Parameter-Dependent Lyapunov Functions for Exact Stability Analysis of Single-Parameter Dependent LTI Systems

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***Abstract*— In this paper, we propose a class of parameter-dependent Lyapunov functions which can be used to assess the stability properties of linear, time-invariant, single-parameter dependent (LTIPD) systems in a non-conservative manner. It is shown that stability of LTIPD systems is equivalent to the existence of a Lyapunov function of a polynomial type (in terms of the parameter) of known, bounded degree satisfying two matrix inequalities. It is also shown that checking the feasibility of these matrix inequalities over a compact set can be cast as a convex optimization problem.**

I. Introduction

For LTI Parameter-Dependent (LTIPD) systems of the form $\dot{x} = A(\rho)x$, $\rho \in \Omega$, stability can be established via the use of constant Lyapunov functions, say, of the form $V(x) = x^T P x$. When the parameter ρ varies in the set Ω or its value is not known a priori, a common (for all ρ) Lyapunov function can be used to check Hurwitz stability of the family of matrices $\{A(\rho), \rho \in \Omega\}$. The resulting notion of stability (quadratic stability) is nonetheless conservative, since the same Lyapunov matrix P is used for the whole parameter space. The conservativeness of quadratic stability is more pronounced for the case of LTIPD systems where the parameter ρ does not vary with time. In order to achieve necessary and sufficient results one then needs to resort to the use of parameter-dependent Lyapunov functions of the form $V(x,\rho) = x^T P(\rho) x$.

Since the explicit dependence of the Lyapunov matrix $P(\rho)$ on the parameter ρ is not known a priori, one typically postulates a convenient functional parameter dependence for $P(\rho)$, and then one proceeds to derive the stability conditions. This approach leads to conditions which are sufficient but not necessary. In order to obtain nonconservative (i.e., necessary and sufficient) conditions it is imperative to know the "correct" parameter dependence for the Lyapunov function. By "correct" we mean a Lyapunov function which depends on the parameter in such a way that for those values of the parameter for which the system is stable the stability conditions are satisfied, while for the values of the parameter for which the system is not stable, the stability conditions fail.

In this paper we show that for LTI systems depending on a single, constant parameter in an affine manner, nonconservative stability tests can be derived by restricting the search over Lyapunov matrices which depend polynomially on the parameter. Therefore, a polynomial-type Lyapunov matrix (of known degree) is suggested in this paper, which can be used to derive necessary and sufficient stability conditions for single-parameter LTIPD systems.

The contributions of the paper are summarized as follows: First, a polynomial-type Lyapunov function of bounded, computable degree is proposed which can be used to derive sufficient *and necessary* stability conditions for single-parameter LTIPD systems. These stability conditions are given in terms of two simultaneous matrix inequalities. The conditions take explicitly into consideration the rank deficiency of the system matrix multiplying the parameter in order to reduce the computational complexity of the proposed algorithm. The main contribution of the paper is therefore the knowledge of the structure of the Lyapunov matrix that leads to nonconservative (i.e., exact) stability results for single-parameter LTIPD systems. Second, the inequalities for checking the stability of an LTIPD system over a compact interval are expressed into computable, non-conservative linear matrix inequalities (LMIs). We thus also provide a non-conservative condition for checking the robust stability of single-parameter LTI systems over compact intervals. We note here that although the stability of LTIPD systems can also be checked using the guardian map techniques of [15], [6] (for similar results see also [16]) nonetheless, it is expected that the Lyapunov-based stability conditions of this paper will be also amenable to synthesis. Such an extension to the synthesis problem is not directly evident from the use of guardian maps.

II. Linear Time-Invariant Parameter-Dependent Systems

The objective of this paper is to find computable, non-conservative conditions for checking the asymptotic stability of LTI systems which depend on a single real parameter of the form

$$\dot{x} = A(\rho)x, \qquad A(\rho) = A_0 + \rho A_1, \qquad \rho \in \Omega \tag{1}$$

where $A_0, A_1 \in \mathbb{R}^{n\times n}$ and $\Omega \subset \mathbb{R}$. At this point we make no a priori assumptions on the set Ω (i.e., connected, bounded, compact, etc.). The parameter ρ is assumed to be constant[1] and it is chosen from the set Ω. It is well known that asymptotic stability of the system (1) is equivalent to the existence of a matrix $P(\rho) \in \mathbb{R}^{n\times n}$ such that

$$P(\rho) > 0, \quad A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0, \quad \rho \in \Omega. \tag{2}$$

Thus, checking the stability of (1) is equivalent to finding a Lyapunov function $P(\rho)$ satisfying the two matrix inequalities (2). By the same token, if for some $\rho' \in \Omega$ the matrix

[1] The results also hold when ρ varies very slowly so that a "quasi-static" point of view is valid.

$A(\rho')$ is not Hurwitz, then there exists no positive-definite matrix that satisfies the second inequality in (2).

Clearly, for single-parameter LTIPD systems, as the one in equation (1), stability can be ensured if there exists a constant Lyapunov function $P(\rho) = P$ for all $\rho \in \Omega$, such that the two inequalities (2) are satisfied. The so-called quadratic stability ensures robust stability against any (arbitrarily fast) variations of the parameter ρ. In case the parameters do not vary with time (such is the case with LTIDP systems), quadratic stability can be very conservative. To reduce this conservatism against slowly-varying or constant parameters, several parameter-dependent Lyapunov functions have been proposed to derive stability conditions. Such conditions, however, provide only sufficiency results which, if fact, may be far from necessary. On the other hand, for the multi- and single-parameter dependent LTI systems, references [5] and [13] provide a class of Lyapunov functions that can be used to derive necessary and sufficient conditions for system (1), assuming that the matrix A_1 has rank one. The Lyapunov function proposed in [5] solves an augmented system and depends multiaffinely on the parameter vector. On the same token, in [13] it is shown that for a single parameter and for $\operatorname{rank} A_1 = 1$, a Lyapunov function which is linear in the parameter can be used to characterize stability of the system (1).

More recently, [2], [3] proposed parameter-dependent Lyapunov functions of polynomial type in the parameter (of sufficiently high degree) which can be used to assess the robust stability of multi-linear systems over a compact set without conservatism. Recall that if $Q(\rho) \in \mathbb{R}^{n\times n}$ is any positive-definite matrix for all $\rho \in \Omega \subset \mathbb{R}$, then the stability of (1) can be established by finding a positive-definite solution to the following Lyapunov *equation*

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) + Q(\rho) = 0. \tag{3}$$

The solution $P(\rho)$ to this equation is given explicitly as [17]

$$P(\rho) = \int_0^\infty e^{A(\rho)t} Q(\rho) e^{A^T(\rho)t} \mathrm{d}t. \tag{4}$$

When $Q(\rho)$ is analytic in ρ, $P(\rho)$ in Eq. (4) is also analytic in ρ and thus it can be expressed in terms of power series in ρ as

$$P(\rho) = P_0 + \rho P_1 + \rho^2 P_2 + \ldots = \sum_{i=0}^{\infty} \rho^i P_i. \tag{5}$$

Starting from this simple observation, and using the uniform convergence of the integral in (4) at $t = +\infty$ with respect to ρ when Ω is compact, Bliman recently showed in [3] that the previous power series can be truncated and thus, a polynomial type Lyapunov matrix of the form

$$P(\rho) = P_0 + \rho P_1 + \rho^2 P_2 + \ldots + \rho^m P_m = \sum_{i=0}^{m} \rho^i P_i \tag{6}$$

of sufficiently high degree m in ρ solves the Lyapunov *inequality* $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$, and thus it can be used to provide necessary and sufficient conditions for the robust stability of (1) over the set Ω. In [3] however no a priori bound on the degree of the truncated polynomial is given. The main contribution of the present paper is to give an explicit bound for the polynomial dependence m of $P(\rho)$ in ρ and to provide a computable algorithm for checking the associated linear matrix inequalities (2). In particular, we show that the existence of a Lyapunov matrix of the form (6) with $m \le \min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\}$ is necessary and sufficient for the stability of (1) *for each* $\rho \in \Omega$, where r denotes the rank of A_1. In other words, for every $\rho \in \Omega$ if $P(\rho)$ in (6) satisfies (2), then the matrix $A(\rho)$ is Hurwitz. Most importantly, if for some $\rho \in \Omega$ the matrix $A(\rho)$ is not Hurwitz, then the matrix $P(\rho)$ is either non-positive definite, or the second inequality in (2) does not hold. Finally, we show how the two matrix inequalities (2) involved in checking the stability of $A(\rho)$ can be cast into computable LMIs without conservatism in case Ω is a compact interval.

III. Main Result

Definition 3.1 ([11]): Given a symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, define

$$\overline{\mathrm{vec}}(P) := \begin{bmatrix} P_{11} \\ \vdots \\ P_{n1} \\ P_{22} \\ \vdots \\ P_{n2} \\ \vdots \\ P_{nn} \end{bmatrix} \in \mathbb{R}^{\frac{1}{2}n(n+1)} \tag{7}$$

Note that the usual $\mathrm{vec}(P)$ operator [4] that stacks the columns of a matrix P one on top of the other consists of all the elements of $\overline{\mathrm{vec}}(P)$ with some repetitions. For every symmetric matrix $P = P^T \in \mathbb{R}^{n\times n}$, there exists a unique full column rank matrix $D_n \in \mathbb{R}^{n^2 \times \frac{1}{2}n(n+1)}$ called *the duplication matrix* [10], [11], which is independent of the matrix P and which depends only on the dimension n of the matrix P, and which satisfies

$$\mathrm{vec}(P) = D_n \overline{\mathrm{vec}}(P). \tag{8}$$

The pseudo-inverse of D_n satisfies the following properties [10], [11]

$$\begin{aligned} &\overline{\mathrm{vec}}(P) = D_n^+ \mathrm{vec}(P), \quad D_n^+ D_n = I_{\frac{1}{2}n(n+1)}, \\ &\mathrm{rank}(D_n) = \mathrm{rank}(D_n^+) = \tfrac{1}{2}n(n+1). \end{aligned}$$

Notice, in particular, that D_n is always full column rank. Consequently, $D_n^+ = (D_n^T D_n)^{-1} D_n^T$.

Definition 3.2 ([11]): Given $A \in \mathbb{R}^{n\times n}$, let $\widehat{A} \in \mathbb{R}^{\frac{1}{2}n(n+1)\times\frac{1}{2}n(n+1)}$ be defined by

$$\widehat{A} := D_n^+ (A \oplus A) D_n = D_n^+ \bar{A} D_n. \tag{9}$$

where $\bar{A} := A \oplus A = I_n \otimes A + A \otimes I_n$ is the Kronecker sum of matrix A with itself.

The matrix $\widehat{A}$ is often called the lower Schlaeflian form of or the power form of the matrix A. It is clear from the definition that $\widehat{A}(\rho) = \widehat{A_0 + \rho A_1} = \widehat{A}_0 + \rho \widehat{A}_1$.

The main result in the paper can be stated as follows:

Theorem 3.1: Given the matrices $A_0,\, A_1 \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} A_1 = r$, let

$$m := \begin{cases} \frac{1}{2}(2nr - r^2 + r), & \text{if } \; r < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if } \; r = n. \end{cases} \tag{10}$$

Then the following two statements are equivalent:

(i) $A_0 + \rho A_1$ is Hurwitz for all $\rho \in \Omega$.

(ii) There exists a set of $m+1$ matrices $\{P_i\}_{0\le i\le m}$, such that

$$(A_0 + \rho A_1)^T P(\rho) + P(\rho)(A_0 + \rho A_1) < 0, \tag{11}$$

$$P(\rho) = \sigma(\rho)\left(\sum_{i=0}^{m} \rho^i P_i\right) > 0, \tag{12}$$

where $\sigma(\rho) = -\operatorname{sign}(\det(\widehat{A}_0 + \rho\widehat{A}_1))$ and $\rho \in \Omega$.

Remark 1 Note that if the domain Ω is connected then $\sigma(\rho)$ is constant via Corollary 3.1 (see below) and the Lyapunov matrix (12) is given simply by $P(\rho) = \sum_{i=0}^{m} \rho^i P_i$ for all $\rho \in \Omega$.

In order to provide the proof of Theorem 3.1 we need first to introduce a few mathematical preliminaries.

Lemma 3.1: Let matrices $A, B \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} B = r$ and let $\rho \in \mathbb{R}$. Then $\deg(\det(A + \rho B)) \le r$.

The following lemma will play a major role in the results of this paper. It states that the adjoint of the parameter-dependent matrix $A + \rho B$ is a matrix polynomial in ρ of a certain maximal degree which depends on the rank of the matrix B. Recall that given an invertible matrix $A \in \mathbb{R}^{n\times n}$, its inverse can be calculated from $A^{-1} = \operatorname{Adj} A / \det(A)$ where $\operatorname{Adj} A$ is the adjoint of A.

Lemma 3.2: Given matrices $A, B \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} B = r$ and $\rho \in \mathbb{R}$, the adjoint of the matrix $A + \rho B$ is a matrix polynomial in ρ of degree at most $\min\{r, n-1\}$, i.e.,

$$\operatorname{Adj}(A + \rho B) = \sum_{i=0}^{\min\{r,n-1\}} \rho^i N_i. \tag{13}$$

We note here that the matrices N_i in (13) can be calculated explicitly from the matrices A_0 and A_1. The details are left to the reader.

Lemma 3.3 ([10], [11]): Given $A \in \mathbb{R}^{n\times n}$ and $\widehat{A}$ as in Definition 3.2, the eigenvalues of $\widehat{A}$ are the $\frac{1}{2}n(n+1)$ numbers $\lambda_i + \lambda_j$, $(1 \le j \le i \le n)$ where λ_i, λ_j are the eigenvalues of A.

The following is immediate from Lemma 3.3.

Corollary 3.1: Suppose the parameter-dependent matrix $A_0 + \rho A_1 \in \mathbb{R}^{n\times n}$ is Hurwitz for all $\rho \in \Omega$. Then

$$\det(\widehat{A_0 + \rho A_1}) = \det(\widehat{A}_0 + \rho\widehat{A}_1) \ne 0, \qquad \forall \rho \in \Omega \tag{14}$$

Central to our results is the following lemma which provides a bound for the rank of the Schlaeflian form of a matrix.

Lemma 3.4: Given a matrix $A \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} A = r$, then $\operatorname{rank} \widehat{A} \le \frac{1}{2}(2nr - r^2 + r)$.

We are now ready to provide the proof of Theorem 3.1.

Proof: [Of Theorem 3.1] $(ii) \Rightarrow (i)$: This is obvious.

$(i) \Rightarrow (ii)$: Since $A_0 + \rho A_1$ is Hurwitz for all $\rho \in \Omega$, from Corollary 3.1 it follows that $\det(\widehat{A}_0 + \rho\widehat{A}_1) \ne 0$. Let the parameter-dependent matrix

$$Q(\rho) := |\det(\widehat{A}_0 + \rho\widehat{A}_1)| I_n > 0, \quad \rho \in \Omega. \tag{15}$$

Note that $Q(\rho)$ is positive definite for each $\rho \in \Omega$. Since $A_0 + \rho A_1$ is Hurwitz for all $\rho \in \Omega$, the following Lyapunov equation has a unique, positive definite-solution $P(\rho) > 0$ for each $\rho \in \Omega$

$$(A_0 + \rho A_1)P(\rho) + P(\rho)(A_0 + \rho A_1)^T + Q(\rho) = 0. \tag{16}$$

Solving this equation for $P(\rho)$ one obtains

$$\begin{aligned} \overline{(A_0 + \rho A_1)}\operatorname{vec}(P(\rho)) &= -|\det(\widehat{A}_0 + \rho\widehat{A}_1)|\operatorname{vec}(I_n) \\ (\bar{A}_0 + \rho\bar{A}_1)\operatorname{vec}(P(\rho)) &= -|\det(\widehat{A}_0 + \rho\widehat{A}_1)|\operatorname{vec}(I_n) \\ D_n^+(\bar{A}_0 + \rho\bar{A}_1)D_n\overline{\operatorname{vec}}(P(\rho)) &= -|\det(\widehat{A}_0 + \rho\widehat{A}_1)|\,\overline{\operatorname{vec}}(I_n) \\ (\widehat{A}_0 + \rho\widehat{A}_1)\overline{\operatorname{vec}}(P(\rho)) &= -|\det(\widehat{A}_0 + \rho\widehat{A}_1)|\overline{\operatorname{vec}}(I_n) \end{aligned}$$

and thus,

$$\begin{aligned} \overline{\operatorname{vec}}(P(\rho)) &= -|\det(\widehat{A}_0 + \rho\widehat{A}_1)|\frac{\operatorname{Adj}(\widehat{A}_0+\rho\widehat{A}_1)}{\det(\widehat{A}_0+\rho\widehat{A}_1)}\overline{\operatorname{vec}}(I_n) \\ &= \sigma(\rho)\operatorname{Adj}(\widehat{A}_0 + \rho\widehat{A}_1)\overline{\operatorname{vec}}(I_n) \end{aligned} \tag{17}$$

where $\sigma(\rho) := -\operatorname{sign}(\det(\widehat{A}_0 + \rho\widehat{A}_1))$.

Let $\hat{r} := \operatorname{rank} \widehat{A}_1$. According to Lemma 3.4 we have that $\hat{r} \le \frac{1}{2}(2nr - r^2 + r)$. Moreover, according to Lemma 3.2 there exist constant matrices N_i such that $\operatorname{Adj}(\widehat{A}_0 + \rho\widehat{A}_1) = \sum_{i=0}^{m} \rho^i N_i$ where $m = \min\{\hat{r}, \frac{1}{2}n(n+1) - 1\} \le \min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\}$. Notice, in particular that $\min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\} = \frac{1}{2}(2nr - r^2 + r)$ for $r < n$ and $\min\{\frac{1}{2}(2nr - r^2 + r), \frac{1}{2}n(n+1) - 1\} = \frac{1}{2}n(n+1) - 1$ if $r = n$.

Moreover, since the mapping $\overline{\operatorname{vec}}(\cdot)$ is one-to-one, its inverse mapping $\overline{\operatorname{vec}}^{-1}(\cdot)$ exists. Therefore, (17) finally yields

$$P(\rho) = \sigma(\rho)\left(\sum_{i=0}^{m} \rho^i P_i\right) \tag{18}$$

where $P_i \in \mathbb{R}^{n\times n}$ are constant matrices given by $P_i = \overline{\operatorname{vec}}^{-1}\Big(N_i\overline{\operatorname{vec}}(I_n)\Big)$ for $0 \le i \le m$. ∎

A. Numerical Examples

Example 1 Consider the matrix $A(\rho) = A_0 + \rho A_1$, where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The largest stability domain for this example is $(-1, 2)$. To compute $P(\rho)$, first note that

$$\widehat{A}_0 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \widehat{A}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

and $\det(\widehat{A}_0 + \rho\widehat{A}_1) = (-4 + 2\rho)(6 + 6\rho) = -24 - 12\rho + 12\rho^2$. Therefore,

$$\operatorname{Adj}(\widehat{A}_0 + \rho\widehat{A}_1) = \begin{bmatrix} 6\rho + 6 & 0 & 0 \\ 0 & -4\rho^2 + 4\rho + 8 & 0 \\ 0 & 0 & -6\rho + 12 \end{bmatrix},$$

and thus,

$$P(\rho) = \begin{bmatrix} 6\rho+6 & 0 \\ 0 & -6\rho+12 \end{bmatrix}.$$

Moreover,

$$A(\rho)P(\rho) + P(\rho)A^T(\rho) = \begin{bmatrix} -24-12\rho+12\rho^2 & 0 \\ 0 & -24-12\rho+12\rho^2 \end{bmatrix}.$$

The eigenvalues of $P(\rho)$ are given by $\lambda_1 = 6\rho+6$ and $\lambda_2 = -6\rho+12$. Notice that $\lambda_{1,2}(\rho) > 0$ for $\rho \in (-1, 2)$ and therefore $P(\rho) > 0$ for $\rho \in (-1, 2)$. Furthermore, $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$ for $\rho \in (-1, 2)$.

Example 2 Consider the parameter-dependent matrix $A(\rho) = A_0 + \rho A_1$, where

$$A_0 = \begin{bmatrix} 0.7493 & -2.4358 & -1.6503 \\ -2.0590 & -3.3003 & -1.4833 \\ -1.5019 & 1.2149 & -4.8737 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1.2149 & 1.6640 & -2.2091 \\ 0.7542 & -0.1501 & 0.2109 \\ 2.1990 & 0.6493 & -0.2214 \end{bmatrix} \quad (19)$$

The exact stability domain for $A(\rho)$ is $(-18.3861, -1.2729) \cup (2.1538, 3.7973)$, which is computed with the method presented in [16]. With the method introduced in the proof of Theorem 3.1, one can first compute $\widehat{A}_0$, $\widehat{A}_1$ and then $\mathrm{Adj}(\widehat{A}_0 + \rho\widehat{A}_1)$. The matrix $P(\rho)$, is a polynomial in ρ of degree $m = \frac{1}{2}n(n+1) - 1 = 5$ since $r = \mathrm{rank}(A_1) = n = 3$.

In this example the stability domain is composed of two disjoint intervals. The parameter-dependent Lyapunov function is given by

$$P(\rho) = \sigma(\rho)\Big(P_0 + \rho P_1 + \rho^2 P_2 + \rho^3 P_3 + \rho^4 P_4 + \rho^5 P_5\Big)$$

It can be checked numerically that $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$ for all $\rho \in \mathbb{R}$. However, $P(\rho)$ is positive definite only for $\rho \in (-18.3861, -1.2729) \cup (2.1538, 3.7973)$.

Remark 2 Theorem 3.1 can be used to determine the whole stability domain of a parameter-dependent LTI system, even if this domain is composed of several disjoint intervals of $\mathbb{R}$ as is the case in Example 2. The approach of [1], [12], [11], [6], [14], [15] without modification, on the other hand, can only be used to check the stability over a connected domain which includes the origin.

IV. A Convex Characterization of the Stability Conditions

The previous analysis shows that the parameter-dependent matrix $A(\rho) = A_0 + \rho A_1$ is Hurwitz for any $\rho \in \Omega$, if and only if there exists a Lyapunov matrix which depends polynomially on the parameter ρ, of the form

$$P(\rho) := P_0 + \rho P_1 + \ldots + \rho^m P_m, \quad (20)$$

such that the corresponding two matrix inequalities

$$A(\rho)P(\rho) + P(\rho)A(\rho)^T < 0, \quad (21)$$
$$P(\rho) > 0, \quad (22)$$

are satisfied. In order to be able to use the stability criterion of Theorem 3.1 in practice, we need a relatively simple method to determine the feasibility of the matrix inequalities (21) and (22).

In this section we provide computable, non-conservative, conditions to test the matrix inequalities (21) and (22) over any compact interval Ω. Without loss of generality, in the sequel we assume that $\Omega := [-1, 1]$.

To this end, let the vector $\rho^{[q]} \in \mathbb{R}^q$ be defined by

$$\rho^{[q]} := \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{q-1} \end{pmatrix}^T, \quad (23)$$

and notice that the parameter-dependent matrix in (20) can be rewritten as

$$P(\rho) = \left(\rho^{[k]} \otimes I_n\right)^T P_\Sigma \left(\rho^{[k]} \otimes I_n\right) \quad (24)$$

where $k = \lceil \frac{m}{2} \rceil + 1$ and $P_\Sigma = P_\Sigma^T \in \mathbb{R}^{nk \times nk}$ is a constant symmetric matrix (here $\lceil \frac{m}{2} \rceil$ denotes the smallest integer which is larger than or equal to $m/2$). Note that the matrix P_Σ is not unique. On the other hand, for any given symmetric matrix P_Σ one can also get a unique polynomial Lyapunov matrix $P(\rho)$ in the form (20) using the expression (24).

The following lemma provides a convenient expression for the matrix $R(\rho) = A(\rho)P(\rho) + P(\rho)A^T(\rho)$ which will be useful for providing a convex characterization of inequality (21).

Lemma 4.1 ([3]): Given a matrix $A(\rho) = A_0 + \rho A_1 \in \mathbb{R}^{n\times n}$ and a symmetric, parameter-dependent matrix $P(\rho) \in \mathbb{R}^{n\times n}$ as

$$P(\rho) = \left(\rho^{[k]} \otimes I_n\right)^T P_\Sigma \left(\rho^{[k]} \otimes I_n\right),$$

let $R(\rho) := A^T(\rho)P(\rho) + P(\rho)A(\rho)$. Then

$$R(\rho) = \left(\rho^{[k+1]} \otimes I_n\right)^T R_\Sigma \left(\rho^{[k+1]} \otimes I_n\right) \quad (25)$$

where,

$$R_\Sigma = H_\Sigma^T P_\Sigma F_\Sigma + F_\Sigma^T P_\Sigma H_\Sigma \quad (26)$$
$$H_\Sigma = \hat{J}_k \otimes I_n \quad (27)$$
$$F_\Sigma = \hat{J}_k \otimes A_0 + \check{J}_k \otimes A_1 \quad (28)$$

and $\hat{J}_k := \begin{bmatrix} I_k & 0_{k\times 1} \end{bmatrix}$ and $\check{J}_k := \begin{bmatrix} 0_{k\times 1} & I_k \end{bmatrix}$.

Notice that the matrix R_Σ depends linearly upon each of the matrices P_Σ, A_0 and A_1.

The following lemma is instrumental in casting the matrix feasibility problem (21)-(22) to a convex optimization problem. It is an extension of a result given in [8].

Lemma 4.2: Let the matrices $\Theta = \Theta^T \in \mathbb{R}^{n\times n}$ and $J, C \in \mathbb{R}^{k\times n}$ be given. The following statements are equivalent.

(i) The inequality $\zeta^T \Theta \zeta < 0$ holds for all nonzero vectors $\zeta \in \mathbb{R}^n$ which satisfy $(J - \delta C)\zeta = 0$, for some real scalar δ such that $|\delta| \leq 1$.

(ii) There exist matrices $D \in \mathbb{R}^{k\times k}$ and $G \in \mathbb{R}^{k\times k}$ such that

$$D = D^T > 0, \quad G + G^T = 0,$$
$$\Theta < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}.$$

A. LMI Conditions for Checking the Stability of LTIPD Systems

It is desirable to find computable, convex, non-conservative conditions to test the stability conditions (21)-(22). Using (24) and (25), the inequalities (21)-(22) can be rewritten as

$$\left(\rho^{[k]} \otimes I_n\right)^T P_\Sigma \left(\rho^{[k]} \otimes I_n\right) > 0, \quad \forall \rho \in \Omega, \tag{29}$$

$$\left(\rho^{[k+1]} \otimes I_n\right)^T R_\Sigma \left(\rho^{[k+1]} \otimes I_n\right) < 0, \quad \forall \rho \in \Omega. \tag{30}$$

Lemma 4.3: Given the matrices $J = \check{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1)\times nk}$ and $C = \hat{J}_{k-1} \otimes I_n \in \mathbb{R}^{n(k-1)\times nk}$, the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ below are equal.

$$\mathcal{C}_1 := \{\zeta \in \mathbb{R}^{nk} : (J - \delta C)\zeta = 0, \text{ some } \delta \in [-1, 1]\},$$

$$\mathcal{C}_2 := \{\zeta \in \mathbb{R}^{nk} : \zeta = (\rho^{[k]} \otimes I_n)z, \ \rho \in [-1, 1], \ z \in \mathbb{R}^n\}.$$

Lemma 4.4: Let $J := \check{J}_{k-1} \otimes I_n = [0 \ I] \in \mathbb{R}^{n(k-1)\times nk}$ and $C := \hat{J}_{k-1} \otimes I_n = [I \ 0] \in \mathbb{R}^{n(k-1)\times nk}$. Then the matrix inequality

$$\left(\rho^{[k]} \otimes I\right)^T \Theta \left(\rho^{[k]} \otimes I\right) < 0 \tag{31}$$

holds for all $\rho \in [-1, 1]$ if and only if there exist matrices $D \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $G \in \mathbb{R}^{n(k-1)\times n(k-1)}$ such that

$$D = D^T > 0, \quad G + G^T = 0,$$
$$\Theta < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix}.$$

Example 3 Let $P(\rho) = (1+\epsilon)I_n - \rho^2 I_n$. It is clear that if $\epsilon > 0$, $P(\rho)$ is positive definite for all $\rho \in [-1, 1]$. If, on the other hand, $\epsilon < 0$, $P(\rho)$ is not positive definite for all $\rho \in [-1, 1]$. Rewriting $P(\rho)$ in the form (24),

$$\begin{aligned} P(\rho) &= \left(\rho^{[2]} \otimes I_n\right)^T P_\Sigma \left(\rho^{[2]} \otimes I_n\right) \\ &= \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix}^T \begin{bmatrix} (1+\epsilon)I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n \\ \rho I_n \end{bmatrix} \end{aligned} \tag{32}$$

and applying Lemma 4.4, with $k = 2$, the condition $P(\rho) > 0$ for all $\rho \in [-1, 1]$ is equivalent to the existence of matrices $D = D^T > 0$ and $G + G^T = 0$ such that

$$-P_\Sigma < \begin{bmatrix} C \\ J \end{bmatrix}^T \begin{bmatrix} -D & G \\ G^T & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} \tag{33}$$

where $J = [0_{n\times n} \ I_n]$ and $C = [I_n \ 0_{n\times n}]$. The matrix inequality (33) is equivalent to the existence of matrices $D = D^T > 0$ and $G + G^T = 0$ such that

$$\begin{bmatrix} D - (1+\epsilon)I_n & -G \\ -G^T & I_n - D \end{bmatrix} < 0. \tag{34}$$

A necessary condition for the existence of D in (34) is $I_n < D < (1+\epsilon)I_n$. When $\epsilon > 0$, such a D exists and along with $G = 0$ the LMI (33) is feasible. When $\epsilon < 0$, no D can satisfy (34) and the LMI (34) has no solution. For both cases, the result of Lemma 4.4 agrees with the direct stability analysis.

The following is a direct consequence of Lemma 4.4. It provides convex conditions in terms of LMIs for checking the robust stability of the parameter dependent matrix $A(\rho) = A_0 + \rho A_1$ for $\rho \in [-1, +1]$.

Theorem 4.1: Let the parameter-dependent matrix $A(\rho) = A_0 + \rho A_1$, where $A_0, A_1 \in \mathbb{R}^{n\times n}$ with $\operatorname{rank} A_1 = r$ and let $k = \lceil \frac{m}{2} \rceil + 1$ where

$$m := \begin{cases} \frac{1}{2}(2nr - r^2 + r), & \text{if } \ r < n, \\ \frac{1}{2}n(n+1) - 1, & \text{if } \ r = n. \end{cases} \tag{35}$$

Let $J_1 = [0 \ I_{n(k-1)}] \in \mathbb{R}^{n(k-1)\times nk}$, $C_1 = [I_{n(k-1)} \ 0] \in \mathbb{R}^{n(k-1)\times nk}$, $J_2 = [0 \ I_{nk}] \in \mathbb{R}^{nk\times n(k+1)}$ and $C_2 = [I_{nk} \ 0] \in \mathbb{R}^{nk\times n(k+1)}$. Then, $A(\rho)$ is Hurwitz for all $\rho \in [-1, 1]$ if and only if there exist symmetric matrices $P_\Sigma \in \mathbb{R}^{nk\times nk}$, $D_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$ and $D_2 \in \mathbb{R}^{nk\times nk}$ and skew-symmetric matrices $G_1 \in \mathbb{R}^{n(k-1)\times n(k-1)}$, $G_2 \in \mathbb{R}^{nk\times nk}$, such that

$$\begin{aligned} &D_1 = D_1^T > 0, \quad G_1 + G_1^T = 0, \\ &-P_\Sigma < \begin{bmatrix} C_1 \\ J_1 \end{bmatrix}^T \begin{bmatrix} -D_1 & G_1 \\ G_1^T & D_1 \end{bmatrix} \begin{bmatrix} C_1 \\ J_1 \end{bmatrix}, \end{aligned} \tag{36}$$

$$\begin{aligned} &D_2 = D_2^T > 0, \quad G_2 + G_2^T = 0, \\ &R_\Sigma < \begin{bmatrix} C_2 \\ J_2 \end{bmatrix}^T \begin{bmatrix} -D_2 & G_2 \\ G_2^T & D_2 \end{bmatrix} \begin{bmatrix} C_2 \\ J_2 \end{bmatrix}, \end{aligned} \tag{37}$$

where $R_\Sigma = R_\Sigma(P_\Sigma)$ as in (26)-(28).

Example 4 Let $A(\rho) = -(1+\epsilon)I_2 + \rho I_2$. Here $A_0 = -(1+\epsilon)I_2$ and $A_1 = I_2$. It is clear that if $\epsilon > 0$, $A(\rho)$ is Hurwitz for all $\rho \in [-1, 1]$ whereas if $\epsilon < 0$, $A(\rho)$ is not Hurwitz for all $\rho \in [-1, 1]$. Applying Theorem 4.1 with $n = 2$ and $m = \frac{1}{2}n(n+1) - 1 = 2$ one has

$$\begin{aligned} P(\rho) &= P_0 + \rho P_1 + \rho^2 P_2 \\ &= \left(\rho^{[2]} \otimes I_2\right)^T P_\Sigma \left(\rho^{[2]} \otimes I_2\right) \\ R(\rho) &= A^T(\rho)P(\rho) + P(\rho)A(\rho) \\ &= \left(\rho^{[3]} \otimes I_2\right)^T R_\Sigma \left(\rho^{[3]} \otimes I_2\right) \end{aligned}$$

where P_Σ is

$$\begin{bmatrix} P_0 & 0.5P_1 \\ 0.5P_1 & P_2 \end{bmatrix}$$

and R_Σ is

$$\begin{bmatrix} A_0^T P_0 + (*) & \star & \star \\ 0.5A_0^T P_1 + 0.5A_1^T P_0 + (*) & A_0^T P_2 + A_1^T P_1 + (*) & \star \\ 0 & 0.5A_1^T P_2 + (*) & 0 \end{bmatrix}.$$

Let $J_1 = [0_{2\times 2} \ I_2]$, $C_1 = [I_2 \ 0_{2\times 2}]$, $J_2 = [0_{4\times 2} \ I_4]$ and $C_2 = [I_4 \ 0_{4\times 2}]$ as in Theorem 4.1. Using the MATLAB$^{\text{TM}}$ LMI Toolbox [7] one can solve (36) and (37) with $\epsilon = 0.001$. On the other hand, for $\epsilon = -0.001$ no solution to the inequalities (36) and (37) exists. Theorem 4.1 thus gives the same results as the direct stability analysis.

Example 5 Let $A(\rho) = A_0 + \rho A_1$ where

$$A_0 = \begin{bmatrix} 1.1132 & 1.6802 & -1.8252 & -0.5279 \\ 1.2328 & -0.8224 & -0.3503 & -0.8995 \\ 2.8858 & 1.9407 & -3.1417 & -1.1186 \\ 1.5929 & 0.1522 & -0.4807 & -2.0469 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & -7.7372 & 0 & 0 \\ 7.7372 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the method of [16], one can show that the matrix $A(\rho)$ is Hurwitz if and only if $\rho \in (-0.9688, 0.5024)$.

In this example, $n = 4$, $r = \mathrm{rank}(A_1) = 2$ and $m = \frac{1}{2}(2nr - r^2 + r) = 7$. The parameter-dependent Lyapunov matrix $P(\rho) = \sum_{i=0}^{7} \rho^i P_i$ satisfies the matrix inequality $A(\rho)P(\rho) + P(\rho)A^T(\rho) < 0$ for all $\rho \in \mathbb{R}$ but it is positive-definite only when $\rho \in (-0.9688, 0.5024)$. For this special example, $P_7 = 0_{4\times 4}$ which shows that the upper bound of the degree of the polynomial Lyapunov matrix is not tight. On the other hand, the matrix inequalities (36) and (37), have no solution. This is expected, since $[-1, 1]$ is not a subset of $(-0.9688, 0.5024)$.

Let now $A(\rho) = A_0 + \rho A_1'$ and $A_1' = 0.5A_1$. The exact stability domain for this system is $(-1.9376, 1.0048)$. Applying the algorithm of Theorem 4.1, and using the MATLAB$^{\mathrm{TM}}$ LMI Toolbox [7], it can be verified that the inequalities (36) and (37) are indeed feasible. This result agrees with the direct analysis, since $[-1, 1] \subset (-1.9376, 1.0048)$ and thus $A_0 + \rho A_1'$ is Hurwitz for all $\rho \in [-1, 1]$.

Remark 3 Notice that when $A(\rho)$ is nominally stable, i.e., when the matrix A_0 is Hurwitz, the inequality (36) is not necessary. This is due to the fact that A_0 Hurwitz along with inequality (37) guarantees that $P(0) > 0$. Also, (37) ensures that $P(\rho) > 0$ for all $\rho \in [-1, 1]$; see [9]. Assuming therefore nominal stability, one can discard the inequality (36), thus reducing considerably the number of variables in the convex feasibility problem of Theorem 4.1.

V. Conclusions

In this paper we propose a class of parameter-dependent Lyapunov matrices $P(\rho)$ which can be used to test the stability of linear, time-invariant, parameter-dependent (LTIPD) systems of the form $\dot{x} = (A_0 + \rho A_1)x$ where $\rho \in \Omega$. The proposed Lyapunov matrix has polynomial dependence on the parameter ρ of a known degree and can be used to derive exact (i.e., necessary and sufficient) conditions for the stability of LTIPD systems. We show that checking these conditions over a compact interval can be cast as a convex programming problem in terms of linear matrix inequalities without conservatism. Finally, it should be pointed out that the results of [13] as well as of the Example 5 indicate that the degree of the polynomial dependence given in Theorem 3.1 is only an upper bound (not tight) and the question of the lowest degree polynomial Lyapunov matrix is still open.

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