\tau s}-1}{\bar{\tau} s} & s \in \mathbf{C}, s \neq 0 \\ -\frac{\tau}{\bar{\tau}} & s = 0 \end{cases},$$
$$\phi_2(s,\tau) = e^{-\tau s}$$

let $\Omega_1(j\omega)$ and $\Omega_2(j\omega)$ be subsets of $\mathbf{C}$ such that $\lambda\phi_1(j\omega,\tau) \in \Omega_1(j\omega), \lambda\phi_2(j\omega,\tau) \in \Omega_2(j\omega)$, $\forall \omega \in \Re_e^+$, $\tau \in [0,\bar{\tau}]$, $\lambda \in [0,1]$. Then the system (1) is asymptotically stable for all $\tau \in [0,\bar{\tau}]$, if there exists a constant matrix $M \in \Re^{n\times n}$ such that the comparison system

$$\begin{aligned} sX(s) &= (A + MA_d)X(s) \\ &\quad +\delta_2(s)(I_n - M)A_dX(s) \\ &\quad +\delta_1(s)\bar{\tau}MA_d sX(s) \end{aligned} \tag{3}$$

is robustly stable for all $\delta_1(s) \in \mathbf{E}_{\Omega_1}$ and $\delta_2(s) \in \mathbf{E}_{\Omega_2}$.
Proof. *Equation (2) can be rewritten as*

$$\begin{aligned} sX(s) &= AX(s) + (I_n - M)A_d e^{-\tau s}X(s) \\ &\quad +MA_d e^{-\tau s}X(s) \\ &= (A + MA_d)X(s) \\ &\quad +\phi_2(s,\tau)(I_n - M)A_dX(s) \\ &\quad +\phi_1(s,\tau)\bar{\tau}MA_d sX(s) \end{aligned} \tag{4}$$

It follows that any solutions of (1) satisfy (4), and (4) is a particular case of the uncertain system (3). Thus, the robust stability of (3) implies that (1) is asymptotically stable for all $\tau \in [0,\bar{\tau}]$. ■

Remark 1 *The sets $\mathbf{E}_{\Omega_1}$ and $\mathbf{E}_{\Omega_2}$ are said to be the covering sets for the delay elements $\phi_1(s,\tau)$ and $\phi_2(s,\tau)$, respectively. In general, the sets Ω_1 and Ω_2 are frequency-dependent, that is, they may move in the complex plane with frequency. In a special case, Ω_1 or Ω_2 may become frequency-independent, that is, a fixed set for all frequencies.*

The simplest choice for Ω_1 and Ω_2 in the previous lemma is $\Omega_1 = \Omega_2 = \mathbf{B}$. In this case, the stability of (1) may be directly determined via $\mu-$analysis, since the small-μ theorem applies even to the case where the uncertainty is non-rational [17]. In particular, a sufficient condition for stability is

$$\sup_{\omega\in\Re} \mu_{\underline{\Delta}}[G(j\omega)] < 1 \tag{5}$$

where $G(s)$ is a state space realization of the comparison system (3) with δ_1 and δ_2 "pulled out" to form an uncertainty block $\underline{\Delta} = \{\text{diag}[\delta_1, \delta_2]\}$ [20]. To analyze the condition of (5), a frequency sweep is typically employed. Because the calculation of μ is NP-hard in general [2], its upper bound is typically used in determining robust stability instead. Alternatively, the analysis of robust stability may be performed without the frequency sweep by solving an LMI. The following lemma states this result.

Lemma 2 *[15](**Scaled Small Gain Lemma**) Consider a system with uncertainty as shown in Figure 1. Let*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

and suppose that the uncertainty $\Delta = \text{diag}\{\delta_1 I_{n_1}, \delta_2 I_{n_2}, \cdots, \delta_r I_{n_r}\}$, where $\delta_i \in \mathbf{E_B}$, $i = 1, 2, \cdots, r$. Then the closed loop system is robustly stable if there exist matrices $X > 0$ and $Q = \text{diag}\{Q_1, Q_2, \cdots, Q_r\} > 0$, $Q_i \in \Re^{n_i\times n_i}$, $i = 1, 2, \cdots, r$, satisfying the following LMI:

$$\begin{bmatrix} A^TX + XA & XB & C^TQ \\ B^TX & -Q & D^TQ \\ QC & QD & -Q \end{bmatrix} < 0. \tag{6}$$

It was shown in [19] that analysis of the comparison system (3) using the scaled-small gain LMI condition (6) is equivalent to the analysis of (1) using the Lyapunov-based conditions of [18, 9, 11, 13]. Furthermore, it was demonstrated that these results may be very conservative and that the conservatism arises mainly from choosing Ω_1 and Ω_2 as the unit ball $\Omega_1 = \Omega_2 = \mathbf{B}$.[2] Choosing other appropriate sets for Ω_1 or Ω_2 may reduce conservatism. Unfortunately, in this case analysis may not be performed directly using either the traditional μ approach or scaled small gain LMI. Herein, we explore the degree to which conservatism can be reduced with standard analysis tools and without a frequency sweep. We employ loop transformations on the comparison system to transform the set $\Omega_1 \neq \mathbf{B}$ into a new set $\bar{\Omega}_1 = \mathbf{B}$. Then, the transformed comparison system is analyzed via the scaled small-gain lemma to derive new, less conservative delay-dependent conditions. Three different loop transformations are employed to obtain the new stability conditions. These transformations are depicted in Figure 2, where

$$P(s) = \left[\begin{array}{c|c} A + MA_d & \begin{bmatrix} \bar{\tau}MA_d & (I_n - M)A_d \end{bmatrix} \\ \hline \begin{bmatrix} I_n \\ I_n \end{bmatrix} & 0 \end{array}\right]$$

3 A Comparison System with Shifted Disk

In this section, we exploit the phase information of the delay element $\phi_1(s,\tau)$ in (4), and establish a less conservative stability condition. Define the set $\Phi_1 :=$

[2] Although the sets Ω_1 and Ω_2 are not explicitly used in [19], this choice is implicit by the analysis techniques.

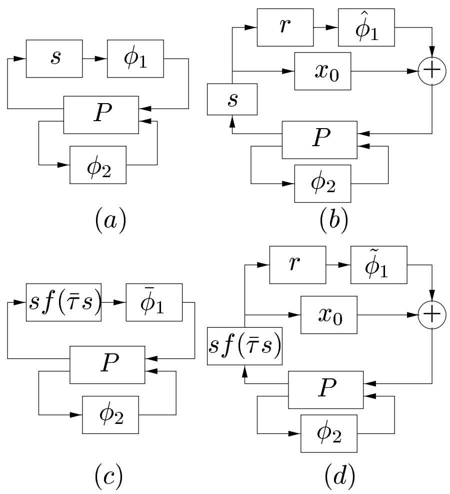

Figure 2: Loop transformations. (a) System (4). (b) With shifted disk. (c) With filter. (d) With both filter and shifted disk.

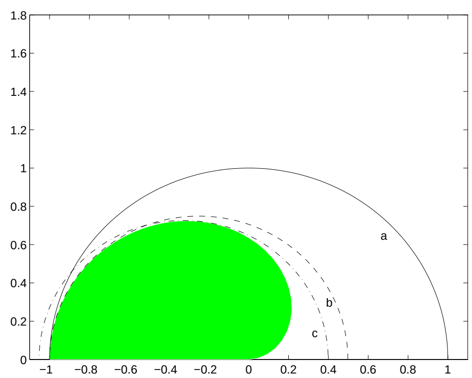

Figure 3: Delay element value set Φ_1 (shaded) and covering disks. (a) Unit disk $\mathbf{B}$. (b) $\mathbf{D}(-0.251; 0.749)$, the disk with minimum radius among all contained within $\mathbf{B}$. (c) $\mathbf{D}(-0.327; 0.726)$, the minimum radius covering disk.

$\{\lambda\phi_1(j\omega,\tau) | \omega \in \Re_e^+, \tau \in [0,\bar{\tau}], \lambda \in [0,1]\}$. As shown in Figure 3, Φ_1 is not symmetric with respect to the imaginary axis but, rather, most points of Φ_1 are located in the left half plane. We may use a disk $\mathbf{D}(q;r)$ to cover this set, that is,

$$|\lambda\phi_1(j\omega,\tau) - q| \le r, \quad \forall\omega \in \Re_e^+, \tau \in [0,\bar{\tau}], \lambda \in [0,1].$$

The valid covering disks, of course, are not unique. Three covering disks are shown in Figure 3. The stability of (1) can now be analyzed by examining the robust stability of (3) with $\Omega_1 = \mathbf{D}(q;r)$ and $\Omega_2 = \mathbf{B}$. To this end, we first employ the transformation

$$\hat{\phi}_1(s,\tau) = \frac{\phi_1(s,\tau) - q}{r}$$

upon (4) which, in view of (2), yields

$$\begin{aligned} sX(s) &= (A + MA_d + q\bar{\tau}MA_dA)X(s) \\ &\quad + \phi_2[(I_n - M)A_d + q\bar{\tau}MA_dA_d]X(s) \\ &\quad + \hat{\phi}_1\bar{\tau}rMA_dAX(s) \\ &\quad + \hat{\phi}_1\phi_2\bar{\tau}rMA_dA_dX(s) \end{aligned}$$

Since $\hat{\phi}_1(s,\tau) \in \mathbf{E_B}$ and $\phi_2(s,\tau) \in \mathbf{E_B}$, we obtain a comparison system which can be realized as the interconnection system

$$\begin{aligned} \dot{x} &= (A + MA_d + q\bar{\tau}MA_dA)x + \bar{\tau}rMA_du_1 \\ &\quad + [(I_n - M)A_d + q\bar{\tau}MA_dA_d]u_2 \\ y_1 &= Ax + A_du_2 \\ y_2 &= x \\ u_1 &= \delta_1 y_1 \\ u_2 &= \delta_2 y_2 \end{aligned} \tag{7}$$

with uncertainties $\delta_1(s) \in \mathbf{E_B}$ and $\delta_2(s) \in \mathbf{E_B}$. Therefore, applying Lemma 2 and defining $W = XM$, we obtain the following delay-dependent stability condition.

Theorem 1 *System (1) is asymptotically stable for any constant time-delay $\tau \in [0,\bar{\tau}]$, if there exist matrices $X > 0$, $U > 0$, $V > 0$ and W such that*

$$\begin{bmatrix} H_1 & H_2 & H_3 & A^TV \\ H_2^T & -V & 0 & 0 \\ H_3^T & 0 & -U & A_d^TV \\ VA & 0 & VA_d & -V \end{bmatrix} < 0 \tag{8}$$

where

$$\begin{aligned} H_1 &= A^TX + XA + WA_d + A_d^TW^T \\ &\quad + \bar{\tau}qWA_dA + \bar{\tau}qA^TA_d^TW^T + U \\ H_2 &= \bar{\tau}rWA_d \\ H_3 &= XA_d - WA_d + \bar{\tau}qWA_dA_d \end{aligned}$$

Remark 2 *It is obvious from Figure 3 that the unit disk $\mathbf{B} = \mathbf{D}(0;1)$ is a valid covering disk, but no useful phase information of Φ_1 is used in this case. In fact, this is the implicit choice made by the Lyapunov-based results of [11, 9, 13]. In particular, if $q = 0$ and $r = 1$, (8) reduces to the result of [13]. Hence, with $\mathbf{D}(-0.251; 0.749)$ which belongs to the unit disk $\mathbf{B}$, the condition (8) is less conservative than the result of [13], in general. In addition, with minimum radius disk $\mathbf{D}(-0.327; 0.726)$, (8) may be even less conservative. However, this is not guaranteed and the results depend upon A and A_d since this disk does not belong to $\mathbf{D}(-0.251; 0.749)$ or $\mathbf{B}$.*

Remark 3 *For the special case when $M = 0$, since $W = 0$, (8) reduces to the delay-independent stability condition.*

4 A Comparison System with Filter

In this section, we analyze the stability of (1) by exploiting the frequency-dependent gain information of the delay element $\phi_1(s,\tau)$ with a filter $f(s)$. Using this filter, we derive a new stability criterion.

4.1 Filter $f(s)$ and its realization

In the sequel, we will use a filter $f(s)$ to capture magnitude and phase information of $\phi_1(s,\tau)$. Suppose that $f(s)$ is real rational, stable and has minimum phase, the order of its denominator is greater than that of the numerator by 1, and $|\phi_1(j\omega,\tau)| \leq |f(j\bar{\tau}\omega)|, \quad \forall\omega \in \Re_e^+$, $\tau \in [0,\bar{\tau}]$. The filters satisfying these conditions are obviously not unique. One choice, given in [10], is

$$f(s) = \frac{2s + 7.0711}{s^2 + 4.5434s + 7.0711}. \tag{9}$$

Its frequency response is shown in Figure 4, where

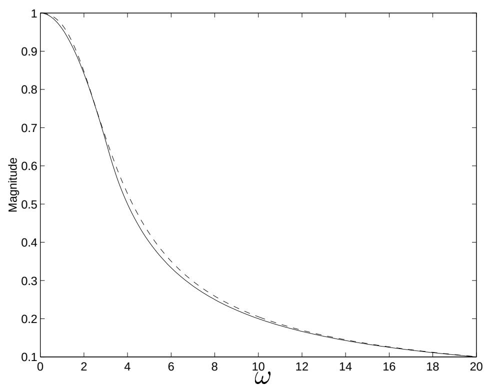

Figure 4: Frequency response of filter (9) $|f(j\omega)|$ (dashed) and $g(\omega)$ (solid).

$$g(\omega) = \max_{0\leq\tau\leq\bar{\tau}} |\phi_1(j\omega,\tau)| = \begin{cases} 1 & \omega = 0 \\ \frac{2\sin\frac{\omega\bar{\tau}}{2}}{\omega\bar{\tau}} & 0 < \omega \leq \frac{\pi}{\bar{\tau}} \\ \frac{2}{\omega\bar{\tau}} & \omega > \frac{\pi}{\bar{\tau}} \end{cases}$$

Define $F(s) = f(s)I_n$, and suppose that a minimum realization of $F(s)$ is given by $F(s) = \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & 0 \end{array}\right]$. Then it is straightforward to verify that

$$F(\bar{\tau}s) = \left[\begin{array}{c|c} \bar{\tau}^{-1}A_f & \bar{\tau}^{-1}B_f \\ \hline C_f & 0 \end{array}\right], \tag{10}$$

where A_f, B_f and C_f are constant matrices and independent of $\bar{\tau}$.

4.2 Stability Analysis

Using the above filter, the stability of (1) can be analyzed by choosing $\Omega_1(j\omega) = \{z \in \mathbf{C} |\ |z| \leq |f(j\bar{\tau}\omega)|\}$ and $\Omega_2 = \mathbf{B}$ and applying Lemma 1. Here, we consider the simplest case[3] where $M = I_n$. Employing the transformation $\bar{\phi}_1(s,\tau) = \frac{\phi_1(s,\tau)}{f(\bar{\tau}s)}$ upon (4) and using (2), we have

$$\begin{aligned} sX(s) &= (A + A_d)X(s) \\ &\quad + \bar{\phi}_1 A_d s\bar{\tau} f(\bar{\tau}s)X(s) \end{aligned}$$

with $\bar{\phi}_1(s,\tau) \in \mathbf{E_B}$. Defining $Y_f(s) = F(\bar{\tau}s)X(s)$, then from (10), we have the following realization of this filter

$$\begin{aligned} \dot{z} &= \bar{\tau}^{-1}A_f z + \bar{\tau}^{-1}B_f x \\ y_f &= C_f z \end{aligned}$$

Therefore,

$$\begin{aligned} s\bar{\tau} f(\bar{\tau}s)X(s) &= s\bar{\tau}Y_f(s) \\ &= s\bar{\tau}C_f Z(s) \\ &= C_f[A_f Z(s) + B_f X(s)] \end{aligned}$$

yielding the comparison system

$$\begin{aligned} sX(s) &= (A + A_d)X(s) \\ &\quad + \delta_1(s)A_d C_f[A_f Z(s) + B_f X(s)] \\ sZ(s) &= \bar{\tau}^{-1}A_f Z(s) + \bar{\tau}^{-1}B_f X(s) \end{aligned} \tag{11}$$

with $\delta_1(s) \in \mathbf{E_B}$. (11) can be realized as

$$\begin{aligned} \dot{x} &= (A + A_d)x + A_d u_1 \\ \dot{z} &= \bar{\tau}^{-1}B_f x + \bar{\tau}^{-1}A_f z \\ y_1 &= C_f B_f x + C_f A_f z \\ u_1 &= \delta_1 y_1 \end{aligned} \tag{12}$$

where only a single uncertainty block appears. Hence applying Lemma 2 we obtain the following delay-dependent criterion.

Theorem 2 *The system (1) is asymptotically stable for any constant time-delay $\tau \in [0,\bar{\tau}]$, if there exist matrices $X > 0$ and $Q > 0$, $Q \in \Re^{n\times n}$ such that*

$$\begin{bmatrix} \hat{A}^T X + X\hat{A} & X\hat{B} & \hat{C}^T Q \\ \hat{B}^T X & -Q & 0 \\ Q\hat{C} & 0 & -Q \end{bmatrix} < 0 \tag{13}$$

where $\hat{A} = \begin{bmatrix} A + A_d & 0 \\ \bar{\tau}^{-1}B_f & \bar{\tau}^{-1}A_f \end{bmatrix}$, $\hat{B} = \begin{bmatrix} A_d \\ 0 \end{bmatrix}$, *and* $\hat{C} = \begin{bmatrix} C_f B_f & C_f A_f \end{bmatrix}$.

The idea of using frequency filtering for time-delay systems can also be found in [7]. However, the above result is not equivalent to that of [7], which was based upon an integral quadratic constraint (IQC). As a matter of fact, it turns out that[4], when the same filter is used, the result of [7] is essentially the scaled small gain condition for a comparison system which involves two $n \times n$ diagonal perturbation blocks as the result of introducing an

[3]The general case is presented in the full version of this paper at *http://www.people.virginia.edu/˜crk4y*.

[4]See the full version of this paper for details.

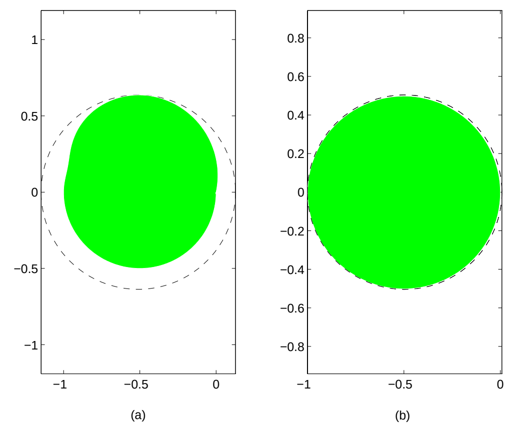

Figure 5: Delay element value set and covering disk. (a) Set $\bar{\Phi}_1$ (shaded) with filter (9) and covering disk $\mathbf{D}(-0.51; 0.637)$ (dashed). (b) Set $\bar{\Phi}_1$ (shaded) with filter (14) and covering disk $\mathbf{D}(-0.496; 0.504)$ (dashed).

additional delay element into the comparison system. In (12), however, only a single such diagonal perturbation block is used. By increasing the total number of uncertainty blocks, additional conservatism may be introduced. Therefore, we expect that the condition (13) is less conservative than the condition of [7] in general. This conjecture has been confirmed in all numerical examples tried by the authors.

5 A Comparison System with Both Filter and Shifted Disk

In this section, we demonstrate that the conservatism of the stability analysis tests may be further reduced by better capturing the gain and phase information of $\phi_1(s,\tau)$ via the use of both a filter and shifted disk. To this end, let $\bar{\phi}_1(s,\tau) = \frac{\phi_1(s,\tau)}{f(\bar{\tau}s)}$ and define a value set $\bar{\Phi}_1$ as $\bar{\Phi}_1 := \{\lambda\bar{\phi}_1(j\omega,\tau) | \, \omega \in \Re_e^+, \, \tau \in [0,\bar{\tau}], \, \lambda \in [0,1]\}$. Consider a covering disk $\mathbf{D}(q;r)$ for $\bar{\Phi}_1$, given by

$$\left|\lambda\bar{\phi}_1(j\omega,\tau) - q\right| \leq r, \quad \forall \omega \in \Re_e^+, \; \tau \in [0,\bar{\tau}], \; \lambda \in [0,1].$$

Two examples which satisfy the above property (see Figure 5) are the filter (9) with the covering disk $\mathbf{D}(-0.51;\, 0.637)$, and the filter

$$f(s) = \frac{2s + 7.0711}{s^2 + 3.15s + 7.0711} \tag{14}$$

along with the covering disk $\mathbf{D}(-0.496; 0.504)$. The stability condition of (1) can be obtained by choosing $\Omega_1(j\omega) = \{\, z \in \mathbf{C} | \, |z - qf(j\bar{\tau}\omega)| \leq r|f(j\bar{\tau}\omega)|\}$ and $\Omega_2 = \mathbf{B}$, and applying Lemma 1. Consider the case where $M = I_n$. With the transformation $\tilde{\phi}_1(s,\tau) = \frac{1}{r}\left(\bar{\phi}_1(s,\tau) - q\right)$, the procedure similar to that of the previous section leads to the following theorem.

Theorem 3 *System (1) is asymptotically stable for any constant time-delay $\tau \in [0,\bar{\tau}]$, if there exist matrices $X > 0$ and $Q > 0$, $Q \in \Re^{n\times n}$ such that*

$$\begin{bmatrix} \tilde{A}^T X + X\tilde{A} & X\tilde{B} & \tilde{C}^T Q \\ \tilde{B}^T X & -Q & 0 \\ Q\tilde{C} & 0 & -Q \end{bmatrix} < 0$$

where $\tilde{A} = \begin{bmatrix} A + A_d + qA_dC_fB_f & qA_dC_fA_f \\ \bar{\tau}^{-1}B_f & \bar{\tau}^{-1}A_f \end{bmatrix}$, $\tilde{B} = \begin{bmatrix} rA_d \\ 0 \end{bmatrix}$, *and* $\tilde{C} = \begin{bmatrix} C_fB_f & C_fA_f \end{bmatrix}$.

Remark 4 *If $q = 0$ and $r = 1$, the above theorem reduces to Theorem 2.*

6 Numerical Example

In this section we compare the stability tests derived in this paper with similar ones published elsewhere [13, 9, 11, 18, 7] through an example motivated by the dynamics of machining chatter. The matrices A and A_d are given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(10.0+K) & 10.0 & 0 & 0 \\ 5.0 & -15.0 & 0 & -.25 \end{bmatrix}$$
$$A_d = \begin{bmatrix} 0 & 0 & K & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

For this case, the generalized Nyquist criterion [5] can be used to obtain the exact stability delay margin. The maximal guaranteed delay margins based on several criteria are shown in Figure 6 as a function of K. We can see that Theorem 3 provides a delay margin quite close to the exact value from Nyquist Criterion. Also, choosing different filters and shifted disks will affect the result as shown in the figure. For this example, the results from Theorem 1, and Theorems 2 and 3 are much better than the previous results for $K > K^*$ when the stability of the system is delay-dependent. In addition, Theorem 1 is less conservative than the result of [13], because, as pointed out earlier, the latter is a special case of the former. They both indicate that for $K < K^*$, the system is stable independent of delay. This is the result of introducing the free matrix M in the comparison system (7). However, when $K < K^*$, Theorems 2 and 3 and the results of [11, 9] can only provide finite delay margins although the correct margin is infinite.

7 Conclusions

Several new conditions, formulated in terms of LMIs, are derived for the stability analysis of linear, time-delay systems. These conditions are based on the ro-

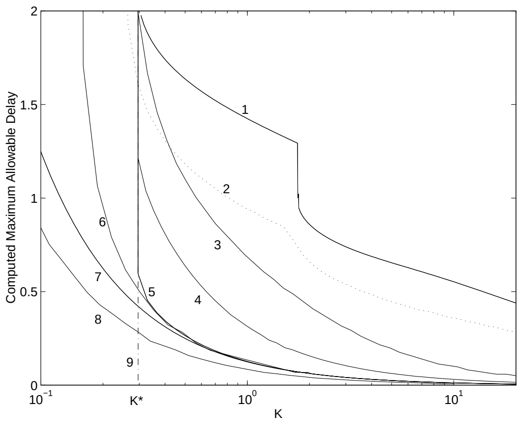

Figure 6: Maximum delay margin vs. K. (1) Actual value from Nyquist Criterion. (2) Theorem 3 with filter (14) and covering disk $\mathbf{D}(-0.496; 0.504)$. (3) Theorem 3 with filter (9) and covering disk $\mathbf{D}(-0.51; 0.637)$. (4) Theorem 1 with covering disk $\mathbf{D}(-0.327; 0.726)$. (5) Result of [13]. (6) Theorem 2 with filter (9). (7) Result of [9]. (8) Result of [11]. (9) Delay-independent result [18].

bust stability analysis of uncertain delay-free comparison systems via the scaled small-gain lemma. The comparison systems are obtained by embedding the non-rational delay elements within norm bounded uncertainty sets and employing loop transformations to reduce conservatism. An example shows that these results can be significantly less conservative than existing criteria in the literature. Finally, we point out that the proposed approach can be easily extended to the robust stability or $\mathcal{H}_\infty$ performance analysis of linear time-delay systems with dynamic or parametric uncertainty.

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