Peer-to-Peer Refuelling within a Satellite Constellation
Part I: Zero-Cost Rendezvous Case

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***Abstract*— In this paper, we study the scheduling problem arising from refuelling multiple satellites in a constellation. The satellites in the constellation are assumed to be capable of refuelling each other. The cost of the rendezvous maneuver between two satellites exchanging fuel is assumed to be negligible. The goal of this refuelling problem is to equalize the fuel stored among all satellites in the constellation after a given period. It is shown that the problem of equalizing the fuel among the satellites can be formulated and solved as a maximum-weight matching problem.**

I. Introduction

The current practice when the fuel on-board a satellite is exhausted is to simply replace the satellite with a new one. Replacing old satellites with new ones incurs a significant cost in production and launching of satellites, not to mention the addition of space debris. An alternative to replacing a satellite when its fuel is depleted is to create a satellite architecture having the capability of refuelling the satellites when needed. Under this new concept, when a satellite runs low on fuel, it can be refuelled and thus, the satellite may continue its service. Satellites in a constellation can be refuelled either from a vehicle launched from the earth for that purpose, or by other satellites in the same constellation.

Much of the previous work on satellite refuelling has been limited to hardware design and the feasibility of transferring liquid in space. A good brief overall conceptual study of this topic can be found in [5]. Only recently the scheduling problem arising from refuelling multiple satellites has gained some attention. Shen and Tsiotras in [10] studied the optimal scheduling for refuelling or servicing multiple satellites in a circular orbit using one single servicing spacecraft. Integer programming was used in [10] to obtain the best schedule of refuelling the satellites in a given order. A heuristic study suggested that the best sequence to visit all satellites can be chosen from the sequences which assume the minimum total sweep angle. In [1], Alfriend et al considered the optimal scheduling for servicing multiple satellites in a geosynchronous orbit with small inclination. It is shown that the fuel consumption is proportional to the distance between the projections on the equatorial plane of the angular momentum vectors of the orbits of the two satellites. Thus, the minimum-fuel ordering is transformed into a classical travelling salesman problem, for which numerous algorithms exist [6].

In this paper, we study the scheduling problem arising from the need to redistribute fuel within a satellite constellation. It is assumed that there is no extra fuel delivered to the constellation by an external spacecraft. Instead, all satellites in the constellation are capable of refuelling each other. This allows satellites that have excess fuel to deliver fuel to the satellites which are depleted of, or are low on, fuel. This refuelling scenario will be hereto called to *Peer-to-Peer (P2P) refuelling*. The goal of the P2P refuelling problem is to achieve fuel equalization among all satellites in the constellation after a given refuelling period.

II. The P2P Refuelling Problem

Since P2P refuelling problem has combinatorial complexity, here we formulate and solve the P2P refuelling problem as a sequence of *fuel transactions*. A fuel transaction involves two satellites, a *buyer* and a *seller*. Under this framework, a seller satellite rendezvous with a buyer to deliver fuel, or a buyer rendezvous with a seller to receive fuel. We say that this satellite constellation “market” reaches an equilibrium state when the fuel distributed among all satellites is equal.

Assume there are $n \geq 3$ satellites within a constellation. Let $\mathcal{I} = \{1, 2, \cdots, n\}$ denote the index set of the n satellites. Let f_i^M and f_i^m, $i \in \mathcal{I}$, denote the maximum fuel capacity and minimum required fuel for each satellite. At time t_k, let $f_i(t_k)$, $i \in \mathcal{I}$, be the amount of fuel stored in satellite i. Satellite i is considered operational at time t_k if and only if $f_i^m \leq f_i(t_k) \leq f_i^M$, $i \in \mathcal{I}$. We are given a time period h within which the fuel transactions must take place. We assume that within the time interval h, each satellite can deliver fuel to no more than one other satellite. In addition, each satellite can receive fuel from no more than one other satellite. That is, no two seller-buyer pairs share a common satellite during one refuelling period. Let $g_i^j(t_k)$ denote the amount of fuel transferred from satellite i to satellite j after they rendezvous (i.e., satellite i is the seller, and satellite j is the buyer). After the transaction (exchange of fuel), we therefore have $f_i(t_{k+1}) = f_i(t_k) - g_i^j(t_k)$ and $f_j(t_{k+1}) = f_j(t_k) + g_i^j(t_k)$ where $t_{k+1} = t_k + h$.

It is assumed that for a pair of satellites, say i and j, only one can be the active satellite which initiates the fuel transaction. For example, if satellite i is active, it applies impulses to travel to j and conducts a fuel transaction with j, before travelling back to its originally designated orbital slot. During the whole process, satellite j remains at its preassigned orbital slot. Thus, only the active satellite consumes fuel during the rendezvous maneuver. At time t_k, let $p_i^j(t_k)$ be the fuel consumed by satellite i to rendezvous with satellite j and then return to its designated orbital slot. Note that since, in general $p_i^j(t_k) \neq p_j^i(t_k)$, it is possible that less fuel is consumed if the buyer initiates the fuel transaction. If this is the case, the buyer can be selected to be active provided that the buyer has enough fuel to complete the go-and-return rendezvous maneuvers.

In the following, we assume that $f_i^M = f_j^M$ and $f_i^m = 0$ for all $1 \leq i,\, j \leq n$. For ease of notation, in the following, we will use the superscript '-' to denote values at t_k and superscript '+' to denote values at t_{k+1}. For example, f_i^- will be used to denote the fuel owned by satellite i before refuelling, and f_i^+ will be used to denote the fuel owned by satellite i after refuelling.

III. Formulation of the Maximum-Weight Matching Problem

In cases where the fuel consumption for a rendezvous maneuver is much smaller than the amount of fuel to be transferred, we can simplify the refuelling problem by assuming that the rendezvous cost is zero. This assumption is generally valid when the satellites are in a close formation or when the total time allowed to complete the fuel transactions is sufficiently large (it has been shown in [8] that for a circular constellation orbit the rendezvous cost decreases monotonically when time-of-flight increases). The case when the rendezvous cost is not zero is treated in [9].

Under the assumption of zero rendezvous cost, the total fuel among all satellites is conserved. Thus, the average fuel stored among all satellites in the constellation is

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i^- = \frac{1}{n}\sum_{i=1}^{n} f_i^+, \quad i \in \mathcal{I}$$

We further assume that each time two satellites rendezvous, the fuel transaction results in the two satellites having the same amount of fuel. That is, if satellite i and satellite j conduct a fuel transaction, then $f_i^+ = f_j^+ = (f_i^- + f_j^-)/2$.

A. The Constellation Graph

The *constellation graph* $\mathcal{G}$ is a graph with the n satellites being the n vertices. An edge exists between two vertices if either of the two vertex-satellites can initiate the rendezvous and carry out a fuel transaction with the other. However, there could be restrictions on the satellite pairs due to operational requirements of the satellite constellation or formation. For example, in order to maintain the normal operation of the constellation, a subset of satellites may be required to remain in their orbital slots while others are engaged into fuel transactions. Reflected in the constellation graph, this implies that there are no edges between those satellites. Obviously, if these satellites are involved in fuel transactions, they can only be inactive. Clearly, if there are no restrictions on the satellite pairs, the constellation graph is a complete graph [3].

In this paper, we will use the difference of the onboard fuel stored in each satellite from the average amount of fuel, as a measure of fuel equalization among all satellites. That is, we wish to maximize

$$\max \quad z = \sum_{i\in\mathcal{I}} \Big(-\left| f_i^+ - \bar{f} \right| \Big). \tag{1}$$

Recall that two satellite pairs do not share a common satellite during a refuelling period. Thus, the edges associated with the satellite pairs can be considered as a matching in the constellation graph. Therefor, the search for satellite pairs to achieve the fuel equalization is equivalent to the search for a matching in the constellation graph such that z in Eq. (1) is maximized. In the following, the search for satellite pairs to conduct fuel transactions will be modelled and solved in the framework of the maximum-weight matching problem (MP) in the constellation graph.

B. The Maximum-Weight Matching Problem in the Constellation Graph

Suppose there are m edges in the constellation graph, and let $\mathcal{L} = \{1,\, 2,\, \cdots,\, m\}$ be the index set of the edges. Let us associate a binary variable x_ℓ with each edge e_ℓ for all $\ell \in \mathcal{L}$, where x_ℓ is defined by

$$x_\ell = \begin{cases} 1 & \text{if edge } \mathrm{e}_\ell \text{ is in the matching,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = (x_1,\, x_2,\, \cdots,\, x_m)$ denote the vector of the binary variables. Then, the conditions for a matching can be written as follows [4].

$$\text{(MC):} \quad \sum_{\mathrm{e}_\ell \in \mathcal{Q}(\mathrm{v}_i)} x_\ell \leq 1, \quad \forall\, i \in \mathcal{I}$$
$$x_\ell \in \{0,\, 1\}, \quad \forall\, \ell \in \mathcal{L}$$

where $\mathcal{Q}(\mathrm{v}_i)$ denotes the set of edges that are incident with the vertex v_i.

Now, let us elaborate on the objective function in Eq. (1). In general, not every satellite is involved in a fuel transaction. For example, if the number of satellites is odd, at least one satellite will be left unmatched. Suppose that satellite i is matched with satellite j. Then after the fuel transaction, the fuel stored between the two is averaged out. Therefore, the contribution of satellite i to the objective function is $-\left|(f_i^- + f_j^-)/2 - \bar{f}\right|$, which is the same as the contribution of satellite j. On the other hand, if a satellite, say satellite k, is not matched with any other satellite, then its fuel remains the same throughout the refuelling period. Thus, its contribution to the objective function is $-\left|f_k^- - \bar{f}\right|$. Utilizing the binary variables in the vector x, we can write the contribution to the objective function of all matched satellites as [7]

$$z_1 = \sum_{i \in \mathcal{I}} \sum_{\mathrm{e}_\ell \in \mathcal{Q}(\mathrm{v}_i)} (-c_\ell x_\ell), \tag{3}$$

where $c_\ell = \left|(f_i^- + f_j^-)/2 - \bar{f}\right|$, and i and j are such that $\mathrm{e}_\ell = \langle \mathrm{v}_i,\, \mathrm{v}_j \rangle$; i.e., e_ℓ is the edge between vertices v_i and v_j. Similarly, we can write the contributions of all unmatched satellites to the objective function as [7]

$$z_2 = \sum_{i \in \mathcal{I}} \left(1 - \sum_{\mathrm{e}_\ell \in \mathcal{Q}(\mathrm{v}_i)} x_\ell \right) \Big(-\left|f_i^- - \bar{f}\right|\Big). \tag{4}$$

Then, the objective function in Eq. (1) can be written as $z = z_1 + z_2$. Thus,

$$z = \sum_{i \in \mathcal{I}} \sum_{\mathrm{e}_\ell \in \mathcal{Q}(\mathrm{v}_i)} \Big(\left|f_i^- - \bar{f}\right| - c_\ell\Big) x_\ell - \sum_{i \in \mathcal{I}} \Big(\left|f_i^- - \bar{f}\right|\Big).$$

Since the last term in the previous equation is constant, we can remove it from z without affecting the optimal solution for maximizing z. Therefore, and rewriting z as a sum over all edges, one obtains

$$z = \sum_{\ell \in \mathcal{L}} \Big(\left|f_i^- - \bar{f}\right| + \left|f_j^- - \bar{f}\right| - \left|f_i^- + f_j^- - 2\bar{f}\right|\Big) x_\ell$$

In the following, we will use π_ℓ to denote the coefficients of x_ℓ, i.e.,

$$\pi_\ell = \left|f_i^- - \bar{f}\right| + \left|f_j^- - \bar{f}\right| - \left|f_i^- + f_j^- - 2\bar{f}\right|. \tag{5}$$

Therefore, the P2P refuelling problem can be formulated as the following maximum matching problem in terms of the a zero-one integer program:

$$\text{(MP-IP):} \quad \text{Maximize} \quad z = \sum_{\ell=1}^{m} \pi_\ell \, x_\ell$$
$$\text{Subject to} \quad \text{(MC)}$$

From the definition of the weights π_ℓ in Eq. (5), we can see that for the edge $\mathrm{e}_\ell = \langle \mathrm{v}_i,\, \mathrm{v}_j \rangle$, if either $f_i^- = \bar{f}$ or $f_j^- = \bar{f}$, then $\pi_\ell = 0$. The same holds for the case when $f_i^- \le \bar{f}$ and $f_j^- \le \bar{f}$, and the case when $f_i^- \ge \bar{f}$ and $f_j^- \ge \bar{f}$. For all other cases, $\pi_\ell > 0$. Therefore, each edge with $\pi_\ell > 0$ has one end at a fuel-sufficient satellite (i.e., a satellite with more fuel than average) and the other end at a fuel-deficient satellite (i.e., a satellite with less fuel than average).

The edges with $\pi_\ell = 0$ do not contribute to the objective of the (MP-IP). In other words, having the two end-vertices of an edge with zero-weight conduct a fuel transaction does not improve the equalization of the fuel in the constellation. Therefore, the optimal cost of the (MP-IP) remains the same if the edges with $\pi_\ell = 0$ are removed from the constellation graph $\mathcal{G}$. Let $\mathcal{G}_r$ denote the remaining graph after removing the edges with $\pi_\ell = 0$ from $\mathcal{G}$. We call $\mathcal{G}_r$ the *reduced constellation graph*. It can be shown [7] that $\mathcal{G}_r$ is a bipartite graph, with the buyer group and the seller group being the two classes of vertex partitions.

Let the vertex set in the seller group be denoted by $\mathcal{V}^s$, and the vertex set in the buyer group $\mathcal{V}^b$. Denote the index set of the seller satellites $\mathcal{I}_s$, and the index set of the buyer satellites $\mathcal{I}_b$. Let $\mathcal{E}_r$ denote the edge set of $\mathcal{G}_r$, and let $\mathcal{N}(\mathrm{v}_i)$ denote the set of vertices that are adjacent to v_i. For each edge $\mathrm{e}_\ell = \langle \mathrm{v}_i,\, \mathrm{v}_j \rangle$ with $i \in \mathcal{I}_s$ and $j \in \mathcal{I}_b$, let x_{ij} denote x_ℓ, and π_{ij} denote π_ℓ. Therefore, the (MP-IP) for the reduced constellation graph can be rewritten as

$$\text{(MP-IP):} \quad \text{Maximize} \quad z = \sum_{i \in \mathcal{I}_s} \sum_{\mathrm{v}_j \in \mathcal{N}(\mathrm{v}_i)} \pi_{ij} \, x_{ij}$$
$$\text{Subject to} \quad \sum_{\mathrm{v}_j \in \mathcal{N}(\mathrm{v}_i)} x_{ij} \le 1, \quad \forall\, i \in \mathcal{I}_s$$
$$\sum_{\mathrm{v}_i \in \mathcal{N}(\mathrm{v}_j)} x_{ij} \le 1, \quad \forall\, j \in \mathcal{I}_b$$

where $x_{ij} \in \{0, 1\}$ and for all $i \in \mathcal{I}_s, j \in \mathcal{I}_b$ and $\langle \mathrm{v}_i, \mathrm{v}_j \rangle \in \mathcal{E}_r$.

IV. The Solution to the (MP-IP)

It has been shown in [2] that the integrality conditions in the (MP-IP) are redundant, and they can be replaced by the nonnegativity conditions of the binary variables x_{ij}. By doing so, one converts the (MP-IP) into a linear programming problem which can be solved by general methods such as the simplex method [11]. However, more efficient methods have been developed over the last three decades specifically for the maximum-weight matching problem. A well-known method is the one proposed by Edmonds and Johnson, described in [4].

However, there is one special case where the solution can be readily obtained without having to refer to the above algorithm. To make this matter clear, and without loss of generality, let us assume that the fuel owned by the n satellites before refuelling is ordered such that $f_1^- \ge f_2^- \ge \cdots \ge f_n^-$. This can always be achieved by rearranging the index set of the vertices according to the order of the fuel stored in each satellite. Suppose that there are n_s sellers among the n satellites; i.e., $f_{n_s}^- \ge \bar{f} > f_{n_s+1}^-$. Then, there are $n_b = n - n_s$ buyers. Let us assume that $n_b \le n_s$. The case with more buyers than sellers can be treated similarly.

A *symmetric matching* consists of such seller-buyer pairs as the seller with the most onboard fuel and the buyer with the least onboard fuel, the seller with the second most onboard fuel and the buyer with the second least onboard fuel, and so on. That is, the symmetric matching, denoted by $\mathcal{M}_s$, is defined as the following collection of edges, $\mathcal{M}_s = \Big\{ \langle \mathrm{v}_1,\, \mathrm{v}_n \rangle,\ \langle \mathrm{v}_2,\, \mathrm{v}_{n-1} \rangle,\ \cdots,\ \langle \mathrm{v}_{n_b},\, \mathrm{v}_{n-n_b+1} \rangle \Big\}$.

Proposition 4.1: If the reduced constellation graph $\mathcal{G}_r$ contains all the edges in the symmetric matching, then the symmetric matching is an optimal solution to the (MP-IP).

Proof: We will only prove the case when $\mathcal{G}_r$ is a complete bipartite graph. The other case follows immediately.

Since $\mathcal{G}_r$ is a complete bipartite graph, and the edge weights are all positive, the maximum-weight matching is a maximum matching. Thus, it suffices to show that the symmetric matching is better than any other maximum matchings. Let $\mathcal{V}_1^s = \{\mathrm{v}_i \mid i = 1, 2, \cdots, n_b\}$ and $\mathcal{V}_2^s = \{\mathrm{v}_i \mid i = n_b+1, n_b+2, \cdots, n_s\}$ denote a decomposition of $\mathcal{V}^s$. Notice that $\mathcal{V}_1^s$ has the same cardinality as $\mathcal{V}^b$. This decomposition of $\mathcal{V}^s$ is shown in Figure 1. First it will be shown that given any maximum matching $\mathcal{M}'$ in $\mathcal{G}_r$ there is a matching $\mathcal{M}$ between $\mathcal{V}_1^s$ and $\mathcal{V}^b$ whose total weight is at least as large as that of $\mathcal{M}'$.

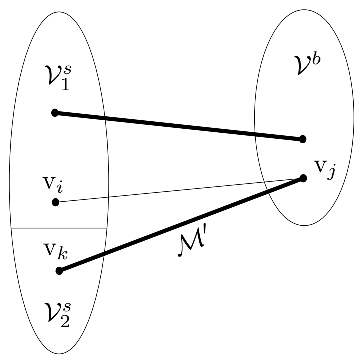

Fig. 1. Demonstration of $\mathcal{V}_1^s$, $\mathcal{V}_1^s$, $\mathcal{V}^b$, and $\mathcal{M}'$.

To this end, suppose $\mathcal{M}'$ consists of edges between $\mathcal{V}_1^s$ and $\mathcal{V}^b$ as well as edges between $\mathcal{V}_2^s$ and $\mathcal{V}^b$. Let there be r edges in $\mathcal{M}'$ which are between $\mathcal{V}_2^s$ and $\mathcal{V}^b$. Then there are precisely r vertices in $\mathcal{V}_1^s$ that are not matched by $\mathcal{M}'$. Let $\mathrm{v}_i \in \mathcal{V}_1^s$ be a vertex that is not matched by $\mathcal{M}'$. Let $\langle \mathrm{v}_k, \mathrm{v}_j\rangle$ be any edge in $\mathcal{M}'$ where $\mathrm{v}_k \in \mathcal{V}_2^s$ and $\mathrm{v}_j \in \mathcal{V}^b$. This scenario is shown in Figure 1. It will be shown that $\pi_{ij} \geq \pi_{kj}$. According to Eq. (5), π_{ij} and π_{kj} are given by $\pi_{ij} = f_i^- - f_j^- - \left|f_i^- + f_j^- - 2\bar{f}\right|$ and $\pi_{kj} = f_k^- - f_j^- - \left|f_k^- + f_j^- - 2\bar{f}\right|$, where we have used the facts that $f_i^- \geq \bar{f}$, $f_k^- \geq \bar{f}$, and $f_j^- \leq \bar{f}$. Utilizing the fact that $f_i^- \geq f_k^-$ and the triangular inequality, we have $\pi_{ij} - \pi_{kj} = f_i^- - f_k^- + \left|f_k^- + f_j^- - 2\bar{f}\right| - \left|f_i^- + f_j^- - 2\bar{f}\right| \geq \left|f_i^- - f_k^- + f_k^- + f_j^- - 2\bar{f}\right| - \left|f_i^- + f_j^- - 2\bar{f}\right| = 0$. Thus, $\pi_{ij} \geq \pi_{kj}$.

From the previous analysis, we conclude that the new matching obtained by replacing the edge $\langle \mathrm{v}_k, \mathrm{v}_j\rangle$ with $\langle \mathrm{v}_i, \mathrm{v}_j\rangle$ in $\mathcal{M}'$ has a total weight no less than that of $\mathcal{M}'$. By replacing all edges in $\mathcal{M}'$ between $\mathcal{V}_2^s$ and $\mathcal{V}^b$ by edges between the unmatched vertices in $\mathcal{V}_1^s$ and $\mathcal{V}^b$, and retaining the original edges of $\mathcal{M}'$ between $\mathcal{V}_1^s$ and $\mathcal{V}^b$, we can construct a matching $\mathcal{M}$ which consists only of edges from $\mathcal{V}_1^s$ to $\mathcal{V}^b$, and its total weight is no less than that of $\mathcal{M}'$. Therefore, it suffices to prove the proposition for the case when $n_b = n_s$. This will be shown by induction.

If $n_b = n_s = 1$, then the proposition is trivial because there is only one edge. Consider the case $n_b = n_s = 2$. Then there are two candidate maximum matchings, $\mathcal{M}_s = \left\{\langle \mathrm{v}_1, \mathrm{v}_4\rangle, \langle \mathrm{v}_2, \mathrm{v}_3\rangle\right\}$ and $\mathcal{M}_c = \left\{\langle \mathrm{v}_1, \mathrm{v}_3\rangle, \langle \mathrm{v}_2, \mathrm{v}_4\rangle\right\}$. It is trivial to show that the symmetric matching $\mathcal{M}_s$ has a total weight no less than that of $\mathcal{M}_c$; i.e., $z_s \geq z_c$, where z_s and z_c denote the total weights of $\mathcal{M}_s$ and $\mathcal{M}_c$, respectively.

Let us now assume that the proposition holds for $n_s = n_b = k \geq 2$. We need to show that the proposition holds for the case with $k+1$ sellers and $k+1$ buyers. In the following, given any maximum matching $\mathcal{M}_m$, with z_m being its total weight, we will show that $z_s \geq z_m$. Again, z_s is the total weight for the symmetric matching $\mathcal{M}_s$.

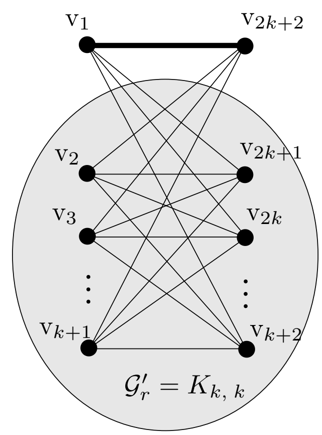

Fig. 2. Illustration of the case when $\langle \mathrm{v}_1, \mathrm{v}_{n+2}\rangle \in \mathcal{M}_m$.

First, if $\langle \mathrm{v}_1, \mathrm{v}_{2k+2}\rangle \in \mathcal{M}_m$, then the graph $\mathcal{G}_r'$ obtained by removing the vertices v_1 and v_{2k+2} from $\mathcal{G}_r$ is a complete bipartite graph with k seller vertices and k buyer vertices. This is illustrated in Figure 2. Therefore, from the induction hypothesis, the symmetric matching $\mathcal{M}_s' = \left\{\langle \mathrm{v}_2, \mathrm{v}_{2k+1}\rangle, \langle \mathrm{v}_3, \mathrm{v}_{2k}\rangle, \cdots, \langle \mathrm{v}_{k+1}, \mathrm{v}_{k+2}\rangle\right\}$ in $\mathcal{G}_r'$ has a total weight no less than the total weight of $\mathcal{M}_m \backslash \langle \mathrm{v}_1, \mathrm{v}_{2k+2}\rangle$. Since the symmetric matching $\mathcal{M}_s$ of $\mathcal{G}_r$ consists of precisely the edge $\langle \mathrm{v}_1, \mathrm{v}_{n+2}\rangle$ and $\mathcal{M}_s'$, we conclude that $z_s \geq z_m$.

Next, we consider the case when $\langle \mathrm{v}_1, \mathrm{v}_{2k+2}\rangle \notin \mathcal{M}_m$. Since $\mathcal{M}_m$ is a maximum matching in $\mathcal{G}_r$, vertices v_1 and v_{2k+2} are matched by $\mathcal{M}_m$. Suppose v_1 and v_{2k+2} are matched by the edges $\langle \mathrm{v}_1, \mathrm{v}_j\rangle$ and $\langle \mathrm{v}_i, \mathrm{v}_{2k+2}\rangle$, respectively, as shown in Figure 3(a).

Consider the bipartite graph induced by the vertices v_1, v_i, v_j, and v_{2k+2}. This is a complete bipartite graph with seller vertices v_1 and v_i and buyer vertices v_j and v_{2k+2}. According to the induction, the total weight of the matching $\left\{\langle \mathrm{v}_1, \mathrm{v}_j\rangle, \langle \mathrm{v}_i, \mathrm{v}_{2k+2}\rangle\right\}$ is no larger than that of the matching $\left\{\langle \mathrm{v}_1, \mathrm{v}_{2k+2}\rangle, \langle \mathrm{v}_i, \mathrm{v}_j\rangle\right\}$. By removing $\langle \mathrm{v}_1, \mathrm{v}_j\rangle$ and $\langle \mathrm{v}_i, \mathrm{v}_{2k+2}\rangle$ from $\mathcal{M}_m$, and adding $\langle \mathrm{v}_1, \mathrm{v}_{2k+2}\rangle$ and $\langle \mathrm{v}_i, \mathrm{v}_j\rangle$ to $\mathcal{M}_m$, we can transform $\mathcal{M}_m$ into a new matching $\mathcal{M}_m'$. This transformation is illustrated in Figure 3. By doing so, we guarantee that the total weight of $\mathcal{M}_m'$ is no smaller than that of $\mathcal{M}_m$; i.e., $z_m' \geq z_m$, where z_m' is the total weight of $\mathcal{M}_m'$. Finally, we have that the edge $\langle \mathrm{v}_1, \mathrm{v}_{n+2}\rangle \in \mathcal{M}_m'$ and thus $z_s \geq z_m'$. Therefore, it follows that $z_s \geq z_m$. ■

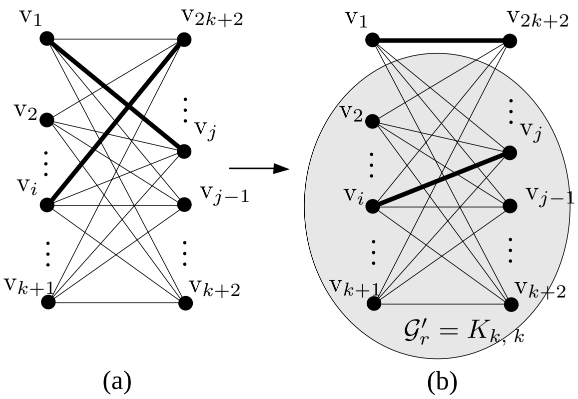

Fig. 3. Illustration of the case when $\langle v_1, v_{n+2}\rangle \notin \mathcal{M}_m$. (a) contains matching $\mathcal{M}_m$, (b) contains matching $\mathcal{M}_m'$, and $z_m' \geq z_m$.

V. Numerical Examples

In this section, numerical examples are presented to demonstrate the above formulation and show some characteristics of the P2P refuelling problem. To this end, we consider a fourteen-satellite constellation which is shown in Figure 4. The fuel initially stored is shown next to each satellite as

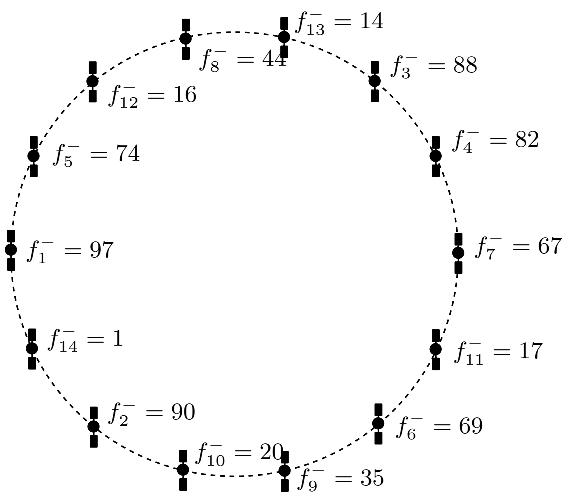

Fig. 4. A refuelling scenario with fourteen satellites.

a percentage of maximum fuel. The average fuel storage in this scenario is $\bar{f} = 51$. As shown in Figure 4, we have seven sellers (satellites 1 to 7) and seven buyers (satellites 8 to 14). Thus, the bipartite reduced constellation graph $\mathcal{G}_r$ can be constructed with the seller satellites in one bipartition and the buyer satellites in the other.

The weights assigned to the edges defined in Eq. (5) can be represented by a weight matrix Π. Each row corresponds to a seller, and each column corresponds to a buyer. For example, the (i, j) element of Π represent the weight between seller satellite i and buyer satellite $j + 7$; i.e., $\Pi(i, j) = \pi_{ij+7}$. Therefore, for the constellation in Figure 4, the weight matrix

TABLE I
Onboard fuel of satellites after fuel transactions in symmetric matching.

Sellers i	1	2	3	4	5	6	7
f_i^+	49	52	52	49.5	47	52	55.5
Buyers j	8	9	10	11	12	13	14
f_j^+	55.5	52	47	49.5	52	52	49

can be calculated as

$$\begin{array}{c} \begin{array}{ccccccc} v_{14} & v_{13} & v_{12} & v_{11} & v_{10} & v_9 & v_8 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \\ \Pi = \begin{pmatrix} \boxed{92} & 74 & 70 & 68 & 62 & 32 & 14 \\ 78 & \boxed{74} & 70 & 68 & 62 & 32 & 14 \\ 74 & 74 & \boxed{70} & 68 & 62 & 32 & 14 \\ 62 & 62 & 62 & \boxed{62} & 62 & 32 & 14 \\ 46 & 46 & 46 & 46 & \boxed{46} & 32 & 14 \\ 36 & 36 & 36 & 36 & 36 & \boxed{32} & 14 \\ 32 & 32 & 32 & 32 & 32 & 32 & \boxed{14} \end{pmatrix} \begin{array}{l} \leftarrow v_1 \\ \leftarrow v_2 \\ \leftarrow v_3 \\ \leftarrow v_4 \\ \leftarrow v_5 \\ \leftarrow v_6 \\ \leftarrow v_7 \end{array} \end{array} \quad (7)$$

The weight matrix in Eq. (7) assumes that each seller is allowed to conduct a fuel transaction with any buyer. Therefore, $\mathcal{G}_r$ is a complete bipartite graph, and each element of Π represents an edge in $\mathcal{G}_r$. Then, a matching can thus be represented in Π by a collection of elements with no two elements appearing in one row or column. A maximum-weight matching is then a collection of independent elements such that the sum of the elements is maximized. For example, the symmetric matching is shown in Eq. (7) with the collection of elements in squares. As proved in Proposition 4.1, the symmetric matching is a maximum-weight matching if all edges are available. After the fuel transaction between the satellite pairs in the symmetric matching, the fuel of each satellite is summarized in Table V. The total deviation of the onboard fuel from the average after the refuelling is $\sum_{i\in\mathcal{I}} \left|f_i^+ - \bar{f}\right| = 30$, while the total deviation of the onboard fuel from the average before the refuelling is $\sum_{i\in\mathcal{I}} \left|f_i^+ - \bar{f}\right| = 420$.

If constraints are imposed on the satellite pairs, $\mathcal{G}_r$ may no longer be a complete bipartite graph, and all edges in the symmetric matching may not exist anymore. In this case, the solution to the maximum-weight matching problem is no longer trivial, and the algorithm mentioned in Section IV has to be applied to obtain the optimal solution.

In general, if not all edges in the symmetric matching exist in the reduced constellation graph, then the edges of the symmetric matching that do exist in the reduced constellation graph are not necessarily part of the optimal solution. To this end, a *partial symmetric matching* will denote the edges in the symmetric matching that do exist in the reduced constellation graph. For example, suppose that in the constellation, satellites 2 and 13 and satellites 6 and 9 are not allowed to have a fuel transaction. Then, the partial symmetric matching is given by $\Big\{\langle v_1, v_{14}\rangle, \langle v_3, v_{12}\rangle, \langle v_4, v_{11}\rangle, \langle v_5, v_{10}\rangle, \langle v_7, v_8\rangle\Big\}$. The reduced constellation graph is not a complete bipartite graph, and the weight matrix has changed as shown in (8) below, where the symbol '$\times$' implies that the two corresponding satellites are not allowed to have a fuel transaction.

$$
\Pi = \begin{array}{c}
\begin{array}{ccccccc}
\mathrm{v}_{14} & \mathrm{v}_{13} & \mathrm{v}_{12} & \mathrm{v}_{11} & \mathrm{v}_{10} & \mathrm{v}_{9} & \mathrm{v}_{8} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left(\begin{array}{ccccccc}
\boxed{92} & 74 & 70 & 68 & 62 & 32 & 14 \\
78 & \times & \boxed{70} & 68 & 62 & 32 & 14 \\
74 & \boxed{74} & 70 & 68 & 62 & 32 & 14 \\
62 & 62 & 62 & 62 & \boxed{62} & 32 & 14 \\
46 & 46 & 46 & \boxed{46} & 46 & 32 & 14 \\
36 & 36 & 36 & 36 & 36 & \times & \boxed{14} \\
32 & 32 & 32 & 32 & 32 & \boxed{32} & 14
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\
\leftarrow \mathrm{v}_1 \\ \leftarrow \mathrm{v}_2 \\ \leftarrow \mathrm{v}_3 \\ \leftarrow \mathrm{v}_4 \\ \leftarrow \mathrm{v}_5 \\ \leftarrow \mathrm{v}_6 \\ \leftarrow \mathrm{v}_7
\end{array}
\tag{8}
$$

Two matchings are shown in the weight matrix Π in Eq. (8). The first one is an optimal matching for the (MP-IP), with the corresponding matrix elements shown in squares. The second one is an optimal matching obtained with the restriction that the partial symmetric matching be part of the optimal matching, with the corresponding matrix elements shown in circles. It can be verified that the total weight of the first solution is 390, much greater the total weight of the second solution which is 352. Therefore, the solution which includes the partial symmetric matching is not optimal.

Notice that the optimal solution to the refuelling problem is not unique. For example, the optimal solution in the last example and the symmetric matching both have a total weight of 390. In addition, in the optimal matching for the last example, if the edges $\langle \mathrm{v}_4, \mathrm{v}_{10} \rangle$ and $\langle \mathrm{v}_5, \mathrm{v}_{11} \rangle$ are replaced by $\langle \mathrm{v}_4, \mathrm{v}_{11} \rangle$ and $\langle \mathrm{v}_5, \mathrm{v}_{10} \rangle$, the resulting matching is also an optimal matching.

Finally, we will show that the optimal matching need not to be a maximum matching. To this end, assume that all satellite pairs in the symmetric matching are not allowed to have fuel transactions. In addition, let us assume that the following satellite pairs are not allowed to have fuel transactions, $\langle \mathrm{v}_1, \mathrm{v}_{10} \rangle$, $\langle \mathrm{v}_1, \mathrm{v}_{12} \rangle$, $\langle \mathrm{v}_3, \mathrm{v}_{10} \rangle$, $\langle \mathrm{v}_3, \mathrm{v}_{14} \rangle$, $\langle \mathrm{v}_5, \mathrm{v}_8 \rangle$, $\langle \mathrm{v}_5, \mathrm{v}_{12} \rangle$, $\langle \mathrm{v}_5, \mathrm{v}_{14} \rangle$, $\langle \mathrm{v}_7, \mathrm{v}_{10} \rangle$, $\langle \mathrm{v}_7, \mathrm{v}_{12} \rangle$, $\langle \mathrm{v}_7, \mathrm{v}_{14} \rangle$. As mentioned earlier, these restrictions may be imposed from satellite or constellation operational requirements. In this case, the weight matrix is changed as follows.

$$
\Pi = \begin{array}{c}
\begin{array}{ccccccc}
\mathrm{v}_{14} & \mathrm{v}_{13} & \mathrm{v}_{12} & \mathrm{v}_{11} & \mathrm{v}_{10} & \mathrm{v}_{9} & \mathrm{v}_{8} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\left(\begin{array}{ccccccc}
\times & \boxed{74} & \times & 68 & \times & 32 & 14 \\
\boxed{78} & \times & 70 & 68 & 62 & 32 & 14 \\
\times & 74 & \times & \boxed{68} & \times & 32 & 14 \\
62 & 62 & 62 & \times & \boxed{62} & 32 & 14 \\
\times & 46 & \times & 46 & \times & 32 & \times \\
36 & 36 & \boxed{36} & 36 & 36 & \times & 14 \\
\times & 32 & \times & 32 & \times & \boxed{32} & \times
\end{array}\right)
\end{array}
\begin{array}{l}
\\ \\
\leftarrow \mathrm{v}_1 \\ \leftarrow \mathrm{v}_2 \\ \leftarrow \mathrm{v}_3 \\ \leftarrow \mathrm{v}_4 \\ \leftarrow \mathrm{v}_5 \\ \leftarrow \mathrm{v}_6 \\ \leftarrow \mathrm{v}_7
\end{array}
\tag{9}
$$

An optimal matching is shown in Eq. (9) with the corresponding matrix elements in squares. Notice that this matching consists of only six edges. Satellites 5 and 8 are left unmatched. The total weight for this matching is 350. Moreover, a maximum-weight maximum matching is obtained and shown in Eq. (9) as well, with the corresponding matrix elements in circles. This matching has seven edges, and it is optimal among all maximum matchings. The total weight for this matching is 342, less than that of the optimal maximum-weight matching which contains only six edges. This shows that the optimal solution is not necessarily a maximum matching.

VI. Conclusion

We have investigated the P2P refuelling problem with negligible rendezvous cost. The problem is formulated as a maximum-weight matching problem in a bipartite graph. It is shown that the symmetric matching is optimal if all edges in the symmetric matching exist. Otherwise, a partial symmetric matching is not guaranteed to be included in the optimal solution. It is also shown that the optimal solution is not unique, and that the optimal solution is not necessarily a maximum matching.

VII. References

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