2016 IEEE 55th Conference on Decision and Control (CDC)
ARIA Resort & Casino
December 12-14, 2016, Las Vegas, USA

Partial Attitude Consensus for Underactuated Satellite Clusters

John Matthew Brewer and Panagiotis Tsiotras

***Abstract*— In this paper, we consider two problems dealing with the partial attitude synchronization of a multi-satellite system in which the satellites, modeled as rigid bodies, are underactuated. Having control authority about only two of the three principal axes, the objective is to align the uncontrolled third axis of all satellites such that they point along the same direction. Two control laws are proposed, one which aligns two satellites towards the same direction in inertial space, and another one which synchronizes an N-satellite system such that the satellites align their underactuated axes towards a fixed inertial direction. The synchronization discrepancy between the satellites is expressed in a unique parameterization that describes the uncontrolled axis of one satellite in the frame of the other via a stereographic projection, while the velocity discrepancy is described using a partial angular velocity difference.**

I. INTRODUCTION

Cooperative spacecraft synchronization has been explored in numerous works. Much of the current literature covers complete attitude convergence of a spacecraft cluster, and various approaches have been used to achieve this objective [1], [2], [3], [4], [5], [6], [7]. In [7], [5], the idea of maintaining attitude alignment and performing synchronized spacecraft maneuvers was explored, although in [5] the communication graph was restricted to a bidirectional ring. Simultaneously aligning the orientations of rigid bodies in a network, and stabilizing each body, while spinning about their unstable intermediate axis, was shown in [6]. Successful attitude convergence was proven using graph-theoretic tools in [1] and [2] with undirected and leader-follower communication graphs. Attitude synchronization of a group of satellites was shown in [4] using relative information with neighboring satellites or information of their orientation with respect to a common reference frame. A decentralized controller was presented in [3], which showed attitude convergence even in the presence of uncertainties, disturbances, and time-varying communication delays.

Control of single spacecraft subject to only two control torques has been investigated in [8], [9], [10], [11] using the same novel attitude parameterization used in this paper.

The first objective of this paper is to develop a control law for the partial attitude synchronization between two under-actuated satellites, where control torques about only two of the three principle body axes for each satellite are available. The control objective is to align the third (uncontrolled) axes of the two satellites. As a result, the two underactuated satellites will eventually point their *uncontrolled* axes along the same direction in the inertial space, given any initial orientation. The relative or absolute orientation of the satellites about these axes is inconsequential. The second objective is to develop a control law for the attitude synchronization between all satellites in an N-satellite cluster, where the control law again influences two of the three principle body axes in each satellite, with the purpose of aligning the third uncontrolled axis. As a result of this control law, all satellites will eventually point their uncontrolled axes along the same *fixed* direction in the inertial space given any initial orientation.

Controlling a single axis (as opposed to three-axis control) may be of interest in many applications. For example, this can be the situation when the synchronized pointing of the sensor boresights (e.g., telescope) or high-gain antennas for a multi-satellite system is desired, without the need for complete full attitude control of the satellites. As shown in this paper, such pointing requirements can be achieved using only two control torques per satellite. Even if full actuation is available for most satellites, the proposed control laws can be utilized in cases of actuator failures.

J. M. Brewer is a graduate student at the D. Guggenheim School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332, USA, Email: jbrewer30@gatech.edu

P. Tsiotras is a Professor with the D. Guggenheim School of Aerospace Engineering and the Institute for Robotics and Intelligence Machines, Georgia Institute of Technology, Atlanta, GA 30332, USA, Email: tsiotras@gatech.edu

II. NOTATION AND ATTITUDE KINEMATICS

In this paper, the relative orientation of a satellite with respect to another satellite is given using the w-z attitude parameters introduced in [12], [13] by Tsiotras and Longuski. The precise definition of the z and the w parameters is given below.

A. The w-z Attitude Parameters

Let there be two reference frames in the three-dimensional space consisting of the unit vectors $\{\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ and $\{\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ with the notation $\hat{\mathbf{i}}$ representing the inertial frame and $\hat{\mathbf{b}}$ a body-fixed frame. The rotation matrix that transforms vectors from frame $\hat{\mathbf{i}}$ to frame $\hat{\mathbf{b}}$ will be denoted by R_i^b and can be decomposed into two successive rotations according to two parameters w and z as follows [12], [13]

$$R_i^b(w, z) = R_{i'}^b(w) R_i^{i'}(z), \tag{1}$$

where $\hat{\mathbf{i}}'$ represents the intermediate frame after the first rotation by z. The matrix $R_i^{i'}(z)$ is the initial rotation about the $\hat{\mathbf{i}}_3$ axis in the positive direction by an angle z, resulting in the intermediate frame $\hat{\mathbf{i}}'$. It is therefore given by

$$R_i^{i'}(z) = \begin{bmatrix} \cos(z) & \sin(z) & 0 \\ -\sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2}$$

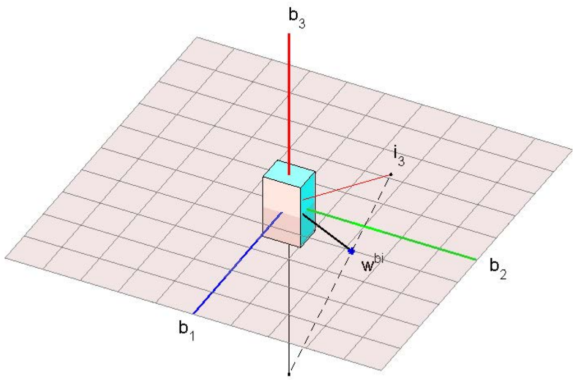


Fig. 1. Visualization of the stereographic projection of $\hat{\mathbf{i}}_3$ in the $\hat{\mathbf{b}}$ frame.

From this intermediate frame, we desire the third axis of the inertial frame (also the third axis of the intermediate frame) to be described with respect to the final body frame. This is accomplished by describing the $\hat{\mathbf{i}}_3$ vector in the $\hat{\mathbf{b}}$ frame as $\hat{\mathbf{i}}_3 = \hat{\mathbf{i}}'_3 = a\hat{\mathbf{b}}_1 + b\hat{\mathbf{b}}_2 + c\hat{\mathbf{b}}_3$. This vector is then stereographically projected onto the $\hat{\mathbf{b}}_1$–$\hat{\mathbf{b}}_2$ plane as a new vector using the variables w_1 and w_2 defined below.

$$w_1 = \frac{b}{1+c}, \qquad w_2 = \frac{-a}{1+c}. \tag{3}$$

The parameters w_1 and w_2 can then be used to describe how far to rotate or "tilt" the $\hat{\mathbf{b}}_3$ axis away from the $\hat{\mathbf{i}}_3$ axis about the vector $\hat{\mathbf{h}} = \hat{\mathbf{i}}_3 \times \hat{\mathbf{b}}_3$ as depicted in Fig. 1. The rotation matrix $R^b_{i'}(w)$ in (1), where $w = [w_1, w_2]^\mathsf{T} \in \mathbb{R}^2$, then describes the rotation about the unit vector $\hat{\mathbf{h}}$, and is given by [13]

$$R^b_{i'}(w) = \frac{1}{1+\|w\|^2}\begin{bmatrix} 1+w_1^2-w_2^2 & 2w_1w_2 & -2w_2 \\ 2w_1w_2 & 1-w_1^2+w_2^2 & 2w_1 \\ 2w_2 & -2w_1 & 1-\|w\|^2 \end{bmatrix}. \tag{4}$$

The angle θ between the unit vectors $\hat{\mathbf{i}}_3$ and $\hat{\mathbf{b}}_3$ can be easily computed as

$$\theta = \arccos\left(\frac{1-w_1^2-w_2^2}{1+w_1^2+w_2^2}\right). \tag{5}$$

Clearly, $\theta = 0$ when $w_1 = w_2 = 0$.

Next, consider the kinematics of the w parameters. To this end, let the angular velocity of the $\hat{\mathbf{b}}$ frame with respect to the $\hat{\mathbf{i}}$ frame, expressed in the $\hat{\mathbf{b}}$ frame, be denoted as $\overrightarrow{\omega}^{bi} = \omega_1\hat{\mathbf{b}}_1 + \omega_2\hat{\mathbf{b}}_2 + \omega_3\hat{\mathbf{b}}_3$, and let $R^b_{\hat{i}}$ be the rotation matrix that transforms elements from the $\hat{\mathbf{i}}$ frame to the $\hat{\mathbf{b}}$ frame of reference as in (1). Taking the derivative of the $\hat{\mathbf{i}}$ unit vectors expressed in the $\hat{\mathbf{b}}$ frame, yields [14], [15], [16]

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{6}$$

Using equations (3) and (6), the differential equations for the parameters w_1 and w_2 can be calculated as

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \frac{1+w_1^2+w_2^2}{2}[\breve{R}^b_{i'}(w)]^\mathsf{T}\omega = \frac{1}{2}\begin{bmatrix} 1+(w_1)^2-(w_2)^2 & 2w_1w_2 & 2w_2 \\ 2w_1w_2 & 1-(w_1)^2+(w_2)^2 & -2w_1 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \tag{7}$$

where $\breve{R}^b_{i'}(w)$ is defined as the matrix consisting of the two first columns of the rotation matrix $R^b_{i'}(w)$ given in (4), and $\omega = [\omega_1, \omega_2, \omega_3]^\mathsf{T} \in \mathbb{R}^3$.

III. UNDERACTUATED TWO SATELLITE SYNCHRONIZATION

A. Definitions and Preliminaries

The first problem we consider in this paper (which we will henceforth refer to as Problem 1) involves the synchronization of two satellites, say, i and j. Associated with satellite i is the body frame $\hat{\mathbf{b}}^i = (\hat{\mathbf{b}}^i_1, \hat{\mathbf{b}}^i_2, \hat{\mathbf{b}}^i_3)$. Satellite i is assumed to have the principle inertia tensor $I^i = \mathrm{diag}(I^i_1, I^i_2, I^i_3)$, with inertias I^i_1, I^i_2, and I^i_3 along the $\hat{\mathbf{b}}^i_1$, $\hat{\mathbf{b}}^i_2$, and $\hat{\mathbf{b}}^i_3$ axes, respectively. We adopt the notation

$$I^i\dot{\omega}^i = S(\omega^i)I^i\omega^i + u^i, \tag{8}$$

to refer to the rigid body dynamics specific to satellite i, where $u^i = [u^i_1, u^i_2, u^i_3]^\mathsf{T} \in \mathbb{R}^3$ represents the control torque influencing satellite i and $\omega^i = [\omega^i_1, \omega^i_2, \omega^i_3]^\mathsf{T} \in \mathbb{R}^3$ represents the inertial angular velocity of satellite i expressed in the $\hat{\mathbf{b}}^i$ frame.

Satellite i is underactuated, that is, a control torque acts only along the $\hat{\mathbf{b}}^i_1$ and $\hat{\mathbf{b}}^i_2$ axes, leaving the $\hat{\mathbf{b}}^i_3$ axis uncontrolled, i.e., $u^i_3 \equiv 0$. The notation and definitions used for satellite j are similar to satellite i and are thus omitted for the sake of brevity. The desired objective is to synchronize the uncontrolled $\hat{\mathbf{b}}^i_3$ and $\hat{\mathbf{b}}^j_3$ axes of the two satellites. This is to be achieved by applying the necessary torques on each satellite that depend only upon the partial *relative* angular velocities and the partial attitude discrepancy, between the two satellites i and j, described by the corresponding attitude parameter w.

To this end, let (a^{ij}, b^{ij}, c^{ij}) be the components of the third body axis of satellite j expressed in the frame of satellite i, that is, let $\hat{\mathbf{b}}^j_3 = a^{ij}\hat{\mathbf{b}}^i_1 + b^{ij}\hat{\mathbf{b}}^i_2 + c^{ij}\hat{\mathbf{b}}^i_3$. A similar form exists for describing the third body axis of satellite i expressed in the frame of satellite j which is found by simply switching the indices, i and j. This allows us to write the w parameters defining the unit vectors $\hat{\mathbf{b}}^j_3$ in the frame $\hat{\mathbf{b}}^i$ as

$$w^{ij}_1 = \frac{b^{ij}}{1+c^{ij}}, \qquad w^{ij}_2 = \frac{-a^{ij}}{1+c^{ij}}, \tag{9}$$

and similarly for $\hat{\mathbf{b}}^i_3$ in the frame $\hat{\mathbf{b}}^j$ in which we now let $w^{ij} = [w^{ij}_1, w^{ij}_2]^\mathsf{T} \in \mathbb{R}^2$ define the stereographic projection of the vector $\hat{\mathbf{b}}^j_3$ onto the $\hat{\mathbf{b}}^i_1$–$\hat{\mathbf{b}}^i_2$ plane, and similarly for $w^{ji} = [w^{ji}_1, w^{ji}_2]^\mathsf{T} \in \mathbb{R}^2$. Note that the two satellites are synchronized when $\hat{\mathbf{b}}^i_3 = \hat{\mathbf{b}}^j_3$, equivalently, when $\hat{\mathbf{b}}^i_3 \times \hat{\mathbf{b}}^j_3 = 0$. Thus, $a^{ij} = b^{ij} = a^{ji} = b^{ji} = 0$, and hence $w^{ij}_1 = w^{ij}_2 =$

0 indicates synchronization. Note that the relative angular velocity of satellite j with respect to satellite i expressed in the frame of satellite i is given by

$$\omega^{ij} = \omega^i - R^i_j \omega^j, \tag{10}$$

where R^i_j is the rotation matrix describing elements of frame $\hat{\mathbf{b}}^j$ in the frame $\hat{\mathbf{b}}^i$. A similar expression holds for the relative angular velocity of satellite i with respect to satellite j expressed in the frame of satellite j.

From (7) it follows that the differential equations for the parameters w^{ij} are given by

$$\dot{w}^{ij} = \frac{1+\|w^{ij}\|^2}{2}[\breve{R}^i_{j'}(w^{ij})]^\mathsf{T}\omega^{ij}, \tag{11a}$$

and similarly for w^{ji}

Next, we present two lemmas that will help us solve Problem 1.

Lemma 1: Let $\hat{\mathbf{b}}^i$ and $\hat{\mathbf{b}}^j$ denote the two body reference frames for satellites i and j, respectively, and let $\tilde{w}^{ij} = [w^{ij}_1, w^{ij}_2, 0]^\mathsf{T} \in \mathbb{R}^3$ be the parameters defining the third axis of the reference frame $\hat{\mathbf{b}}^j$ expressed in the frame $\hat{\mathbf{b}}^i$ as in (9), and similarly let $\tilde{w}^{ji} = [w^{ji}_1, w^{ji}_2, 0]^\mathsf{T} \in \mathbb{R}^3$ be the parameters defining the third axis of the reference frame $\hat{\mathbf{b}}^i$ expressed in the frame of reference $\hat{\mathbf{b}}^j$. Then

$$\tilde{w}^{ij} = -R^i_j \tilde{w}^{ji}, \qquad \tilde{w}^{ji} = -R^j_i \tilde{w}^{ij}. \tag{12}$$

Proof: The rotation matrix that transforms vectors from frame $\hat{\mathbf{b}}^j$ to frame $\hat{\mathbf{b}}^i$ is R^i_j and depends on the values of w^{ij} and z^{ij} as follows

$$R^i_j = R^i_{j'}(w^{ij}) R^{j'}_j(z^{ij}) \tag{13}$$

with $R^i_{j'}(w^{ij})$ and $R^{j'}_j(z^{ij})$ as in (1) and similarly for $R^j_i = R^j_{i'}(w^{ji}) R^{i'}_i(z^{ji})$. Using the fact that $R^i_j = [R^j_i]^\mathsf{T}$, the following two identities result immediately.

$$\begin{aligned} -w^{ji}_1 &= w^{ij}_1 \cos(z^{ij}) - w^{ij}_2 \sin(z^{ij}) \\ -w^{ji}_2 &= w^{ij}_1 \sin(z^{ij}) + w^{ij}_2 \cos(z^{ij}) \end{aligned} \tag{14}$$

By substituting equations (14) and the rotation matrix expression in equation (13) into equation (12) proves the desired result. ■

Lemma 2: Let Q_i represent the rotation matrix that transforms elements from the body frame $\hat{\mathbf{b}}^i$ to the inertial frame, let $\breve{Q}_i \in \mathbb{R}^{3\times 2}$ be the matrix consisting of the first two columns of Q_i, and let Q_j and $\breve{Q}_j$ be defined similarly for the body frame $\hat{\mathbf{b}}^j$. Then, $\breve{Q}_i^\mathsf{T}\breve{Q}_i = I_2$ and $\breve{Q}_i^\mathsf{T}\breve{Q}_j = \tilde{R}^i_j$, where $\tilde{R}^i_j$ represents the upper left 2×2 sub-block of R^i_j.

Proof: The proof follows immediately by performing the matrix multiplications, and thus it is omitted. ■

B. Main Result

In this section we provide the explicit feedback control law that solves Problem 1.

Proposition 1: Assume that satellites i and j are in communication. The feedback control law for satellite i

$$\begin{aligned} u^i_1 &= -k^i_3(I^i_2 - I^i_3)\omega^i_2\omega^i_3 - k_1 w^{ij}_1 - k_2 \delta\eta^{ij}_1, \\ u^i_2 &= -k^i_3(I^i_3 - I^i_1)\omega^i_1\omega^i_3 - k_1 w^{ij}_2 - k_2 \delta\eta^{ij}_2, \end{aligned} \tag{15}$$

where the control gains are $k_1 > 0$, $k_2 > 0$, and

$$k^i_3 = \begin{cases} 1 & \text{for } I^i_1 \neq I^i_2 \\ \alpha_i & \text{for } I^i_1 = I^i_2 \end{cases} \tag{16}$$

where $\alpha_i \in \mathbb{R}$ and $\delta\eta^{ij}_1$ and $\delta\eta^{ij}_2$ are defined as

$$\delta\eta^{ij} = \eta^i - \tilde{R}^i_j \eta^j, \tag{17}$$

with $\delta\eta^{ij} = [\delta\eta^{ij}_1, \delta\eta^{ij}_2]^\mathsf{T} \in \mathbb{R}^2$, and similarly for satellite j, ensures that $\lim_{t\to\infty} w^{ij} = \lim_{t\to\infty} w^{ji} = 0$ and $\lim_{t\to\infty} \delta\eta^{ij} = \lim_{t\to\infty} \delta\eta^{ji} = 0$, and thus solves Problem 1.

Proof: From equation (8) and the control law (15), the closed-loop dynamics of satellite i are given by

$$\begin{aligned} I^i_1\dot{\omega}^i_1 &= (1-k^i_3)(I^i_2 - I^i_3)\omega^i_2\omega^i_3 - k_1 w^{ij}_1 - k_2\delta\eta^{ij}_1, \\ I^i_2\dot{\omega}^i_2 &= (1-k^i_3)(I^i_3 - I^i_1)\omega^i_1\omega^i_3 - k_1 w^{ij}_2 - k_2\delta\eta^{ij}_2, \\ I^i_3\dot{\omega}^i_3 &= (I^i_1 - I^i_2)\omega^i_1\omega^i_2, \end{aligned} \tag{18}$$

where a similar form also exists for satellite j by switching the indices, i and j.

Consider the Lyapunov function candidate

$$\begin{aligned} V =& k_1 \ln(1+\|w^{ij}\|^2) + k_1 \ln(1+\|w^{ji}\|^2) \\ &+ (\eta^i)^\mathsf{T} J^i \eta^i + (\eta^j)^\mathsf{T} J^j \eta^j. \end{aligned} \tag{19}$$

where $J^i = \mathrm{diag}(I^i_1, I^i_2)$ and $J^j = \mathrm{diag}(I^j_1, I^j_2)$. By taking the derivative of V along the trajectories of (18), and after performing several algebraic manipulations, yields the following expression

$$\dot{V} = -2k_2(\eta^i)^\mathsf{T}\delta\eta^{ij} - 2k_2(\eta^j)^\mathsf{T}\delta\eta^{ji} \tag{20}$$

where we have used Lemma 1 along with the fact that

$$2(w^{ij})^\mathsf{T}\dot{w}^{ij} = (1+\|w^{ij}\|^2)\big((w^{ij})^\mathsf{T}\eta^i + (w^{ji})^\mathsf{T}\eta^j\big) \tag{21}$$

which follows from the use of (11), Lemma 1, and (17). To proceed, recognize that $\tilde{R}^i_j = [\tilde{R}^j_i]^\mathsf{T}$ and recall from (17) that

$$\delta\eta^{ij} = \eta^i - \tilde{R}^i_j\eta^j, \qquad \delta\eta^{ji} = \eta^j - \tilde{R}^j_i\eta^i.$$

It then follows that

$$\dot{V} = -2k_2\big(\breve{Q}_i\eta^i - \breve{Q}_j\eta^j\big)^\mathsf{T}\big(\breve{Q}_i\eta^i - \breve{Q}_j\eta^j\big) \leq 0.$$

where we have used Lemma 2. Since $\dot{V} \leq 0$, it follows that the signals w^{ij}, w^{ji}, η^i, and η^j are all bounded. Now, by letting $\dot{V} = 0$, it follows that

$$\breve{Q}_i\eta^i = \breve{Q}_j\eta^j, \tag{22}$$

which after a left multiplication of $\breve{Q}_i^\mathsf{T}$ or $\breve{Q}_j^\mathsf{T}$ using again Lemma 2, we find $\eta^i = \tilde{R}^i_j\eta^j$ and $\eta^j = \tilde{R}^j_i\eta^i$. This along with (22) leads to

$$\delta\eta^{ij} = \eta^i - \tilde{R}^i_j\eta^j = 0, \tag{23a}$$

$$\delta\eta^{ji} = \eta^j - \tilde{R}^j_i\eta^i = 0, \tag{23b}$$

$$(\eta^i)^\mathsf{T}\eta^i = (\eta^j)^\mathsf{T}\eta^j. \tag{23c}$$

Taking advantage of (23), the following is found after some additional algebraic manipulations.

$$0 = \omega^i_1 w^{ij}_2 - \omega^i_2 w^{ij}_1. \tag{24}$$

Assume now that $w^{ij} \neq 0$, otherwise there is nothing to prove. From the previous expression it follows that $w^{ij} = \lambda\eta^i$ where $\lambda \neq 0$ and $\eta^i \neq 0$. It will be shown that this leads to a contradiction. Indeed, assuming $\dot{V} \equiv 0$, it follows from Lemma 1, (18), and (22) that

$$\begin{aligned} 0 &= (\eta^i)^\mathsf{T}\dot{\omega}^i_{12} - (\eta^j)^\mathsf{T}\dot{\omega}^j_{12} \\ &= -k_1(\eta^i)^\mathsf{T}\big[(J^i)^{-1} + \tilde{R}^i_j(J^j)^{-1}\tilde{R}^j_i\big]w^{ij}_{12}. \end{aligned} \tag{25}$$

Letting $P = (J^i)^{-1} + \tilde{R}^i_j(J^j)^{-1}\tilde{R}^j_i > 0$, the previous expression yields $0 = (\eta^i)^\mathsf{T}Pw^{ij} = \lambda(\eta^i)^\mathsf{T}P\eta^i$, which implies that either $\lambda = 0$ or $\eta^i = 0$, leading to a contradiction. Thus, the only point contained within the invariant set $\mathcal{M} \triangleq \{(w^{ij}, \eta^{ij}) : \dot{V} \equiv 0\}$ is $(w^{ij}, \eta^{ij}) = (0, 0)$ and by LaSalle's invariance principle, the proof is complete. ■

IV. N-SATELLITE SYNCHRONIZATION WITH INERTIAL VELOCITY FEEDBACK

In this section we consider the problem of partial attitude synchronization of $N \geq 2$ underactuated satellites. Each satellite, denoted by i, belongs to the set $S = \{1, 2, \ldots, N\}$. Satellite $i \in S$ is in communication with a subset of S denoted by $\mathcal{N}_i \subset S \setminus \{i\}$. We assume that $\mathcal{N}_i \neq \emptyset$ for all $i \in S$. We also assume that each communication link between any satellite pair is bi-directional, that is, $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. As in Problem 1, each satellite in S is underactuated with the desired objective being to synchronize the uncontrolled $\hat{\mathbf{b}}^i_3$ axes for all satellites so that, unlike the objective in Problem 1, the final orientation of $\hat{\mathbf{b}}^i_3$ for each satellite $i \in S$ is along some common, *fixed* inertial direction. This can be achieved by applying the necessary torques on each satellite i that depend only upon its corresponding *inertial* angular velocities and the partial attitude discrepancy, between the satellites i and $j \in \mathcal{N}_i$, described by w^{ij}. Henceforth, we will refer to this as Problem 2. The definitions related to any satellite in S and the parameters defining the relationship between any two satellites i and j in S are identical to those in Problem 1.

For solving Problem 2 in this section, we need to use some tools from algebraic graph theory [17], [18]. Communication between satellites will be encoded using a communication graph $\mathcal{G}$. The graph $\mathcal{G} = \{V, E\}$ is described by the vertex set $V = \{1, \cdots, N\}$ having as vertices the satellites, and the edge set $E = \{(i,j) \in V \times V \mid j \in \mathcal{N}_i\} = \{e_1, \cdots, e_M\}$ consisting of M directed links (i, j), where the set $\mathcal{N}_i \subset V$ consists of all satellites j in communication with satellite i. A graph $\mathcal{G}$ with a directed set of edges is a *directed* graph. The graph $\mathcal{G}$ is *undirected* when $j \in \mathcal{N}_i \iff i \in \mathcal{N}_j$, $\forall i, j \in V$, $i \neq j$. The incidence matrix for a directed graph $\mathcal{G}$, denoted by $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{N\times M}$, is defined as

$$[\mathcal{D}(\mathcal{G})]_{ik} \begin{cases} 1, & \text{if } (i,j) = e_k \in E \\ -1, & \text{if } (j,i) = e_k \in E \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Let the incidence matrix $\mathcal{D}(\mathcal{G})$ for a directed graph be the same for an undirected graph in which each edge in the undirected graph is arbitrarily directed. Notice that each column of $\mathcal{D}(\mathcal{G})$ sums to 0 and contains a 1 and a -1 for each end of the corresponding edge. An undirected graph $\mathcal{G}$ is a tree when there is *exactly* one path between any two vertices in the graph, i.e. it is acyclic. A tree has $M = N-1$ edges and the matrix $\mathcal{D}(\mathcal{G})$ for a tree is full column rank.

Lemma 3: Consider an undirected tree graph $\mathcal{G}$ with n vertices and $m = n-1$ edges and incidence matrix $\mathcal{D}(\mathcal{G}) \in \mathbb{R}^{n\times m}$. Let the matrix $\tilde{\mathcal{D}}(\mathcal{G}) \in \mathbb{R}^{pn\times qm}$ be such that it has the same structure as $\mathcal{D}(\mathcal{G}) \otimes A$, where $A \in \mathbb{R}^{p\times q}, p \geq q$ constructed as follows. For every element in $\mathcal{D}(\mathcal{G})$, there corresponds a $p\times q$ submatrix in $\tilde{\mathcal{D}}(\mathcal{G})$ so that $\tilde{\mathcal{D}}(\mathcal{G})$ is given by

$$[\tilde{\mathcal{D}}(\mathcal{G})]_{a,b} = \begin{cases} \Phi_{ik} & \text{if } (i,j) = e_k \in E \\ \Psi_{ik} & \text{if } (j,i) = e_k \in E \\ 0 & \text{otherwise.} \end{cases} \tag{27}$$

where a and b indicate a range of indices for the submatrix $[\tilde{\mathcal{D}}(\mathcal{G})]_{a,b}$ where $p(i-1)+1 \leq a \leq pi$ and $q(k-1)+1 \leq b \leq kq$, and where $\Phi_{ik}, \Psi_{ik} \in \mathbb{R}^{p\times q}$. Then, if either Φ_{ik} or Ψ_{ik} is full column rank for all $i = 1, \ldots, N$ and $k = 1, \ldots, m$, then $\tilde{\mathcal{D}}(\mathcal{G})$ is also full column rank.

Proof: Begin by noticing that because the graph $\mathcal{G}$ is a tree, then $\mathcal{D}(\mathcal{G})$ is full column rank. Now, notice that an ordering of vertices and edges always exists such that the upper $(n-1)\times m$ submatrix of $\mathcal{D}(\mathcal{G})$ is lower triangular. Since $\mathcal{D}(\mathcal{G})$ may be constructed for an undirected graph through arbitrarily directing each link, then without loss of generality, we choose Φ_{ik}, for all $i = 1, \ldots, N$ and $k = 1, \ldots, m$, to be full column rank and the matrix $\tilde{\mathcal{D}}(\mathcal{G})$ may be constructed such that the upper $p(n-1)\times qm$ submatrix is lower block triangular where the blocks along the diagonal are the appropriate matrices Φ_{ik}. Then it follows directly that $\tilde{\mathcal{D}}(\mathcal{G})$ is full column rank. ■

The next proposition provides the solution to Problem 2.

Proposition 2: Consider a system of N satellites and assume that their communication graph $\mathcal{G}$ is a tree. The feedback control law for each satellite $i \in S$,

$$\begin{aligned} u^i_1 &= -k_{2i}\omega^i_1 - \textstyle\sum_{j\in\mathcal{N}_i} k_1 w^{ij}_1, \\ u^i_2 &= -k_{2i}\omega^i_2 - \textstyle\sum_{j\in\mathcal{N}_i} k_1 w^{ij}_2, \end{aligned} \tag{28}$$

where the control gains are $k_1 > 0$ and $k_{2i} > 0$ for $i \in S$, ensures that $\lim_{t\to\infty} w^{ij} = 0$ and $\lim_{t\to\infty} \eta^i = 0$ for all $i \in S$ and $j \in \mathcal{N}_i$, and thus solves Problem 2.

Proof: From the control law (28) and equations (15), the closed-loop dynamics of satellite i are given by

$$\begin{aligned} I^i_1\dot{\omega}^i_1 &= (I^i_2 - I^i_3)\omega^i_2\omega^i_3 - k_{2i}\omega^i_1 - \textstyle\sum_{j\in\mathcal{N}_i} k_1 w^{ij}_1, \\ I^i_2\dot{\omega}^i_2 &= (I^i_3 - I^i_1)\omega^i_1\omega^i_3 - k_{2i}\omega^i_2 - \textstyle\sum_{j\in\mathcal{N}_i} k_1 w^{ij}_2, \\ I^i_3\dot{\omega}^i_3 &= (I^i_1 - I^i_2)\omega^i_1\omega^i_2. \end{aligned} \tag{29}$$

Consider the Lyapunov function candidate

$$V = \sum_{i\in S}\sum_{j\in\mathcal{N}_i} \frac{k_1}{2}\ln(1 + \|w^{ij}\|^2) + \frac{1}{2}\sum_{i\in S}(\omega^i)^\mathsf{T} I^i\omega^i. \tag{30}$$

Taking the time derivative of V along the trajectories of the system (29), and after combining terms, yields

$$\begin{aligned} \dot{V} &= \sum_{i\in S} -k_{2i}(\eta^i)^{\mathsf{T}}\eta^i + \sum_{i\in S}\sum_{j\in\mathcal{N}_i}(k_1 - k_1)(\eta^i)^{\mathsf{T}}w^{ij} \\ &= -\sum_{i\in S} k_{2i}(\eta^i)^{\mathsf{T}}\eta^i \ \leq\ 0. \end{aligned} \tag{31}$$

It follows that the system is Lyapunov stable and hence the signals w^{ij} and ω^i, for $j \in \mathcal{N}_i$ and all $i \in S$, are bounded. From (29), it follows that $\mathcal{M} \triangleq \{(w^{ij}, \eta^i) : \dot{V} \equiv 0\} = \{(w^{ij}, 0) : \sum_{j\in\mathcal{N}_i} w^{ij} = 0 \ \forall i \in S\}$.

Since a tree with N vertices and $N-1$ edges has an incidence matrix $\mathcal{D} \in \mathbb{R}^{N\times(N-1)}$ for an arbitrarily directed digraph that is full column rank [18], the agreement subspace $\lim_{t\to\infty}\sum_{j\in\mathcal{N}_i} w^{ij} = 0$, for all $i \in S$, can be described as

$$\begin{bmatrix} \sum_{j\in\mathcal{N}_1} w^{1j} \\ \vdots \\ \sum_{j\in\mathcal{N}_N} w^{Nj} \end{bmatrix} = \mathbf{0} = \tilde{\mathcal{D}}E, \tag{32}$$

where $\tilde{\mathcal{D}}$ is a block matrix with the same form of $\mathcal{D} \otimes A$, where $A \in \mathbb{R}^{2\times 2}$, and E represents the column matrix consisting of the edge measurements w^{ij} without the repeat edge measurements w^{ji}. It follows from Lemma 1 and Lemma 3, where we have taken $\Phi_{ik} = \mathrm{I}_{2\times 2}$ and $\Psi_{ik} = \tilde{R}^i_j$ for each corresponding edge $w^{ji} \in E$, that $\tilde{\mathcal{D}}$ is also full column rank. Then the agreement subspace $\mathcal{M}$ consists only of the point where $\lim_{t\to\infty} w^{ij} = 0$ and $\lim_{t\to\infty} \eta^i = 0$ for all $i \in S, j \in \mathcal{N}_i$, and by LaSalle's invariance principle, the proof is complete. ■

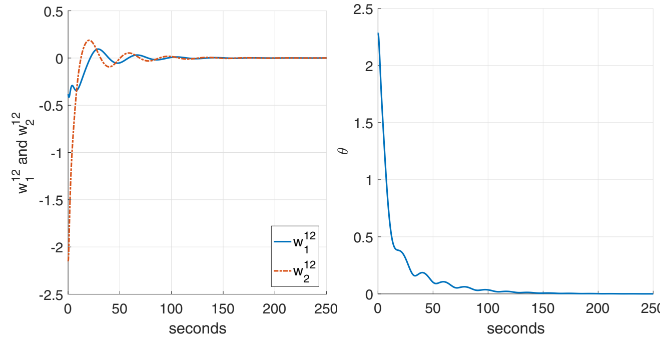
V. SIMULATION RESULTS

To demonstrate the results of this paper, two numerical simulations to validate Proposition 1 and Proposition 2 were performed. For Proposition 1, two satellites, one having generic inertia properties and the other being axisymmetric, were simulated. For Problem 2, four satellites with generic inertia properties were simulated.

A. Simulation 1: Two Satellites in Communication

For Problem 1, we assume two satellites in communication and take $i = 1$ and $j = 2$. The satellites have moments of inertia $I^1 = \mathrm{diag}(10, 25, 20)\ \mathrm{kg}\cdot\mathrm{m}^2$ and $I^2 = \mathrm{diag}(10, 10, 25)\ \mathrm{kg}\cdot\mathrm{m}^2$. The initial angular velocities for Satellites 1 and 2 are $\omega^1(0) = (0.2, -0.1, 0.2)^{\mathsf{T}}$ rad/s and $\omega^2(0) = (-0.2, 0.2, -0.2)^{\mathsf{T}}$ rad/s, respectively. The initial orientations given in unit quaternions for Satellites 1 and 2 are $[0.8805, 0.4402, -0.1761, 0]$ and $[0.2774, 0.5547, 0.5547, 0.5547]$, respectively. The control gains were set to $k_1 = 1.5$ and $k_2 = 6$ for both satellites, $k_3^1 = 1$, and $k_3^2 = 0$.

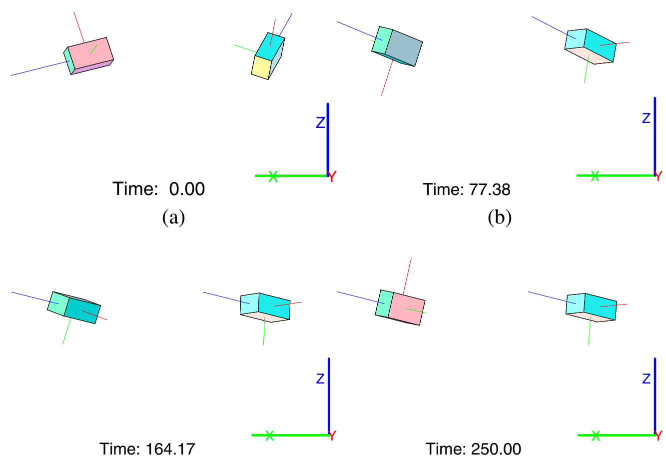
The w parameters describing the $\hat{\mathbf{b}}_3$ axis of Satellite 2 in the frame of Satellite 1 are plotted in Fig. 2. To better visualize the effectiveness and behavior of the controller, an animation depicting the two satellites as they rotate in inertial space was also made. Snapshots at distinct time instances from the animation are shown in Fig. 3. The animation can be found in http://dcsl.gatech.edu/movies/undersatsim1.avi.



(a) Time history of w parameters of sat 2 in the frame of sat 1

(b) Time history of θ between sat 1 and sat 2

Fig. 2. The w parameter state history of simulation 1.



(a) (b) (c) (d)

Fig. 3. Snapshots at different phases of simulation 1.

B. Simulation 2: N-Satellites in Communication

For Problem 2, we assume four satellites communicating based on a given communication topology encoded in the sets $\mathcal{N}_i$, for $i \in S = \{1, 2, 3, 4\}$. For this problem we therefore have $N = 4$. The assumed communication connections are given by $\mathcal{N}_1 \triangleq \{3\}$, $\mathcal{N}_2 \triangleq \{3\}$, $\mathcal{N}_3 \triangleq \{1, 2, 4\}$, $\mathcal{N}_4 \triangleq \{3\}$. The satellites have moments of inertia $I^1 = \mathrm{diag}(45, 25, 15)\ \mathrm{kg}\cdot\mathrm{m}^2$, $I^2 = \mathrm{diag}(15, 35, 45)\ \mathrm{kg}\cdot\mathrm{m}^2$, $I^3 = \mathrm{diag}(84, 24, 15)\ \mathrm{kg}\cdot\mathrm{m}^2$, and $I^4 = \mathrm{diag}(60, 40, 15)\ \mathrm{kg}\cdot\mathrm{m}^2$, and their initial angular velocities are $\omega^1(0) = (0.2, -0.1, 0.1)^{\mathsf{T}}$ rad/s, $\omega^2(0) = (-0.1, -0.1, -0.2)^{\mathsf{T}}$ rad/s, $\omega^3(0) = (0.1, 0.1, 0.1)^{\mathsf{T}}$ rad/s, and $\omega^4(0) = (0.1, 0.3, 0.1)^{\mathsf{T}}$ rad/s, respectively. The initial orientations given in unit quaternions for Satellites 1, 2, 3, and 4 were chosen as $[1, 0, 0, 0]$, $[0.5547, 0.5547, -0.2774, 0.5547]$, $[0, -0.3162, 0, 0.9487]$, and $[0, -0.4472, 0, 0.8944]$, respectively. The control gains were set to $k_1 = 0.2$ and $k_{2i} = 1$ for $i = 1, 2, 3, 4$.

The w parameters describing the $\hat{\mathbf{b}}_3$ axis of Satellite 2 in the frame of Satellite 1 and Satellites 1, 2, and 4 in the frame of Satellite 3 are plotted in Fig. 4. Similar behavior was observed for the other partial attitude parameters as well. Several snapshots of the simulation's animation are shown in Fig. 5. The animation can be found in http://dcsl.gatech.edu/movies/undersatsim2.avi.

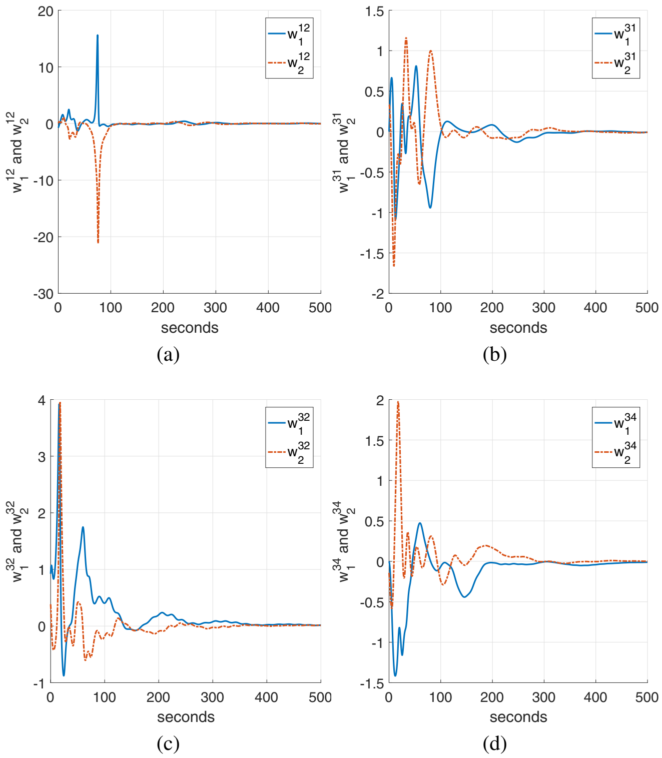


Fig. 4. The w parameter state histories for Problem 2: (a) w parameters of satellite 2 in the frame of satellite 1; (b) w parameters of satellite 1 in the frame of satellite 3; (c) w parameters of satellite 2 in the frame of satellite 3; (d) w parameters of satellite 4 in the frame of satellite 3.

VI. CONCLUSIONS

In this paper we have solved two problems related to the partial attitude stabilization within a system of multiple underactuated satellites. Each of the satellites considered has two control torques acting on two of their principal axes, while the third principal axis is left uncontrolled. Only relative partial attitude is used in the feedback control. Two cases are considered. In the first case the satellites' underactuated axes end up pointing along the same orientation in space and use only relative partial velocities for feedback. In the second case the axes end up pointing along the same fixed inertial orientation and use inertial partial velocity for feedback. In addition to maintaining alignment of the relative attitude among intentionally underactuated spacecraft, the results of this paper can also be used to control originally fully-actuated satellite clusters subject to actuator failures.

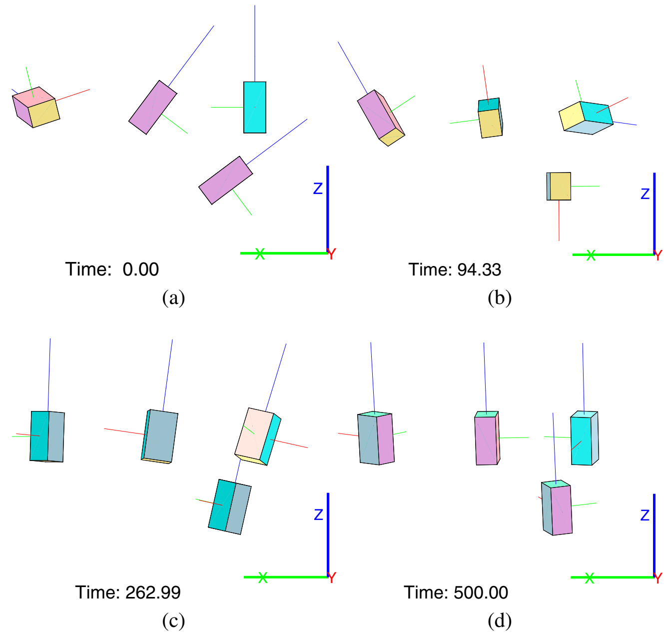


Fig. 5. Snapshots at different phases of the simulation for Problem 2.

REFERENCES

[1] D. V. Dimarogonas, P. Tsiotras, and K. J. Kyriakopoulos, "Laplacian cooperative attitude control of multiple rigid bodies," in *IEEE International Symposium on Intelligent Control*, Munich, Germany, October 4-6 2006, pp. 3064–3069.

[2] D. V. Dimarogonas, P. Tsiotras, and K. J. Kyriakopoulos, "Leader-follower cooperative attitude control of multiple rigid bodies," *Systems & Control Letters*, vol. 58, no. 6, pp. 429 – 435, 2009.

[3] J. Erdong, J. Xiaolei, and S. Zhaowei, "Robust decentralized attitude coordination control of spacecraft formation," *Systems & Control Letters*, vol. 57, no. 7, pp. 567 – 577, 2008.

[4] J. Thunberg, E. Montijano, and X. Hu, "Distributed attitude synchronization control," in *50th IEEE Conference on Decision and Control and European Control Conference*, Orlando, FL, Dec. 12–15, 2011, pp. 1962–1967.

[5] J. R. Lawton and R. W. Beard, "Synchronized multiple spacecraft rotations," *Automatica*, vol. 38, no. 8, pp. 1359 – 1364, 2002.

[6] S. Nair and N. E. Leonard, "Stabilization of a coordinated network of rotating rigid bodies," in *43rd IEEE Conference on Decision and Control*, vol. 5, Atlanta, GA, Dec. 15–17, 2004, pp. 4690–4695.

[7] P. K. C. Wang, F. Y. Hadaegh, and K. Lau, "Synchronized formation rotation and attitude control of multiple free-flying spacecraft," *Journal of Guidance, Control, and Dynamics*, vol. 22, no. 1, pp. 28–35, 1999.

[8] P. Tsiotras and J. M. Longuski, "Spin-axis stabilization of symmetric spacecraft with two control torques," *Systems & Control Letters*, vol. 23, no. 6, pp. 395 – 402, 1994.

[9] P. Tsiotras, M. Corless, and J. Longuski, "A novel approach to the attitude control of axisymmetric spacecraft," *Automatica*, vol. 31, no. 8, pp. 1099 – 1112, 1995.

[10] P. Tsiotras and J. Luo, "Control of underactuated spacecraft with bounded inputs," *Automatica*, vol. 36, no. 8, pp. 1153 – 1169, 2000.

[11] H. Shen and C. M. Roithmayr, "Co-spin with symmetry axis stabilization, and de-spin for asteroid capture," in *American Control Conference*, Portland, OR, June 4–6, 2014, pp. 1599–1604.

[12] P. Tsiotras and J. M. Longuski, "Comments on a new parameterization of the attitude kinematics," in *AIAA/AAS Astrodynamics Specialists Conference*, ser. 96, no. 3627. San Diego, CA: AAS, July 29-31 1996.

[13] P. Tsiotras and J. M. Longuski, "A new parameterization of the attitude kinematics," *The Journal of the Astronautical Sciences*, vol. 43, no. 3, pp. 243–262, 1995.

[14] M. D. Shuster, "A survey of attitude representations," *The Journal of the Astronautical Sciences*, vol. 41, no. 4, pp. 439–517, 1993.

[15] T. R. Kane, P. W. Likins, and D. A. Levinson, *Spacecraft Dynamics*. New York: McGraw-Hill Book Company, August 1983.

[16] B. Wie, *Space Vehicle Dynamics and Control*, Second ed. Reston, VA: AIAA, Inc., August 2008.

[17] C. Godsil and G. Royle, *Algebraic Graph Theory*. Springer New York, 2001.

[18] M. Mesbahi and M. Egerstedt, *Graph Theoretic Methods in Multiagent Networks*. Princeton, New Jersey: Princeton University Press, 2010.