Peer-to-Peer Refuelling within a Satellite Constellation
Part II: Nonzero-Cost Rendezvous Case

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***Abstract*— In this paper, we study the scheduling problem arising from refuelling multiple satellites in a constellation. It is assumed that there is no fuel delivered to the constellation externally. Instead, the satellites in the constellation are assumed to be capable of refuelling each other. The cost of the rendezvous maneuver between two satellites conducting a fuel transfer is taken into consideration in the formulation of the problem. The goal of the refuelling problem is to equalize the fuel stored among all satellites in the constellation after a refuelling period. It is shown that the problem of equalizing the fuel among the satellites can be formulated and solved as a maximum-weight matching problem.**

I. Introduction

In this and a companion paper [4] we address the problem of a satellite constellation in Low Earth Orbit (LEO). It is assumed that no extra fuel is delivered to the constellation by an external spacecraft. In order to keep the constellation operational we require that the fuel is equally distributed between all the satellites in the constellation. We assume that refuelling is performed by the satellites themselves. We call this the Peer-to-Peer (P2P) refuelling scenario. In such a P2P refuelling scenario satellites with excess fuel can deliver fuel to the satellites depleted or low on fuel. Thus, the lifespan of the satellite constellation (dictated by the fuel stored among all satellites) is kept to its full potential. The goal of the P2P refuelling problem is thus to achieve fuel equalization among all satellites in the constellation within a given time period.

In [4] we formulated and solved the P2P refuelling problem when the rendezvous cost (i.e., the fuel expended for the two satellites to meet) is negligible. This is a realistic scenario only when the satellites in the constellation are close together or when the allowed rendezvous periods are long. Otherwise the fuel expenditure to perform the rendezvous cannot be ignored and has to be accounted for in the problem formulation.

The (time-constrained) P2P refuelling problem has combinatorial complexity. In order to obtain a solution we formulate the P2P refuelling problem as a sequence of *fuel transactions* between *sellers* (i.e., satellites high on fuel) and *buyers* (i.e., satellites low on fuel). The activity of two satellites transferring fuel is called a *fuel transaction*. Within a given refuelling period, if two satellites conduct a fuel transaction, these two same satellites cannot conduct fuel transactions with any other satellites. Therefore, the objective of the P2P refuelling problem is to find independent seller-buyer pairs such that the fuel is equalized among all satellites after a fuel transaction is conducted between a pair of satellites. As in [4], the P2P refuelling problem is formulated and solved as a maximum-weight matching problem.

II. Problem Formulation

The following notation is adopted from [4]. We assume that there are $n \geq 3$ satellites within a constellation. Then $\mathcal{I} = \{1, 2, \cdots, n\}$ denotes the index set of the n satellites. f_i^- and f_i^+, $i \in \mathcal{I}$, denote the fuel owned by satellite i before and after refuelling. $\bar{f}^-$ and $\bar{f}^+$ denote the average amount of fuel among all satellites before and after refuelling. g_i^j denotes the amount of fuel that is transferred from satellite i to satellite j when they conduct a fuel transaction. Among all satellites, *fuel-sufficient satellites* (i.e., sellers(are the ones with more than the average amount of fuel, and *fuel-deficient satellites* (i.e., buyers) are the ones with less than the average amount of fuel.

It is assumed that for a pair of satellites, say i and j, only one of the two can be active (i.e., it initiates the fuel transaction). For example, if satellite i is active, it applies thrust to travel to j and conducts a fuel transaction with j, before returning to its original orbital slot. During this process, satellite j remains at its pre-assigned orbital slot. Thus, only the active satellite consumes fuel during a fuel transaction.

Let p_i^j denote the fuel consumed by satellite i to rendezvous with satellite j and then return to its original orbital slot. One may expect the active satellite to always be the seller because a seller has more fuel at its disposal. However, this is not necessarily true because in general it is possible that less fuel can be consumed if the buyer initiates the fuel transaction. If this is the case, the buyer can be selected to be active provided that the buyer has enough fuel to complete the go-and-return rendezvous maneuvers.

Let p_{ij} denote the cost for the fuel transaction between satellites i and j. That is, if satellite i is active, then $p_{ij} = p_i^j$, but if satellite j is active, then $p_{ij} = p_j^i$. In addition, it is assumed that when two satellites perform a fuel transaction, the fuel is redistributed such that the two satellites have the same amount of fuel. That is, if satellites i and j conduct a fuel transaction, then

$$f_i^+ = f_j^+ = \frac{f_i^- + f_j^- - p_{ij}}{2}.$$

As in [4] we define a constellation graph consisting of the satellites in the constellation as the vertices. If two satellites are capable of conducting a fuel transaction, then an edge exists between the two corresponding vertices on the constellation graph. Clearly, if every pair of satellites can conduct a fuel transaction, then the constellation graph is a complete graph [1]. However, in reality, restrictions may exist such that no fuel transactions can be performed between some satellite pairs. For example, if two satellites are both low on fuel, then these two satellites cannot perform a fuel transaction. Consequently, there is no edge between the two on the constellation graph.

In the following, let v_i, $i = 1, 2, \cdots, n$, denote the n vertices in the constellation graph, and let $\langle \mathrm{v}_i, \mathrm{v}_j \rangle$ denote the edge between vertices v_i and v_j.

In order to measure fuel equalization among the satellites after refuelling, a deviation vector is defined as $d_f = (f_1^+ - \bar{f},\ f_2^+ - \bar{f},\ \cdots,\ f_n^+ - \bar{f})$. That is, d_f denotes the difference between the fuel among the satellites and the average amount of fuel. Clearly, if the fuel is equalized, then d_f is a zero vector. Therefore, equalizing fuel is equivalent to minimizing the total variation of the fuel from the average. Thus, the objective can be written as maximizing

$$\max \quad z = -\sum_{i \in \mathcal{I}} \left| f_i^+ - \bar{f} \right| = -\|d_f\|_1. \tag{1}$$

Ideally, $\bar{f}$ should be the average fuel after refuelling; i.e., $\bar{f} = \bar{f}^+$. However, since the fuel consumption associated with the rendezvous maneuvers is not negligible, the total fuel stored in the satellites after refuelling is less than the fuel before refuelling. By defining the objective as in Eq. (1) with $\bar{f} = \bar{f}^+$ the sole concern of the objective function is the equalization of fuel after refuelling, and no consideration is taken in regards to the amount of fuel spent during the rendezvous maneuvers. This could create a situation where too much fuel is spent during the rendezvous maneuvers just to achieve better fuel equalization. Therefore, in order to penalize too much fuel consumption during the refuelling maneuvers we set $\bar{f} = \bar{f}^-$ in Eq. (1). Notice that $\bar{f}^+ < \bar{f}^-$. Thus, setting $\bar{f} = \bar{f}^-$ in Equation (1) imposes the condition that the fuel after refuelling should not deviate from $\bar{f}^-$ too much. That is, it penalizes solutions where too much fuel is consumed during a refuelling period.

As mentioned in [4], the search for satellite pairs to conduct fuel transactions can be formulated as a search for a maximum-weight matching in the constellation graph. To this end, suppose that there are m edges in the constellation graph, and let $\mathcal{L} = \{1, 2, \cdots, m\}$ be the index set of the edges. In order to formulate the maximum-weight matching problem, a binary variable x_ℓ is defined for each edge e_ℓ, $\ell \in \mathcal{L}$, as

$$x_\ell = \begin{cases} 1 & \text{if edge } \mathrm{e}_\ell \text{ is in the matching,} \\ 0 & \text{otherwise.} \end{cases}$$

Assuming $\bar{f} = \bar{f}^-$ and following the derivations in [4], the maximum-weight matching problem considering the rendezvous cost can be written as

$$\text{(MP-IP-WC):} \quad \text{Maximize} \quad z = \sum_{\ell=1}^{m} \pi_\ell \, x_\ell \tag{2a}$$

$$\text{Subject to} \quad \sum_{\ell=1}^{m} a_{i\ell} x_\ell \le 1, \ \forall\ i \in \mathcal{I} \tag{2b}$$

$$x_\ell \in \{0,\ 1\}, \ \forall\ \ell \in \mathcal{L} \tag{2c}$$

Here, the weight π_ℓ for edge e_ℓ is given by

$$\pi_\ell = \left| f_i^- - \bar{f}^- \right| + \left| f_j^- - \bar{f}^- \right| - \left| f_i^- + f_j^- - p_{ij} - 2\bar{f}^- \right|, \tag{3}$$

where ℓ is the index for edge $\langle \mathrm{v}_i, \mathrm{v}_j \rangle$. In addition, $a_{i\ell},\ i \in \mathcal{I}$ and $\ell \in \mathcal{L}$, are the elements of the incidence matrix [1] of the constellation graph. In the incidence matrix, $a_{i\ell} = 1$ if the vertex v_i and the edge e_ℓ are incident, and $a_{i\ell} = 0$ otherwise.

The number of edges in the constellation graph can be reduced according to the signs of the weights π_ℓ. Namely, the edges with the weights $\pi_\ell \le 0$ can be removed from the constellation graph because any optimal solution to the (MP-IP-WC) does not contain those edges. Doing so could potentially reduce the effort to solve the (MP-IP-WC). The resulting graph is called the *reduced constellation graph*. For an edge $\mathrm{e}_\ell = \langle \mathrm{v}_i, \mathrm{v}_j \rangle$, if $f_i^- \le \bar{f}^-$ and $f_j^- \le \bar{f}^-$, then from Eq. (3) we can see that $\pi_\ell \le 0$. Therefore, the reduced constellation graph does not contain edges between satellites which are either fuel-deficient or have precisely the average amount of fuel. This is the same as in the P2P refuelling problems where the rendezvous fuel expenses are not considered [4].

However, for an edge $\mathrm{e}_\ell = \langle \mathrm{v}_i, \mathrm{v}_j \rangle$ with $f_i^- > \bar{f}$ or $f_j^- > \bar{f}$, it is not obvious whether $\pi_\ell > 0$ or $\pi_\ell \le 0$. In general, it could be beneficial to have two fuel-sufficient satellites conduct a fuel transfer. Thus, the reduced constellation graph may contain some edges between fuel-sufficient satellites. This is different from the case when the rendezvous cost is not considered, where all edges between fuel-sufficient satellites are removed from the constellation graph [4]. Therefore, unlike the formulation in [4], where the reduced constellation graph is a bipartite graph, for the underlying P2P refuelling problem, the reduced constellation graph is not a bipartite graph.

After the reduced constellation graph is obtained, the solution to the P2P refuelling problem considering the rendezvous cost can be obtained by solving the (MP-IP-WC) defined on the reduced constellation graph. The solution to the maximum-weight matching problem can be obtained using, for example, the algorithm by Edmonds and Johnson [2].

III. Cost of Fuel Transaction

In order to solve the (MP-IP-WC) we first need to calculate the fuel expenditure required for a satellite to rendezvous with another satellite and then return to its designated orbital slot. To this end, let us consider satellite i and satellite j. Let the weights of the permanent structure of the satellites be m_{si} and m_{sj}, and let I_{spi} and I_{spj} denote the specific impulses of the propulsion systems onboard the two satellites. In order to calculate p_{ij} two cases need to be considered. In the first case, satellite i is the active satellite, and in the second case, satellite j is the active satellite.

Case 1, satellite i is active. In this case, satellite i initiates the fuel transaction. Let ΔV_{ij}^i be the velocity change required for satellite i to rendezvous with satellite j, and ΔV_{ji}^i be the velocity change required for satellite i to depart from satellite j and return to its designated orbital slot. Then the amount of fuel spent for i to rendezvous with j is given by [3]

$$p_{ti} = (m_{si} + f_i^-)\left(1 - e^{-\frac{\Delta V_{ij}^i}{g_0 I_{spi}}}\right)$$

where g_0 is the gravitational acceleration at sea level.

If $p_{ti} > f_i^-$, then satellite i does not have enough fuel to complete the rendezvous with j and thus, it cannot initiate the rendezvous. In this case, satellite j has to be the active one. Otherwise, if $p_{ti} \le f_i^-$, then satellite i can complete the rendezvous with j. In the latter case, after the rendezvous, the fuel owned by the two satellites are

$$f_{i1} = f_i^- - p_{ti}, \quad \text{and} \quad f_{j1} = f_j^- \tag{4}$$

After the rendezvous, fuel is transferred between the two satellites. After the fuel transfer, the fuel stored in the two satellites is

$$f_{i2} = f_{i1} - g_i^j, \quad \text{and} \quad f_{j2} = f_{j1} + g_i^j. \tag{5}$$

Recall that g_i^j denotes the amount of fuel that is transferred from satellite i to satellite j. It has negative value if satellite i receives fuel from satellite j.

Since satellite i has to return to its original designated orbital slot, it has to perform a second rendezvous maneuver for the return trip. As mentioned earlier, the velocity change of the returning maneuver is ΔV_{ji}^i. Thus, the fuel consumption for this returning maneuver is given by

$$p_{bi} = (m_{si} + f_{i1} - g_i^j)\left(1 - e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}\right). \tag{6}$$

Therefore, after satellite i returns, the amount of fuel onboard each satellite is given by

$$f_i^+ = f_{i2} - p_{bi}, \quad \text{and} \quad f_j^+ = f_{j2}. \tag{7}$$

It is required that the two satellites have the same amount of fuel after the fuel transaction; i.e., $f_i^+ = f_j^+$. This, along with Eqs. (4), (5) and (7), allows us to solve for g_i^j as

$$g_i^j = \frac{f_i^- - f_j^- - p_{ti} - p_{bi}}{2}. \tag{8}$$

Substituting g_i^j into Eq. (6), we get the explicit expression for p_{bi} as

$$p_{bi} = (2m_{si} + f_i^- + f_j^- - p_{ti})\,\frac{1 - e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}}{1 + e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}}. \tag{9}$$

Thus, it can be verified that g_i^j is explicitly given by

$$g_i^j = \frac{(f_i^- - p_{ti})e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}} - f_j^- - m_{si}\left(1 - e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}\right)}{1 + e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}}, \tag{10}$$

and the amount of fuel onboard satellite i before it undocks with j is given by

$$f_{i2} = \frac{f_i^- + f_j^- - p_{ti} + m_{si}\left(1 - e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}\right)}{1 + e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}}. \tag{11}$$

If $p_{bi} > f_{i2}$, then satellite i does not have enough fuel to return to its original orbital slot, which also implies that satellite i cannot be the active satellite. On the other hand, if $p_{bi} \le f_{i2}$, satellite i has enough fuel to return, and thus, can be the active satellite. In that case, it can be verified that the fuel stored in the two satellite after refuelling is given by

$$f_i^+ = f_j^+ = \frac{(f_i^- + f_j^- - p_{ti})e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}} - m_{si}\left(1 - e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}\right)}{1 + e^{-\frac{\Delta V_{ji}^i}{g_0 I_{spi}}}}.$$

If satellite i can be the active satellite, then the total fuel expense for the fuel transaction is given by

$$p_i^j = p_{ti} + p_{bi}. \tag{12}$$

Case 2, satellite j is active. Similar results can be derived for the case where satellite j is the active satellite. In this case, let ΔV_{ji}^j be the velocity change required for satellite j to rendezvous with satellite i, and ΔV_{ij}^j be the velocity change required for satellite j to depart from satellite i and return to its designated orbital slot.

Suppose satellite j has enough fuel to complete the go-and-return maneuvers. Then the amount of fuel that is necessary for j to rendezvous with i is given by

$$p_{tj} = (m_{sj} + f_j^-)\left(1 - e^{-\frac{\Delta V_{ji}^j}{g_0 I_{spj}}}\right).$$

The amount of fuel that is transferred from satellite j to i can be calculated as

$$g_j^i = \frac{(f_j^- - p_{tj})e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}} - f_i^- - m_{sj}\left(1 - e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}\right)}{1 + e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}}. \tag{13}$$

In addition, the amount of fuel needed for satellite j to return to its original orbital slot can be calculated as

$$p_{bj} = (2m_{sj} + f_i^- + f_j^- - p_{tj})\, \frac{1 - e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}}{1 + e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}}. \tag{14}$$

It can be verified that the fuel stored in the two satellites after refuelling is given by

$$f_i^+ = f_j^+ = \frac{(f_i^- + f_j^- - p_{tj})e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}} - m_{sj}\left(1 - e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}\right)}{1 + e^{-\frac{\Delta V_{ij}^j}{g_0 I_{spj}}}}.$$

Thus, the total fuel expense for the fuel transaction when j is active is given by

$$p_j^i = p_{tj} + p_{bj}. \tag{15}$$

Finally, the cost of the fuel transaction between satellites i and j are selected as

$$p_{ij} = \begin{cases} p_i^j, & \text{if } i \text{ can be active, but } j \text{ cannot;} \\ p_j^i, & \text{if } j \text{ can be active, but } i \text{ cannot;} \\ \min\{p_i^j,\ p_j^i\}, & \text{if both satellites can be active;} \\ \text{undefined}, & \text{if neither satellite can be active.} \end{cases} \tag{16}$$

IV. Numerical Examples

In this section, we present two numerical examples to demonstrate several characteristics of the P2P refuelling problem.

A. Example 1

In the first example, we consider the circular constellation of fourteen satellites shown in Figure 1. The satellites are evenly distributed along the circular orbit at an altitude of 500 km. The fuel before refuelling is shown next to each satellite in the figure. The value of f_i^- is shown as the weight of the fuel onboard satellite i. It is assumed that a satellite with a full tank of fuel has a weight of 100 units, and a full tank of fuel weighs 40 units in this constellation. The average fuel storage before refuelling is $\bar{f}^- = 20.4$ units.

We assume that all fourteen satellites have the same structure. Specifically, if a satellite has a full tank of fuel, the permanent structure weighs 60 units, and the fuel weighs 40 units. It is also assumed that the specific impulse of the propulsion system onboard each satellite is 300 seconds. In addition, each satellite is allowed to pair up with any other satellites.

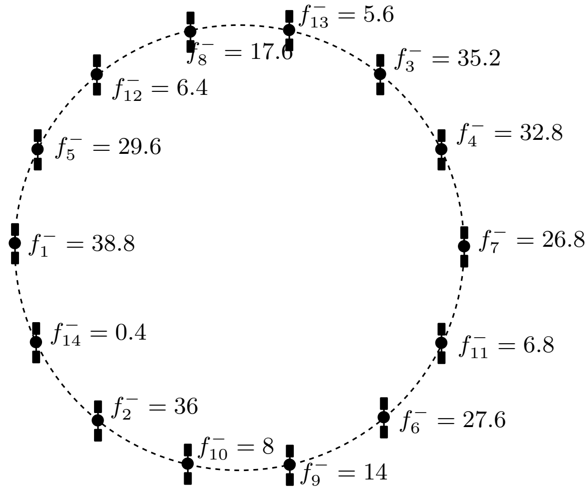

Fig. 1. The refuelling scenario with fourteen satellites in Example 1.

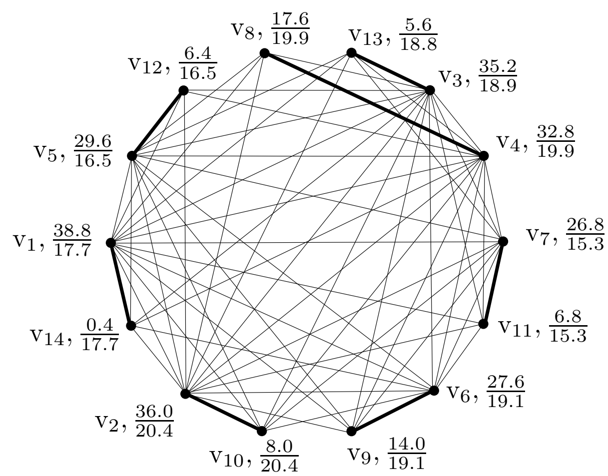

Fig. 2. The reduced constellation graph with the optimal matching for Example 1.

The rendezvous between two satellites is assumed to be a minimum-ΔV two-impulse rendezvous maneuver. Thus, the velocity change of a rendezvous can be calculated according to the method presented in [5]. The total time for a fuel transaction is assumed to be 12 (orbital periods of the circular orbit). A total time of 6 is allotted to the portion where the active satellites rendezvous with the inactive satellites, and the remaining time of 6 is allotted to the maneuvers where the active satellites return to their original locations.

With all these assumptions, p_i^j, the fuel expense between satellites i and j if satellite i is active and satellite j is inactive, can be calculated according to Eqs. (12) and (15). p_i^j can be conveniently represented by a matrix, denoted by P_1. Specifically, P_1 is defined such that $P_1(i,j) = p_i^j$. For the underlying example, P_1 is calculated as follows.

$$
P_1 = \begin{array}{l}
\begin{array}{rrrrrrrr}
\mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 & \mathrm{v}_4 & \mathrm{v}_5 & \mathrm{v}_6 & \mathrm{v}_7 & \mathrm{v}_8 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left(\begin{array}{rrrrrrrr}
\times & 8.65 & 20.75 & 24.47 & 4.11 & 20.36 & 27.73 & 12.19 \\
8.50 & \times & 27.80 & 23.98 & 12.30 & 12.25 & 19.88 & 19.35 \\
20.44 & 27.69 & \times & 4.03 & 16.12 & 16.00 & 8.19 & 7.97 \\
23.80 & 23.57 & 3.98 & \times & 19.54 & 11.92 & 3.89 & 11.59 \\
3.92 & 11.93 & 15.61 & 19.17 & \times & 22.45 & 22.36 & 7.63 \\
19.18 & 11.68 & 15.43 & 11.63 & 22.18 & \times & 7.71 & 21.40 \\
26.13 & 18.90 & 7.86 & 3.76 & 22.07 & 7.65 & \times & 14.51 \\
10.89 & 17.55 & 7.19 & 10.59 & 7.13 & 20.19 & 13.71 & \times \\
13.76 & 6.97 & 16.95 & 13.61 & 16.54 & 3.30 & 10.12 & 21.76 \\
9.84 & 3.18 & \mathrm{CI} & 15.88 & 12.60 & 6.47 & 12.58 & \\
\mathrm{CI} & 12.72 & 9.74 & 6.51 & \mathrm{CI} & 3.06 & 3.07 & \mathrm{CI} \\
6.63 & 12.81 & 9.57 & 12.53 & 3.08 & \mathrm{CI} & \mathrm{CI} & 2.94 \\
12.84 & \mathrm{CI} & 3.10 & 6.34 & 9.42 & \mathrm{CI} & 9.22 & 2.93 \\
\mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI}
\end{array} \right. \\
\begin{array}{rrrrrr}
\mathrm{v}_9 & \mathrm{v}_{10} & \mathrm{v}_{11} & \mathrm{v}_{12} & \mathrm{v}_{13} & \mathrm{v}_{14} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left. \begin{array}{rrrrrr}
15.85 & 11.82 & 22.73 & 8.01 & 15.52 & 3.78 \\
7.94 & 3.78 & 15.16 & 15.19 & 22.14 & 3.71 \\
19.02 & 22.15 & 11.49 & 11.42 & 3.72 & 21.63 \\
15.11 & 18.32 & 7.61 & 14.74 & 7.53 & 24.40 \\
18.15 & 14.41 & 24.22 & 3.57 & 10.90 & 7.20 \\
3.59 & 7.29 & 3.50 & 23.76 & 17.36 & 13.85 \\
10.89 & 14.08 & 3.48 & 17.28 & 10.59 & 20.09 \\
22.24 & 18.91 & 15.86 & 3.18 & 3.18 & 12.55 \\
\times & 3.09 & 6.35 & 18.12 & 18.08 & 9.20 \\
2.96 & \times & 8.84 & 14.33 & \mathrm{CI} & 5.82 \\
6.03 & 8.74 & \times & \mathrm{CI} & 11.44 & \mathrm{CI} \\
\mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \times & 5.82 & \mathrm{CR} \\
\mathrm{CI} & \mathrm{CI} & \mathrm{CI} & 5.80 & \times & \mathrm{CI} \\
\mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \mathrm{CI} & \times
\end{array} \right)
\begin{array}{l}
\leftarrow \mathrm{v}_1 \\ \leftarrow \mathrm{v}_2 \\ \leftarrow \mathrm{v}_3 \\ \leftarrow \mathrm{v}_4 \\ \leftarrow \mathrm{v}_5 \\ \leftarrow \mathrm{v}_6 \\ \leftarrow \mathrm{v}_7 \\ \leftarrow \mathrm{v}_8 \\ \leftarrow \mathrm{v}_9 \\ \leftarrow \mathrm{v}_{10} \\ \leftarrow \mathrm{v}_{11} \\ \leftarrow \mathrm{v}_{12} \\ \leftarrow \mathrm{v}_{13} \\ \leftarrow \mathrm{v}_{14}
\end{array}
\end{array} \tag{17}
$$

In Eq. (17), the symbol 'CI' stands for 'Cannot Initiate', and the symbol 'CR' stands for 'Cannot Return'. Thus, in P_1, an entry of 'CI' implies that the satellite of the row index cannot initiate the fuel transaction with the satellite of the column index; an entry of 'CR' implies that the satellite of the row index can rendezvous with the satellite of the column index, but it cannot return to its original orbital slot. For example, consider satellites 6, 12, and 14. Since satellite 6 has a large amount of fuel, it can both rendezvous with satellite 12 and return to its original location, and the total fuel expense is 23.76. On the other hand, satellite 12 does not have enough fuel to rendezvous with satellite 6, so $P_1(12,\, 6) =$ 'CI'. However, satellite 12 has enough fuel to rendezvous with satellite 14 (notice satellite 14 is closer to satellite 12 than satellite 6), but the rendezvous costs so much that it does not have enough fuel left in its tank. In addition, satellite 14 has little fuel to spare when fuel is transferred between them, so satellite 12 does not have enough fuel to return. Therefore, $P_1(12,\, 14) =$ 'CR'.

The fuel expense of a fuel transaction between two satellites can then be calculated according to Eq. (16). Similar to p_i^j, p_{ij} can also be represented as elements of a matrix, denoted by P_2. Namely, $P_2(i,\, j) = p_{ij}$. Since $p_{ij} = p_{ji}$, P_2 is a symmetric matrix. Using the matrix P_2 we can remove the edges of satellite pairings where both of them are inactive from the constellation graph.

The weight π_ℓ assigned to the edge e_ℓ is calculated according to Eq. (3). The weights can also be represented in a matrix form. Namely, a matrix, denoted by Π, can be created such that $\Pi(i,\, j) = \pi_\ell$ for each edge $\pi_\ell = \langle \mathrm{v}_i,\, \mathrm{v}_j \rangle$.

As mentioned in Section II, edges with negative weights are removed from the constellation graph. After removing all these edges, we have the following reduced constellation graph shown in Figure 2.

Once the reduced constellation graph and the weights on the edges are obtained, we can calculate the maximum-weight matching. The optimal matching is also shown in Figure 2, depicted by thicker dark lines. The optimal matching has seven edges, which implies that every satellite is engaged in a fuel transaction with another satellite. The fuel stored in each satellite before and after refuelling is also shown next to each vertex in the form of a fraction. The value of the numerator is the amount of fuel before refuelling, and the value of the denominator is the amount of fuel after refuelling. It can be seen that the fuel after refuelling is much more evenly distributed than that before refuelling. Indeed, before refuelling, $\sum_{i=1}^{14} |f_i^- - \bar{f}| = 168$, but after refuelling, $\sum_{i=1}^{14} |f_i^+ - \bar{f}| = 30.1$. The total fuel expense for the orbital transfers of the active satellites is 30.1.

Recall that in cases where every two satellites are allowed to conduct a fuel transaction and the rendezvous cost can be neglected, it has been shown in [4] that the symmetric matching is the optimal matching. In the constellation shown in Figure 1, the symmetric matching is the following collection of edges

$$
\begin{aligned}
\big\{ &\langle \mathrm{v}_1,\, \mathrm{v}_{14} \rangle,\, \langle \mathrm{v}_2,\, \mathrm{v}_{13} \rangle,\, \langle \mathrm{v}_3,\, \mathrm{v}_{12} \rangle,\, \langle \mathrm{v}_4,\, \mathrm{v}_{11} \rangle, \\
&\langle \mathrm{v}_5,\, \mathrm{v}_{10} \rangle,\, \langle \mathrm{v}_6,\, \mathrm{v}_9 \rangle,\, \langle \mathrm{v}_7,\, \mathrm{v}_8 \rangle \big\}.
\end{aligned}
$$

However, as can be seen from Figure 2, the optimal matching is not the symmetric matching, even though in Example 1 every two satellites are allowed to conduct a fuel transaction. In fact, one of the edges in the symmetric matching, $\langle \mathrm{v}_7,\, \mathrm{v}_8 \rangle$, does not even exist in the reduced constellation graph. It is removed from the constellation graph because the weight for this edge is negative. Therefore, the fact that the rendezvous cost is considered in the formulation has a big impact on the satellite pairings.

B. Example 2

In the second example, we again consider a constellation with fourteen satellites evenly distributed on a circular orbit with an altitude of 500 kilometers. The constellation is shown in Figure 3.

Figure 4 shows the reduced constellation graph along with the optimal matching, shown by thicker solid lines. The fuel stored in each satellite before and after the fuel transaction is shown as well in the same way as in Figure 2. Before refuelling, $\sum_{i=1}^{14} |f_i^- - \bar{f}| = 103.6$. But after refuelling, $\sum_{i=1}^{14} |f_i^+ - \bar{f}| = 52.6$. Therefore, the fuel is more evenly distributed after refuelling. The total amount of fuel spent on the rendezvous maneuvers is 43.5.

Notice that the optimal matching contains only five edges, so not every satellite is engaged in a fuel transfer. In this

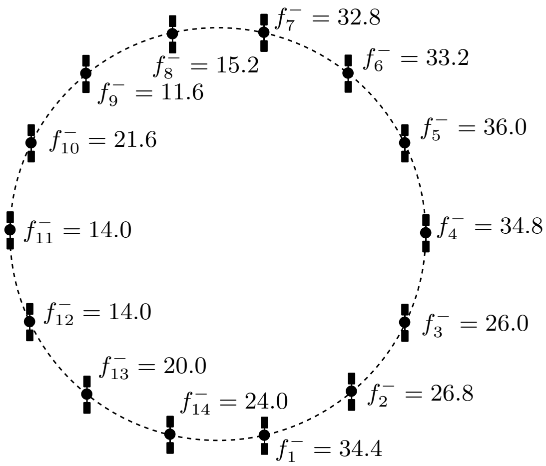

Fig. 3. The fourteen-satellite constellation in Example 2.

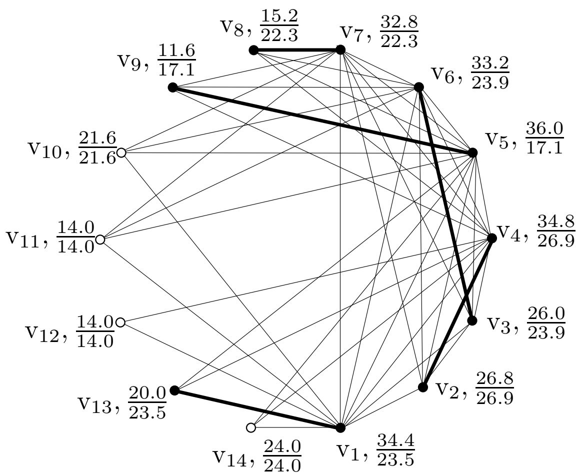

Fig. 4. The reduced subgraph with the optimal matching for Example 2.

example, for instance, satellites 10, 11, 12, and 14 are not matched with any other satellites.

In this example, the satellites with the least amount of fuel before the fuel transaction are left unmatched. They are satellites 11 and 12. The reason is that these satellites are farther away from the fuel-sufficient satellites than other fuel-deficient satellites. Thus, the fuel expenses for the rendezvous maneuvers between satellite 11 or 12 and the fuel-sufficient satellites are high. This prohibits fuel transactions between satellite 11 or 12 and the fuel-sufficient satellites to be beneficial to the refuelling objective. For example, it is beneficial for satellite 12 to have a fuel transaction with only satellite 1 or satellite 4. However, pairing satellites 1 and 13 is more beneficial than pairing satellites 1 and 12, and pairing satellites 4 and 2 is more beneficial than pairing satellites 4 and 12.

As a result, the optimal matching displays a balance between two conflicting objectives, namely, fuel equalization and minimum fuel expense on the rendezvous maneuvers. In the case of vertices v_{11} and v_{12}, the latter objective is the dominant factor. However, if we increase the total time allowed to each rendezvous maneuver, then the rendezvous fuel expense will decrease [5]. Then, the objective of minimum fuel expense will become less significant. Eventually, as the total time is increased, the fuel equalization objective will become the dominating factor, and satellites 11 and 12 will be matched.

V. Conclusion

In this paper, we studied the P2P refuelling problem where the fuel expense for the rendezvous maneuvers is taken into consideration when scheduling the satellite rendezvous pairings. The problem is formulated as a maximum-weight matching problem. It is shown that fuel transactions between any fuel-deficient satellites are not beneficial to the refuelling objective. Other non-beneficial satellite pairs can also be identified before solving the maximum-weight matching problem. By removing the non-beneficial satellite pairs, the computation effort to solve the maximum-weight matching problem can be reduced. It is shown that the optimal solution is not necessarily the symmetric matching, and that not all satellites are involved in fuel transactions.

VI. References

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