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A Novel Approach to the Attitude Control of Axisymmetric Spacecraft*

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A novel approach is proposed for the attitude stabilization and reorientation of an axisymmetric spacecraft subject to two gas jet actuators. The approach is based on a new formulation of the attitude kinematics.

Key Words—Attitude control; stability; nonlinear control systems; feedback control.

Abstract—We consider the problem of attitude stabilization of an axisymmetric spacecraft using two pairs of gas jet actuators. These actuators provide control torques about two axes spanning the two-dimensional plane orthogonal to the axis of symmetry. Using a new kinematic formulation, we derive a stabilizing control law that achieves arbitrary reorientation of the spacecraft under the assumption that the initial spin rate about the symmetry axis is zero.

1. INTRODUCTION

The problem of attitude stabilization of a rotating rigid body has been the subject of active research (see e.g. Mortensen, 1968; Crouch, 1984; Salehi and Ryan, 1985; Wie *et al.*, 1989; Byrnes and Isidori, 1991a). Most of the existing results assume that three torques are available for control purposes, supplied either by gas jet actuators or by momentum exchange devices. In this case the complete stabilization/reorientation problem can be solved using linear (Mortensen, 1968; Wie *et al.*, 1989) or nonlinear controllers (Salehi and Ryan, 1985). On the other hand, the problem of attitude stabilization when less than three independent control torques are available has only recently been addressed (Krishnan *et al.*, 1992, 1994; Tsiotras and Longuski, 1993, 1994b; Walsh *et al.*, 1994). The case with less than three independent control torques is of interest from both theoretical and practical points of view. On the theoretical side, when less than three control torques are available, the linearized system is not stabilizable, so nonlinear control techniques need to be employed for the stabilization problem. Moreover, the nonlinear control is necessarily nonsmooth (see Byrnes and Isidori, 1991a). From a practical point of view, the case of less than three control torques is also of interest. Although most spacecraft are equipped with three control torques, the case with less than three control torques could correspond, say, to the situation when one or more of the actuators fail.

Mortensen (1968) used Euler parameters (quaternions) to describe the kinematics of the attitude motion and derived linear globally asymptotically stabilizing control laws. Salehi and Ryan (1985) derived positively homogeneous nonlinear feedback laws for the attitude stabilization problem; their results include those of Mortensen (1968) as a special case. Wie *et al.* (1989) presented a feedback regulator for eigenaxis rotational maneuvers. The control algorithm consists of linear feedback of error quaternions and body rates, and includes a decoupling control torque that counteracts the gyroscopic coupling. They also discuss the issue of robustness in the presence of initial body rate and inertia matrix uncertainty. Other standard references on the attitude stabilization problem include Meyer (1966) and, more recently, Wen and Kreutz-Delgado (1991). Again, we emphasize that all these references assume three control torques.

The first (and perhaps most complete) mathematical description of the attitude stabilization problem was given by Crouch (1984), who provided necessary and sufficient conditions for the controllability of a rigid body in the case of one, two and three independent control torques. These results can be summarized as follows. For three independent control torques the system is

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completely controllable, although in the case of momentum wheel actuators a certain minimum control effort is required. A necessary and sufficient condition for complete controllability of a symmetric rigid body with control torques, supplied by two pairs of gas jet actuators, about axes spanning a two-dimensional plane, is that the axis orthogonal to this plane must not be a principal axis of symmetry of the spacecraft. In the nonsymmetric case the system is controllable unless certain algebraic criteria hold; i.e. it is generically controllable. These criteria impose certain conditions on the relative magnitudes of the principal inertias, as in the case of stability considerations. For such a system, it is further shown that controllability is equivalent to local controllability at any equilibrium. When a spacecraft is controlled by less than three independent momentum wheel actuators, the system is not controllable, or even accessible at any equilibrium.

More recently, Byrnes and Isidori (1991a) established that a rigid spacecraft controlled by two pairs of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a continuously differentiable, i.e. smooth or $\mathscr{C}^1$, feedback control law. Hence Byrnes and Isidori settled in the negative a longstanding problem concerning the existence of a smooth (static or dynamic) state feedback law that locally asymptotically stabilizes a rigid spacecraft with two controls. It is interesting to note that it was well known for some time, that, relative to a fixed desired reference attitude, this system is locally reachable and locally asymptotically null-controllable. What Byrnes and Isidori have shown is that there is no way to realize such open-loop strategies via smooth feedback. However, a smooth $\mathscr{C}^1$ feedback control law was derived in the same reference, which locally asymptotically stabilizes the spacecraft to a circular attractor, rather than to an isolated equilibrium. In a physical system this circular attractor corresponds to a steady rotation about the axis with no control authority. In the same work they demonstrated a more general result for a class of nonlinear systems, including the rigid spacecraft model of interest, namely that such a system is locally asymptotically stabilizable precisely when it can be linearized via state feedback transformations.

The problem of attitude stabilization of a symmetric rigid spacecraft using only two control torques about axes spanning the two-dimensional plane orthogonal to the symmetry axis was considered by Krishnan et al. (1992, 1994) for both gas jet actuators and momentum wheel actuators. The complete dynamics fail to be controllable or even accessible in this case; thus the methodologies of Crouch (1984) and Byrnes and Isidori (1991a) are not applicable. However, the spacecraft dynamics are strongly accessible and small time locally controllable in a restricted sense, namely when the spin rate remains zero; however, any stabilizing control has to be necessarily nonsmooth. In Krishnan et al. (1992) such a nonsmooth control strategy was developed, which achieves arbitrary reorientation of the spacecraft, for the restricted case of zero spin rate. This nonsmooth control law is based on previous results on the stabilization of nonholonomic mechanical systems (Bloch and McClamroch, 1989).

In this paper we first present a new formulation of the attitude kinematics using the attitude coordinates developed by Tsiotras and Longuski (1993, 1994a, 1995). The kinematic equations in these coordinates have a very simple and compact form, which permits the efficient design of control laws for the attitude motion of a rotating rigid body. As an illustration of the potential of these attitude coordinates in feedback control design, we again consider the problem treated by Krishnan et al. (1992). That is, we seek to asymptotically stabilize via feedback the attitude motion (angular velocity and orientation) of an axisymmetric rigid body (e.g. a spacecraft) when only two body-fixed torques are available. Without loss of generality, we assume that the two control torques are along principal axes perpendicular to the symmetry axis. Note that here the term axisymmetric implies inertial symmetry and not geometric symmetry. A geometrically symmetric uniform rigid body is inertially symmetric, but one may have an inertially axisymmetric body that is not geometrically axisymmetric (e.g. a uniform orthogonal parallelepiped with two equal sides).

Using the new formulation of the kinematics, we first construct a manifold with the property that when the motion of the system is constrained to this manifold, all motions asymptotically approach zero. This manifold actually corresponds to the zero-dynamics manifold (Isidori and Moog, 1988) when one of the new attitude coordinates is taken as the system output. For the case when the spin rate is initially zero we first derive a feedback control law that drives the closed-loop system trajectories to this manifold. We then modify the control law such that, once on this manifold, the closed-loop trajectories go to the origin. The derived feedback control laws thus allow arbitrary reorientation of the spacecraft for the restricted case of zero initial spin rate. When the initial spin rate is not zero, the best one can expect is asymptotic stabilization to a uniform rotation about the symmetry axis. This corresponds to the circular attractor of Byrnes and Isidori (1991b). The results of this paper can be easily extended to handle this case as well.

The control law presented is very simple, and avoids the successive switchings of Krishnan *et al.* (1992, 1994). At the same time, in Krishnan *et al.* (1992) Eulerian angles are used to describe the kinematics of the motion, and, as a result, the control law is only valid within the set of applicability of this particular set of angles. On the other hand, the kinematic variables proposed in this paper do not possess this deficiency, since their domain of applicability is a dense subset of the configuration space—actually, the whole space except an isolated point. (The issue of singularities of the kinematic representations is briefly discussed in the next section.) Moreover, the control strategy presented in Krishnan *et al.* (1992) requires the transformation to a new set of coordinates that do not have any obvious physical interpretation. Thus, some of the insight into the problem is lost in the process. In contrast, the control law presented in this paper is derived in terms of coordinates which are amenable to physical interpretation. This is especially desirable if, in addition to stabilization, one requires some measure of performance from the control law. If this is the case then the performance measure has to be defined in terms of a set of variables or coordinates that have a physical or intuitive appeal, instead of some abstract algebraic quantities. On the other hand, the results in Krishnan *et al.* (1992) do not intend to address the spacecraft stabilization problem per se, but rather to illustrate a more general theory on stabilization of nonholonomic systems (Bloch and McClamroch, 1989); at the same time, they treat both cases of gas jet actuators and momentum wheels in the same framework.

Although in this paper we only consider a specific control problem of practical interest, namely spacecraft stabilization using only two control torques, the main purpose of this work is more general—to expose the new formulation of the kinematics and to illustrate how it can facilitate the design of stabilizing control laws for more general attitude control problems. Other results on stabilization of symmetric and non-symmetric spacecraft subject to two and three control torques using this kinematic formulation are reported by Tsiotras and Longuski (1993, 1994b) and Tsiotras (1994).

The structure of the paper is as follows. In the next section we review the dynamics (equations for the angular velocities) and the kinematics (equations for the orientation) for the rotational motion of a rigid body. After a brief discussion of the configuration space of rigid-body rotations, its structure group and its parameterizations, we introduce a set of kinematic coordinates and discuss their relationship with the Euler angles. Section 3 includes the main results of the paper. In particular, Lemma 3.1 presents a linear feedback control law (in terms of the new coordinates) that achieves asymptotic stabilization of the spin axis when the angular velocity is treated as the control input for the kinematics. In terms of the Eulerian angles, this implies that two of these (ϕ and θ) are zero, while the third angle (ψ) is, in general, nonzero. Using this result, we then construct an integral manifold in terms of the Eulerian angles, which in essence presents a measure of mismatch between the final orientation using the previous control law and the desired (zero) orientation. Theorem 3.1 presents a feedback control law (in terms of ω) that guarantees that this mismatch is zero, i.e. it drives all the trajectories to the manifold, and then to the origin. This control law is nonsmooth (in fact, singular) at the origin; we can choose, however, the gains of the control law in such a way that this nonsmoothness does not create any problems in practice. Finally, Theorem 3.2 implements the control law of Theorem 3.1 through the dynamics, which for this problem turns out to be a simple integrator. In the last section we extend these results, and we show that, even for the general case of a nonsymmetric body or nonzero spin rate, the new formulation is attractive, facilitating the design of stabilizing feedback control laws.

2. EQUATIONS OF MOTION

2.1. *Dynamics*

The equations describing the rotational motion of a rigid body are Euler's equations of motion. If we choose the axes of the body-fixed reference frame along the principal axes of inertia of the rigid body with origin at the center of mass, Euler's equations of motion take the simplified form

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\,\omega_2\omega_3 + \frac{M_1}{I_1}, \tag{1a}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\,\omega_3\omega_1 + \frac{M_2}{I_2}, \tag{1b}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\,\omega_1\omega_2 + \frac{M_3}{I_3}, \tag{1c}$$

where ω_1, ω_2, ω_3 denote the components of the body angular velocity vector with respect to the body principal axes, M_1, M_2, M_3 are the external torques, and the positive scalars I_1, I_2, I_3 are the principal moments of inertia of the body with respect to its center of mass. For an axially symmetric body ($I_1 = I_2$) with no external torques about the symmetry axis (here corresponding to I_3), the equations can be written as

$$\dot{\omega}_1 = a\omega_2\omega_3 + u_1, \tag{2a}$$

$$\dot{\omega}_2 = -a\omega_3\omega_1 + u_2, \tag{2b}$$

$$\dot{\omega}_3 = 0, \tag{2c}$$

where $a \triangleq (I_2 - I_3)/I_1$ is an inertia parameter and $u_1 \triangleq M_1/I_1$, $u_2 \triangleq M_2/I_2$ are the control inputs.

It should be clear from (2c) that, for a symmetric body, the component ω_3 of the angular velocity along the symmetry axis cannot be affected by the control inputs. In fact, ω_3 remains constant for all time. Hence, as already mentioned, the system (2) is not controllable.

Introducing the complex variables

$$\omega \triangleq \omega_1 + i\omega_2, \quad u \triangleq u_1 + iu_2, \tag{3}$$

one can rewrite (2a,b) in the compact form

$$\dot{\omega} = -ia\omega_{30}\omega + u \tag{4}$$

where $\omega_{30} \triangleq \omega_3(0)$.

Equations (2) or (4) describe the rotational dynamics of an axisymmetric rigid body. A complete characterization of the attitude motion also requires a description of the kinematics. In contrast to the dynamics formalism, there is more than one way to describe the rotational kinematics of a rigid body.

2.2. *Kinematics*

In principle, the orientation of a rigid body is described by the matrix relating a body-fixed reference frame and an inertial reference frame (Kane *et al.*, 1983). The set of all such matrices form what is commonly known as the (three-dimensional) rotation group; it is denoted by SO(3) and consists of all matrices that are orthogonal and have determinant 1. That is, SO(3) is the subgroup of all invertible 3×3 matrices, defined by

$$SO(3) = \{R \in GL(3, \mathbb{R}) : RR^t = I, \det R = 1\},$$

where $GL(n, \mathbb{R})$ is the general linear group of all $n \times n$ invertible matrices with real entries. Henceforth we shall refer to SO(3) simply as the rotation group. In fact, SO(3) carries an inherent smooth manifold structure, and thus forms a (continuous and compact) Lie group (Varadarajian, 1984).

The attitude history of a body-fixed reference frame with respect to an inertial reference frame can therefore be described by a curve traced by the corresponding rotation $R(t) \in SO(3)$ for $t \geq 0$. The differential equation satisfied by $R(t)$ while it is moving along this trajectory is given by the system of equations

$$\frac{dR}{dt} = S(\omega_1, \omega_2, \omega_3)R, \tag{5}$$

where $S(\omega_1, \omega_2, \omega_3)$ is the skew-symmetric matrix

$$S(\omega_1, \omega_2, \omega_3) \triangleq \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

and where explicit dependence on time has been suppressed for notational simplicity.

Because of the six constraints associated with the orthogonality of the rotation matrix, there is more than one way to parameterize the rotation group, i.e. to specify a set of parameters such that every element $R \in SO(3)$ is uniquely and unambiguously determined (Stuelpnagel, 1964). Loosely speaking, each parameterization of the rotation group corresponds to a choice of (local) coordinates for the manifold SO(3). Commonly used parameterizations of SO(3) include the three-dimensional parameterization by Eulerian angles and the four-dimensional parameterization by quaternions (Kane *et al.*, 1983). In this section we initially use Eulerian angles to describe the kinematics because of their physical significance, and also for comparison with the results of Krishnan *et al.* (1992). Then we introduce an alternative formulation of the kinematics that facilitates the design of feedback control laws for the attitude problem, and we show how such a formulation can be derived by stereographically projecting one of the columns of R onto the complex plane. As such, the new kinematic formulation does not depend on the particular parameterization of the rotation group, since it is directly derived from the differential equation for the rotation matrix R. In order to elucidate the motivation and the advantages behind the derivation of the new kinematic formulation, we develop in parallel an Eulerian angle description for the kinematics.

Using a 3–2–1 Eulerian angle sequence (ψ, θ, ϕ) for the description of the orientation (see Fig. 1), one has the associated kinematic equations

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta, \tag{6a}$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi, \tag{6b}$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta. \tag{6c}$$

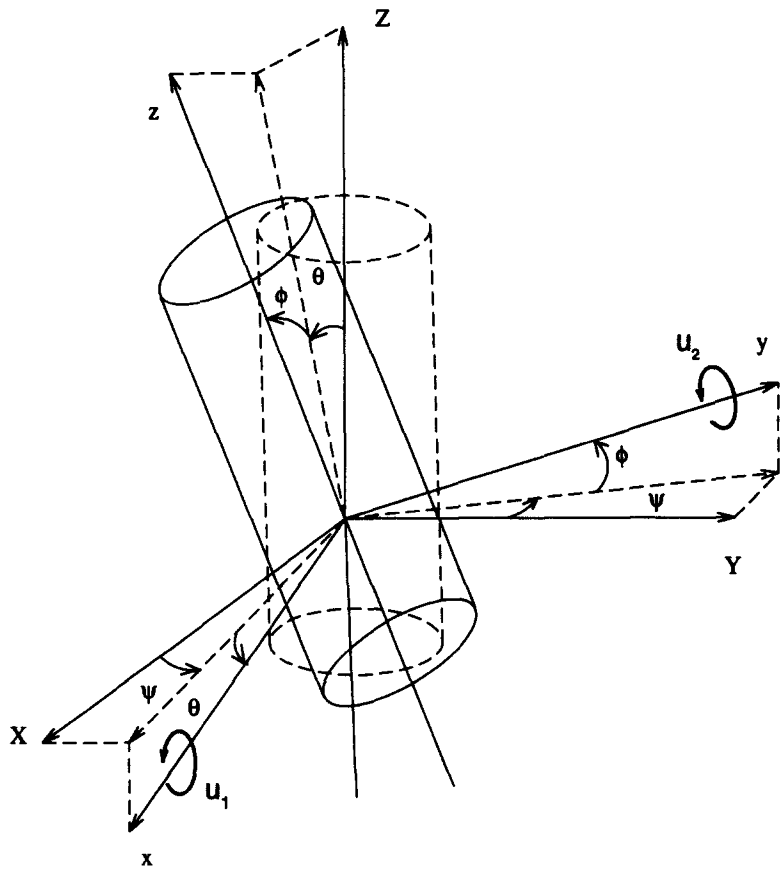

Fig. 1. Eulerian angle sequence 3–2–1.

The Eulerian angles provide a local coordinate system for the rotation group SO(3). Equations (6) exhibit a singularity at $\theta = \pm\frac{1}{2}\pi$. We therefore restrict the subsequent discussion to the set $\mathcal{M}$ defined by

$$\mathcal{M} = \{(\phi, \theta, \psi) \in S^3 : \phi, \psi \in (-\pi, \pi], \theta \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)\},$$

where $S^3 = S^1 \times S^1 \times S^1$ and S^1 is the usual mathematical notation for the unit circle. Using this parameterization of SO(3), the orientation of the body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its 3 axis (or z axis) through an angle ψ, then rotating about its 2 axis (or y axis) by an angle θ and finally rotating about its 1 axis (or x axis) by an angle ϕ (see Fig. 1). Thus ϕ and θ determine the orientation of the local body-fixed 3 axis (the symmetry axis) with respect to the inertial 3 axis (Z axis), and ψ determines (loosely speaking) the relative rotation about this axis.

Every choice of Eulerian angles has the disadvantage of singularity in the associated kinematic equations, and this is also true for every other three-dimensional parameterization of the rotation group. If one requires a nonsingular description of the kinematics, one has to necessarily increase the order of the parameterization, using for example the four-dimensional parameterization of quaternions. This four-dimensional parameterization is not however, 1–1, since SU(2), the unitary group of complex 2×2 matrices with unit determinant (where quaternions naturally live, $S^3 \simeq \mathrm{SU}(2)$) gives a double (universal) covering of SO(3), thus providing a 2–1 way of parameterizing the rotation group. A 1–1 universal parameterization is possible only if we increase the order of the parameterization to five; however, this is rarely done, and in most cases the four-dimensional quaternion representation is enough for a nonsingular description of the kinematics, since this 2–1 correspondence between the quaternions and the elements of SO(3) is a local homeomorphism (Stuelpnagel, 1964).

Tsiotras and Longuski (1994a, 1995) presented an alternative formulation of the kinematics of the rotational motion of a rigid body that simplifies (6). This formulation has been used by Longuski and Tsiotras (1993) to derive analytic solutions for the attitude motion of a rigid body subject to large angular displacements. In addition, this kinematic formulation has proved to be extremely useful in control problems (Tsiotras and Longuski, 1993, 1994b; Tsiotras, 1994).

2.3. *New coordinates*

Introducing the complex kinematic variable, $w = w_1 + \mathrm{i}w_2$, defined by

$$w \triangleq \frac{\sin\phi\cos\theta + \mathrm{i}\sin\theta}{1+\cos\phi\cos\theta}, \tag{7}$$

one can readily show that w satisfies the following complex differential equation:

$$\dot{w} + \mathrm{i}\omega_3 w = \tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega}w^2 \tag{8}$$

where ω is defined in (3) and the over bar denotes complex conjugate. This is a scalar Riccati equation with time-varying coefficients. Equation (7) is derived using stereographic projection of the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ in $\mathbb{R}^3$ onto the complex plane. In fact, (7) represents the stereographic projection of the third column of the rotation matrix (the one that is independent of the cyclic coordinate ψ for this choice of Eulerian angles) onto the complex plane $\mathbb{C}$. An equation similar to (8) first appeared in Darboux (1887) in connection with some problems in classical differential geometry. More recently, Walsh *et al.* (1994) used a kinematic formulation that closely resembles the one in this paper in order to solve the (more difficult) problem of attitude stabilization of a nonsymmetric spacecraft using only two control torques; their results, however, are only local.

From (7), the real and imaginary parts of w are given by

$$w_1 = \frac{\sin\phi\cos\theta}{1+\cos\phi\cos\theta}, \quad w_2 = \frac{\sin\theta}{1+\cos\phi\cos\theta} \tag{9}$$

and they satisfy the differential equations

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + \tfrac{1}{2}\omega_1(1 + w_1^2 - w_2^2),$$

$$\dot{w}_2 = -\omega_3 w_1 + \omega_1 w_1 w_2 + \tfrac{1}{2}\omega_2(1 + w_2^2 - w_1^2).$$

The transformation (7) is not restricted to the particular Eulerian angle set used here, or any other parameterization of the rotation group. In fact, since (8) can be derived directly from (5) using the method of stereographic projection of the unit sphere S^2 onto the complex plane, it is independent of the particular parameterization of SO(3). Note also that the differential equation for ψ in the (ω, w) space is given by

$$\dot{\psi} = \tfrac{1}{2}\mathrm{i}(\omega - \bar{\omega})\frac{(w+\bar{w})(1+|w|^2)}{(1+w^2)(1+\bar{w}^2)}, \tag{10}$$

where $|\cdot|$ denotes the magnitude of a complex number, i.e. $|w|^2 = \bar{w}w$. The variables (w_1, w_2, ψ) can therefore serve as a local coordinate description of SO(3). In the next section we introduce another variable z, such that (w_1, w_2, z) are local coordinates of SO(3), locally diffeomorphic to (ϕ, θ, ψ), in such a way that z will be the 'natural' complement of w in the description of the kinematics.

Note that, by construction, $w_1 = w_2 = 0$ implies that the body 3 axis is aligned with the inertial 3 axis, i.e. it implies that $\phi = \theta = 0$. Therefore stabilization about $w = 0$ in the complex plane corresponds to stabilization of the (symmetry) 3 axis. In Tsiotras and Longuski (1993, 1994b) globally asymptotically stabilizing control laws for this problem, namely for the reduced system of equations (4) and (8) (equivalently, (4) and (6a,b)) were derived, for both the cases of zero and nonzero spin rate ω_3. Consideration of this reduced system is possible because ψ is an ignorable (cyclic) variable for the system of equations (6), and thus does not affect the solutions of these equations. For this reduced system, stabilization about the origin in the $(\omega_1, \omega_2, \phi, \theta)$ state space corresponds to stabilization of the symmetry axis, with the body orientation about this axis being indeterminate. In the extended $(\omega_1, \omega_2, \phi, \theta, \psi)$ state space this implies stabilization about the one-dimensional manifold

$$\mathcal{N} = \{(\omega_1, \omega_2, \phi, \theta, \psi) \in \mathbb{R}^2 \times \mathcal{M} : \omega_1 = \omega_2 = \phi = \theta = 0\}, \tag{11}$$

rather than an isolated equilibrium. Feedback stabilization about a reduced equilibrium manifold has received attention recently, since it appears to be an important extension of stabilization about an equilibrium yielding bounded trajectories (Byrnes and Isidori, 1991a).

Consider now the case of the complete system of equations (4) and (6) assuming that $\omega_3(0) = 0$. As already mentioned, in this case the equations are strongly accessible and small-time locally controllable at any equilibrium. Thus arbitrary reorientation of the spacecraft can be achieved if $\omega_3(0) = 0$; if $\omega_3(0) \neq 0$, reorientation of the spacecraft is not possible. Of course, smooth stabilization about the one-dimensional manifold $\mathcal{N}$ is always possible, regardless of the value of $\omega_3(0)$.

3. FEEDBACK CONTROL STRATEGY

3.1. *Stabilization of the kinematics*

Assuming a priori that $\omega_{30} = 0$, the system to be driven to the origin takes the form

$$\dot{\omega}_1 = u_1, \tag{12a}$$

$$\dot{\omega}_2 = u_2, \tag{12b}$$

with kinematics

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi \tan\theta, \qquad (13a)$$

$$\dot{\theta} = \omega_2 \cos\phi, \qquad (13b)$$

$$\dot{\psi} = \omega_2 \sin\phi \sec\theta, \qquad (13c)$$

or, in the complex notation introduced previously,

$$\dot{\omega} = u, \qquad (14a)$$

$$\dot{w} = \tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega}w^2 \qquad (14b)$$

$$\dot{\psi} = \tfrac{1}{2}\mathrm{i}(\omega - \bar{\omega})\frac{(w+\bar{w})(1+|w|^2)}{(1+w^2)(1+\bar{w}^2)}. \qquad (14c)$$

These equations have the form of a cascade system, so we concentrate first on the problem of stabilization of the subsystem (14b,c), regarding ω as the control input, and then we implement this control law through the integrator (14a).

Stabilization of systems in cascade form has received attention recently as an effective way of stabilizing nonlinear systems (Coron and Praly, 1991; Bacciotti, 1992; Kanellakopoulos *et al.* 1992). However, we cannot use these results here, since our system is not smoothly stabilizable. Instead, we have to derive the stabilizing controller for the system (14) directly. We start with the following result concerning the subsystem (14b) (Tsiotras and Longuski, 1993).

Lemma 3.1. The feedback control law

$$\omega = -\kappa w \qquad (15)$$

($\kappa > 0$) globally exponentially stabilizes the subsystem (14b) with rate of decay $\tfrac{1}{2}\kappa$.

With the control law (15), the closed-loop subsystem corresponding to (14b) takes the form

$$\dot{w} = -\tfrac{1}{2}\kappa(1+|w|^2)w. \qquad (16)$$

In the (ϕ, θ) variables this control law can be written as

$$\omega_1 = -\kappa\frac{\sin\phi\cos\theta}{1+\cos\phi\cos\theta}, \qquad \omega_2 = -\kappa\frac{\sin\theta}{1+\cos\phi\cos\theta}, \qquad (17)$$

and the corresponding closed-loop system in the (ϕ, θ, ψ) variables takes the form

$$\dot{\phi} = -\kappa\frac{\sin\phi}{\cos\theta(1+\cos\phi\cos\theta)}, \qquad (18a)$$

$$\dot{\theta} = -\kappa\frac{\sin\theta\cos\phi}{1+\cos\phi\cos\theta}, \qquad (18b)$$

$$\dot{\psi} = -\kappa\frac{\sin\phi\tan\theta}{1+\cos\phi\cos\theta}. \qquad (18c)$$

As $t \to \infty$, the Eulerian angles ϕ and θ go to zero, but ψ tends to some unspecified value ψ_∞, which is, in general, nonzero. If we could calculate this value ψ_∞, we might be able to devise a feedback control law that would impose $\psi_\infty = 0$, and the problem would be solved. It turns out that such a strategy is possible, and is based on the rather surprising fact that one is able to analytically integrate the system of nonlinear differential equations (18) in closed form. Given any initial angles $\phi_0 = \phi(0)$, $\theta_0 = \theta(0)$ and $\psi_0 = \psi(0)$, we can therefore calculate the final value of ψ. In addition, by considering the initial angles that result in zero for the final value of ψ, we construct an invariant manifold that can be used to derive stabilizing control laws for the complete kinematics (13).

We now proceed to the derivation of this manifold. Eliminating time from (18), we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = \sin\theta, \quad \frac{\mathrm{d}\theta}{\mathrm{d}\phi} = \frac{\sin\theta\cos\theta}{\tan\phi}. \qquad (19)$$

Integrating the last equation yields

$$\ln\left(\frac{\sin\phi}{\sin\phi_0}\right) = \ln\left(\frac{\tan\theta}{\tan\theta_0}\right),$$

or

$$\tan\theta = a_0 \sin\phi,$$

where $a_0 \triangleq \tan\theta_0/\sin\phi_0$. From this equation, along with the first of (19), we have

$$\frac{\mathrm{d}\psi}{\mathrm{d}\phi} = \frac{a_0\sin\phi}{\sqrt{1+a_0^2\sin^2\phi}}.$$

Integrating this equation yields

$$\psi + \arcsin(p_0\cos\phi) = \psi_0 + \arcsin(p_0\cos\phi_0),$$

where $p_0 \triangleq a_0/\sqrt{1+a_0^2}$. If the state of the system (18) asymptotically converges to the origin, we must have

$$\psi_0 + \arcsin(p_0\cos\phi_0) - \arcsin p_0 = 0. \qquad (20)$$

For the case when $\phi_0 = 0$ (and $-\tfrac{1}{2}\pi < \theta_0 < \tfrac{1}{2}\pi$) it can be easily shown that the previous equation simplifies to the statement that $\psi_0 = 0$. The choice of ϕ as the new independent variable is justified by the fact that it decreases monotonically to zero. Indeed, from (18a) and (7), one can write the differential equation for ϕ as

$$\dot{\phi} = -\kappa w_1/\cos^2\theta.$$

Now $\theta \in (-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$, and if $\dot{\phi} = 0$ for some time

$t^* < \infty$, we have $w_1(t^*) = 0$. However, this is not possible, because from (16) the magnitude of w_1 decreases monotonically (exponentially) to zero. The monotonicity of $\phi(t)$ is also evident from the phase portrait of the system of equations (18a,b), which is depicted in Fig. 2.

Equation (20) yields an expression for the initial conditions from which the feedback law $\omega = -\kappa w$ will drive the system (18) to the origin. We now introduce the manifold

$$\mathcal{S} \triangleq \{(\phi, \theta, \psi) \in \mathcal{M}: z(\phi, \theta, \psi) = 0\},$$

where the function $z: \mathcal{M} \to \mathbb{R}$ is defined by

$$z(\phi, \theta, \psi) \triangleq \psi + \arcsin(p \cos\phi) - \arcsin p, \tag{21a}$$

$$p = \frac{a}{\sqrt{1 + a^2}}, \quad a = \frac{\tan\theta}{\sin\phi}. \tag{21b}$$

Clearly, $\mathcal{S}$ is an invariant manifold for the system (18). Moreover, by construction, every trajectory of $\mathcal{S}$ satisfies $\lim_{t\to\infty}(\phi(t), \theta(t), \psi(t)) = 0$. It is therefore advantageous to consider the utility of this manifold in achieving stabilization to the origin $\phi = \theta = \psi = 0$. Note that $\mathcal{S}$ is independent of the control gain κ; therefore, once on this manifold, any positive κ will lead to the origin. The manifold $\mathcal{S}$ is shown in Fig. 3.

The derivative of z along trajectories of (14) can be computed to be

$$\dot{z} = -w_2\omega_1 + w_1\omega_2 = \operatorname{Im}(\omega\bar{w}). \tag{22}$$

As expected, the choice of $\omega = -\kappa w$ maintains $\dot{z} \equiv 0$, and a trajectory reaching $\mathcal{S}$ remains there.

In order to render $\mathcal{S}$ an attracting manifold, we restrict consideration to $w \neq 0$ and propose the following control law

$$\omega = -\kappa w - \mathrm{i}\frac{\mu}{\bar{w}} z \tag{23}$$

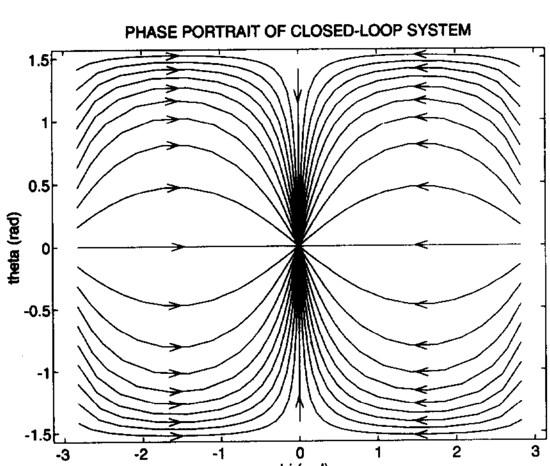

Fig. 2. Phase portrait of reduced system.

with $\mu > 0$. With this control law, the closed-loop system becomes

$$\dot{w} = -\tfrac{1}{2}\kappa(1 + |w|^2)w - \tfrac{1}{2}\mathrm{i}\mu z\left(\frac{1}{\bar{w}} - w\right), \tag{24a}$$

$$\dot{z} = -\mu z, \tag{24b}$$

with $(w, z) \in (\mathbb{C}\backslash\{0\}) \times \mathbb{R}$. Moreover, $\omega + \kappa w = 0$ for $z = 0$. Therefore, as long as $w \neq 0$, the control law (23) drives all trajectories of the subsystem (24) to the origin, for arbitrary initial conditions. Care should be taken in implementing (23), because the control law is not defined at points where $w = 0$. We shall return to this point shortly. We now have the following result regarding the system (24).

Theorem 3.1. Consider the closed-loop system (24) with

$$\mu > \tfrac{1}{2}\kappa$$

and consider any initial condition $(w(0), z(0)) \in \mathbb{C} \times \mathbb{R}$ with $w(0) \neq 0$. Then the following hold:

(i) $w(t) \neq 0$ for all $t \geq 0$;
(ii) the trajectory $(w(\cdot), z(\cdot))$ is bounded and

$$\lim_{t\to\infty}(w(t), z(t)) = 0;$$

(iii) the control history $\omega(\cdot)$ is bounded and has bounded derivative.

Proof. We first show that if $w(0) \neq 0$ then $w(t) \neq 0$ for all $t \geq 0$.

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = 2\operatorname{Re}(\dot{w}\bar{w}),$$

one readily obtains that $|w|^2$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = -\kappa|w|^2(1 + |w|^2). \tag{25}$$

Using the transformation $v \triangleq 1/|w|^2$, one can integrate (25) to obtain

$$|w(t)| = \left(\frac{1}{c_0 \mathrm{e}^{\kappa t} - 1}\right)^{1/2},$$

where $c_0 \triangleq (|w(0)|^2 + 1)/|w(0)|^2$. Clearly, $w(t) \neq 0$ for all $t \geq 0$ and $\lim_{t\to\infty} w(t) = 0$. In fact, the magnitude of $w(t)$ is bounded between two exponential functions as follows:

$$|w(0)|\,\mathrm{e}^{-\kappa t/2} \geq |w(t)| > c_0^{-1/2}\mathrm{e}^{-\kappa t/2} \quad \forall t \geq 0. \tag{26}$$

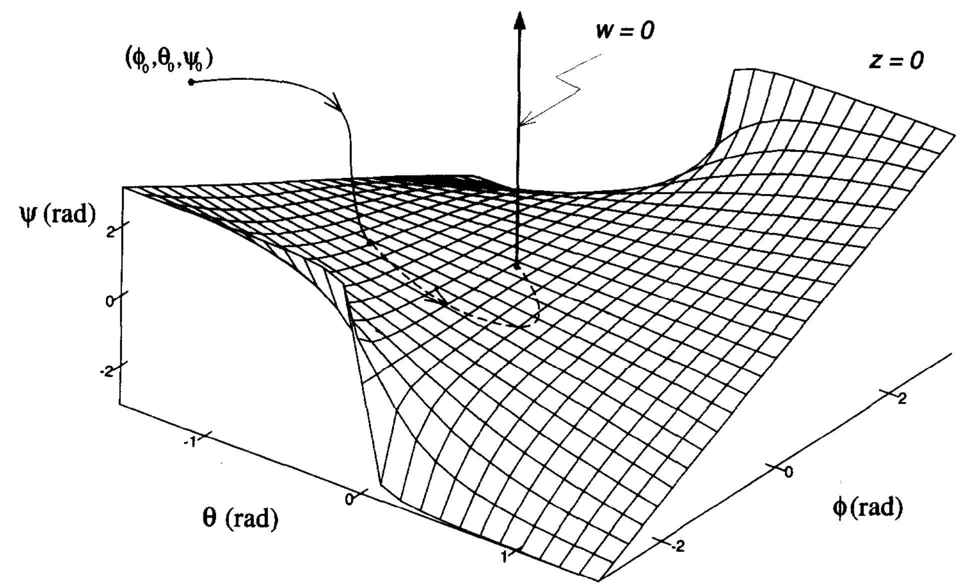

Fig. 3. The two-dimensional manifold $\mathcal{S}$.

From (24b), it should be clear that

$$z(t) = z(0)e^{-\mu t}.$$

Hence $z(\cdot)$ is bounded and $\lim_{t\to\infty} z(t) = 0$.

We now show that $\omega(\cdot)$ and $\dot{\omega}(\cdot)$ are bounded. Let $v \triangleq z/\bar{w}$. Since $c_0 > 1$, one has

$$|v(t)| = \frac{|z(t)|}{|w(t)|} \le |z(0)|\, e^{-\mu t}(c_0 e^{\kappa t} - 1)^{1/2}$$
$$< |z(0)|\, c_0^{1/2} e^{-(\mu - \kappa/2)t} \quad \forall t \ge 0.$$

Since $\lambda \triangleq \mu - \frac{1}{2}\kappa > 0$, v decays exponentially to zero with rate at least equal to λ. Thus $\lim_{t\to\infty} v(t) = 0$, and v is bounded by $|v(t)| < |z(0)|\, c_0^{1/2} \triangleq \beta_1$ for all $t \ge 0$. Hence ω is bounded as

$$|\omega(t)| < \kappa\, |w(0)| + \mu\, |z(0)|\, c_0^{1/2} \triangleq \beta_2 \quad \forall t \ge 0.$$

Direct calculation shows that the derivative $\dot{v}$ is bounded as

$$|\dot{v}(t)| < (\mu + \beta_4)\, |z(0)|\, c_0^{1/2} \triangleq \beta_3 \quad \forall t \ge 0,$$

where $\beta_4 \triangleq \frac{1}{2}\kappa(1 + |w(0)|^2) + \frac{1}{2}\mu(|z(0)| + \beta_1 c_0^{1/2})$, and the derivative of w is bounded as

$$|\dot{w}(t)| \le \tfrac{1}{2}\kappa(1 + |w(0)|^2)\, |w(0)|$$
$$+ \tfrac{1}{2}\mu(\beta_1 + |z(0)|\, |w(0)|) \triangleq \beta_5 \quad \forall t \ge 0.$$

Therefore the derivative of ω is bounded as $|\dot{\omega}(t)| < \kappa\beta_5 + \mu\beta_3$ for all $t \ge 0$. This completes the proof. □

Note that $|w|$ and z decay exponentially fast to zero for all values of $\kappa > 0$ and $\mu > 0$. The condition $\mu > \frac{1}{2}\kappa$ is required in order to ensure that the control ω and its derivative are bounded. We therefore have $\lim_{t\to\infty}(w(t), z(t)) = 0$ which implies that $\lim_{t\to\infty}(\phi(t), \theta(t), \psi(t)) = 0$. If, on the other hand, $w(0) = 0$, the control law has to be slightly modified, as will be shown later.

3.2. *Stabilization of the complete system*

This section contains the main results of the paper. Let $\mathcal{X}$ be the open set $\mathbb{C} \times (\mathbb{C}\backslash\{0\}) \times \mathbb{R}$. Given any compact subset $\mathcal{W} \subset \mathcal{X}$, we present a controller that generates u for the full system (14) and that has the property that, for any initial condition in $\mathcal{W}$, the resulting trajectory converges asymptotically to zero.

The proposed controller is given by

$$u = u_s = -\tfrac{1}{2}\kappa(\omega + \bar{\omega}w^2) - i\mu g(\omega, w, z)$$
$$- \alpha\left(\omega + \kappa w + i\frac{\mu}{\bar{w}} z\right), \quad (27)$$

where

$$g(\omega, w, z) \triangleq \frac{\operatorname{Im}(\omega\bar{w})}{\bar{w}} - \frac{z}{2\bar{w}^2}(\bar{\omega} + \omega\bar{w}^2) \quad (28)$$

and the scalars α, κ and μ are chosen to satisfy

$$\kappa > 0, \quad \mu > \kappa, \quad \alpha > \tfrac{1}{2}(\kappa + \beta), \quad (29)$$

with β satisfying

$$\left|\omega + \kappa w + i\mu\frac{z}{\bar{w}}\right| \frac{(1 + |w|^2)^{1/2}}{|w|} \le \beta \;\; \forall(\omega, w, z) \in \mathcal{W}. \quad (30)$$

The main idea behind the proposed control law is to approximately implement the control law (23) through the integrator (14a), by choosing the gain α 'large enough'. Indeed, introducing the variable

$$s \triangleq \omega + \kappa w + i\mu \frac{z}{\bar{w}}, \tag{31}$$

we can rewrite the closed-loop system in the form

$$\dot{s} = -\alpha s, \tag{32a}$$

$$\dot{w} = -\tfrac{1}{2}\kappa(1+|w|^2)w - \tfrac{1}{2}i\mu z\left(\frac{1}{\bar{w}} - w\right) + \tfrac{1}{2}s + \tfrac{1}{2}\bar{s}w^2, \tag{32b}$$

$$\dot{z} = -\mu z + \mathrm{Im}\,(s\bar{w}). \tag{32c}$$

Note that for large α the s subsystem can be considered as a boundary-layer system for (32). The outer layer, corresponding to $s = 0$, is in fact the system (24), which, as shown in Theorem 3.1, is globally exponentially stable for $\mu > \frac{1}{2}\kappa$ in $(\mathbb{C}\backslash\{0\}) \times \mathbb{R}$. Therefore, for large α and since the boundary-layer subsystem (32a) is globally exponentially stable, one expects the overall system to behave like the system of Theorem 3.1. This statement is made precise in the following theorem.

Theorem 3.2. Consider the closed-loop system (32) with α, μ and κ satisfying (29) and consider any initial condition $(\omega(0), w(0), z(0)) \in \mathcal{W}$. Then the following hold:

(i) $w(t) \neq 0$ for all $t \geq 0$; hence the control law (27)–(30) is well defined for all $t \geq 0$;
(ii) the trajectory $(s(\cdot), w(\cdot), z(\cdot))$ is bounded and
$$\lim_{t\to\infty} (s(t), w(t), z(t)) = 0;$$
(iii) the control history $u(\cdot)$ is bounded and satisfies $\lim_{t\to\infty} u(t) = 0$.

Proof. We first show that if $w(0) \neq 0$ then $w(t) \neq 0$ for all $t \geq 0$. From (32b), the magnitude of w obeys the equation

$$\frac{d}{dt}|w|^2 = -(1+|w|^2)[\kappa|w|^2 - \mathrm{Re}\,(s\bar{w})]. \tag{33}$$

From the definition (30) of β and the condition (29) on α, we have

$$|s(t)| = |s(0)|\,e^{-\alpha t} \leq \beta\left[\frac{|w(0)|^2}{|w(0)|^2+1}\right]^{1/2} e^{-(\kappa+\beta)t/2} \quad \forall t \geq 0.$$

Recalling (25) and (26), the last inequality implies that $|s(t)| \leq \beta\,|\tilde{w}(t)|$ for all $t \geq 0$, where $\tilde{w}$ satisfies the differential equation

$$\frac{d}{dt}|\tilde{w}|^2 = -(\kappa+\beta)(1+|\tilde{w}|^2)\,|\tilde{w}|^2, \quad \tilde{w}(0) = w(0). \tag{34}$$

Since $|\mathrm{Re}\,(s\bar{w})| \leq |s|\,|w|$ and $|s| \leq \beta\,|\tilde{w}|$, it follows from (33) that

$$\frac{d}{dt}|w|^2 \geq -(1+|w|^2)(\kappa\,|w|^2 + \beta\,|w|\,|\tilde{w}|). \tag{35}$$

From (34) and (35), $|w(t)| \leq |\tilde{w}(t)|$ implies that

$$\frac{d}{dt}|w(t)|^2 \geq \frac{d}{dt}|\tilde{w}(t)|^2.$$

Since $|w(0)| = |\tilde{w}(0)|$, it follows that $|w(t)| \geq |\tilde{w}(t)|$ for all $t \geq 0$. Now, according to (34), $\tilde{w}(t) \neq 0$ for $t \geq 0$. Thus $w(t) \neq 0$ for $t \geq 0$.

Next we show that $\lim_{t\to\infty} w(t) = 0$. Since $|w(t)| \geq |\tilde{w}(t)|$ for all $t \geq 0$, where $\tilde{w}$ decays exponentially with rate $\frac{1}{2}(\kappa+\beta)$, we have

$$\frac{|s(t)|}{|w(t)|} \leq \frac{|s(t)|}{|\tilde{w}(t)|} < \beta e^{-[\alpha-(\kappa+\beta)/2]t} \quad \forall t \geq 0. \tag{36}$$

Since $\alpha > \frac{1}{2}(\kappa+\beta)$, this term is exponentially decreasing. Moreover, since $\mathrm{Re}\,(s\bar{w}) \leq |s|\,|w|$, one obtains from (33)

$$\frac{d}{dt}|w|^2 \leq -(1+|w|^2)\,|w|^2\left(\kappa - \frac{|s|}{|w|}\right). \tag{37}$$

From (36), $\lim_{t\to\infty}(|s(t)|/|w(t)|) = 0$; hence one can readily show that $\lim_{t\to\infty} w(t) = 0$. Since w is continuous, it must also be bounded.

We now show that the variable $\eta = z/|w|^2$ is bounded and converges asymptotically to zero. The evolution of η is governed by

$$\dot{\eta} = -(\mu-\kappa)\eta + \kappa\,|w|^2\,\eta - (1+|w|^2)\,\mathrm{Re}\left(\frac{s}{w}\right)\eta + \mathrm{Im}\left(\frac{s}{w}\right). \tag{38}$$

Since $\lim_{t\to\infty} w(t) = 0$, $\lim_{t\to\infty} s(t)/w(t) = 0$ and $\mu - \kappa > 0$, one can now readily show that $\eta(\cdot)$ is bounded and $\lim_{t\to\infty} \eta(t) = 0$.

It now follows that $z(\cdot)$ is bounded and converges asymptotically to zero. We have therefore shown that the solutions of (32) are bounded, and that $\lim_{t\to\infty}(s(t), w(t), z(t)) = 0$.

Since $\lim_{t\to\infty} \eta(t) = 0$, one also has from (31) that $\lim_{t\to\infty} \omega(t) = 0$. It is easy to check that g in (27) is bounded and tends to zero as $t \to \infty$.

Therefore u is bounded and $\lim_{t\to\infty} u(t) = 0$, as claimed. This completes the proof. □

Corollary 3.1. Under the hypotheses of Theorem 3.2, we have

$$\lim_{t\to\infty} (\omega_1(t), \omega_2(t), \phi(t), \theta(t), \psi(t)) = 0.$$

So far, we have demonstrated that for initial conditions with $w(0) \neq 0$, it is possible to construct a control law that drives the system (12), (13) to the origin, with (ϕ, θ, ψ) avoiding the one-dimensional manifold

$$\mathcal{N}' \triangleq \{(\phi, \theta, \psi) \in \mathcal{M}: \phi = \theta = 0, \psi \neq 0\}.$$

The previous methodology cannot be used if the initial condition is such that $\phi(0) = \theta(0) = 0$ and $\psi(0) \neq 0$ (i.e. $w(0) = 0$ and $z(0) \neq 0$). Linearization of the system (14a,b) about $w = 0$, however, results in

$$\dot{\omega} = u, \tag{39a}$$

$$\dot{w} = \tfrac{1}{2}\omega. \tag{39b}$$

This linearized system is completely controllable, and by choosing, for example, a constant control $u = u_c \in \mathbb{C}$, one can move away from $\mathcal{N}'$. Once away from $\mathcal{N}'$, one can use the control (27) to drive the system to the origin. We summarize the control strategy that drives the system (12), (13) to the origin $\omega_1 = \omega_2 = \phi = \theta = \psi = 0$ from arbitrary initial conditions:

$$u = \begin{cases} u_c \in \mathbb{C} & \text{if } w(0) = 0 \text{ and } z(0) \neq 0, \\ u_s(\omega, w, z) & \text{if otherwise.} \end{cases} \tag{40}$$

Note that the difference between the current control strategy and the control strategy of Krishnan *et al.* (1992), where the same problem is investigated, is that in the latter the authors intend to render the one-dimensional manifold $\mathcal{N}'$ attractive, whereas in the current work we intend to avoid $\mathcal{N}'$ and render the invariant manifold $\mathcal{S}$ attractive. We have demonstrated a control strategy that for initial conditions $(\phi_0, \theta_0, \psi_0) \notin \mathcal{N}'$ drives the system to the origin with the proper choice of feedback gains. If the initial conditions are on $\mathcal{N}'$ then one must first move away from $\mathcal{N}'$ in order to go to the origin, i.e. the control strategy is nonlocal in nature. This can be achieved either by the methodology described earlier or by the techniques of Krishnan *et al.* (1992). Physically, this nonlocal control strategy implies that, starting from initial conditions $\phi(0) = \theta(0) = 0$, $\psi(0) \neq 0$, one is not able to reach the origin $\phi = \theta = \psi = 0$ by maintaining $\phi = \theta \equiv 0$. Such a condition would require a pure rotation about the symmetry axis, at the time when no such control torque is available.

Remark 3.1. Equation (7) establishes a smooth change of coordinates (i.e. a diffeomorphism) between (w_1, w_2) and (ϕ, θ), for all (ϕ, θ) that do not correspond to an 'upside-down' configuration of the rigid body $(\theta = 0, \phi = \pi)$; otherwise, $w = \infty$. This permits the use of (8) instead of (6a,b) in stabilization problems, since stability of w implies that $w(t) < \infty$ for all $t > 0$ and thus avoidance of the singularity at $w = \infty$. Of course, one has to take into consideration the case when the rigid body initially has this singular configuration; however, we can always avoid this problem by simply turning the thrusters on to move away from this initial orientation before using (40).

4. SOME EXTENSIONS

Note that when the initial spin rate is not zero $(\omega_{30} \neq 0)$, stabilization about the equilibrium manifold $\mathcal{N}$—the 'circular attractor' of Byrnes *et al.*, (1988) and Byrnes and Isidori (1991a)—can be easily accomplished using the linear control law

$$u = -\kappa_1 \omega - \kappa_2 w, \tag{41}$$

where $\kappa_1 > 0$ and $\kappa_2 > 0$. In this case the closed-loop equations become

$$\dot{\omega} = -\mathrm{i} a \omega_{30} \omega - \kappa_1 \omega - \kappa_2 w, \tag{42a}$$

$$\dot{w} = -\mathrm{i}\omega_{30} w + \tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega} w^2, \tag{42b}$$

and one can show global asymptotic stability using the Lyapunov function

$$V(\omega, w) = \tfrac{1}{2}|\omega|^2 + \kappa_2 \ln(1 + |w|^2)$$

and a LaSalle-type argument. Therefore we have the following theorem.

Theorem 4.1. The linear control law (41) globally asymptotically stabilizes the system (42) about the origin. Equivalently, the control law (41) globally asymptotically stabilizes the system (2), (6) about the equilibrium manifold (11).

One can also think of (7) and (21) as defining a diffeomorphism between $(\phi, \theta, \psi) \in \mathcal{M}$ and $(w_1, w_2, z) \in \mathbb{R}^3$. Under this transformation, one can re-derive the differential equation for z for the general case (when ω_3 is not identically zero) and obtain the result that

$$\dot{z} = \mathrm{Im}(\omega \bar{w}) + \omega_3. \tag{43}$$

We have already seen that, for $\omega_3 = 0$, the differential equation for z reduces to (22); however, there is no a priori reason to expect that for $\omega_3 \neq 0$ the only contribution to the $\dot{z}$ equation is just the last term in (43). This surprisingly simple result implies that (43) along with (8), which we rewrite here as

$$\dot{w} = -i\omega_3 w + \tfrac{1}{2}\omega + \tfrac{1}{2}\bar{\omega}w^2, \qquad (44)$$

can be used as an alternative new description of the kinematics of attitude motion. It should be mentioned here that although (44) has been introduced previously (Tsiotras and Longuski, 1994a), by itself it is not enough to describe the rotation of a rigid body, since it gives information about the time evolution of just a single column of the rotation matrix (i.e. information about the time evolution of one of the body axes in inertial space). Such information is not enough to reconstruct the rotation matrix; knowledge of at least one more element is necessary. (Then one can construct the rotation matrix as follows: first find the second column using the conditions of unit length and the orthogonality to the first column, and subsequently take the third column as the cross-product of the previous two columns.)

Equation (43) has an additional desirable property that 'naturally' complements (44) in the following sense. From (44), one sees that the magnitude of w obeys the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|w|^2 = (1+|w|^2)\,\mathrm{Re}\,(\omega\bar{w}). \qquad (45)$$

The duality in which the product $\omega\bar{w}$ appears in (43) and (45) can be used in stabilization problems; the two equations (43) and (45) are effectively 'decoupled' from ω, and the linear control law

$$\omega = -w, \quad \omega_3 = -z \qquad (46)$$

renders the closed-loop system exponentially stable. Implementation of this linear control law through the dynamics is easy in the case of three control torques, but a very challenging problem when less than three independent control torques are available (Walsh *et al.*, 1994). These and other issues related to the attitude stabilization of general (nonsymmetric) rigid bodies will be addressed in a forthcoming paper. Note, however, that if we allow three independent control torques, one feedback control law that achieves global asymptotic stability for the complete attitude equations described by (1), (43) and (44) is given as follows.

Theorem 4.2. The feedback control law

$$M = -\omega - w - izw, \quad M_3 = -\omega_3 - z, \qquad (47)$$

where $M = M_1 + iM_2$, globally asymptotically stabilizes the system (1), (43), (44).

Proof. Consider the following candidate Lyapunov function for the closed-loop system corresponding to (47):

$$V(\omega, \omega_3, w, z) = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) + \ln(1+|w|^2) + \tfrac{1}{2}z^2. \qquad (48)$$

This is a positive-definite function. Computing the derivative of (48) along the trajectories of the closed-loop system, one obtains

$$\dot{V} = -|\omega|^2 - \omega_3^2. \qquad (49)$$

Since $\dot{V} \equiv 0$ if and only if $\omega_1 = \omega_2 = \omega_3 = w_1 = w_2 = z \equiv 0$, global asymptotic stability follows directly from a straightforward application of LaSalle's Theorem (Hahn, 1967). □

The complete derivation of (43) and (44), their relation to the classical attitude representations, as well as their physical significance are discussed in Tsiotras and Longuski (1995) and Tsiotras (1994). Other results on the rigid-body stabilization problem using the (w, z) coordinates and another set of stereographic coordinates, derived from Euler parameters, can be found in Tsiotras (1994).

5. NUMERICAL EXAMPLE

We illustrate the results of this paper with a numerical example. We consider the same example as that in Krishnan *et al.* (1992), mainly for the sake of comparison. Specifically, we consider a large-angle maneuver of an axisymmetric spacecraft that is initially at rest ($\omega_1(0) = \omega_2(0) = 0$) and with initial orientation given by $\phi(0) = \pi$, $\theta(0) = 0.25\pi$, $\psi = -0.5\pi$. These large initial conditions correspond physically to an almost upside-down configuration.

The purpose of the controller is to reorient the spacecraft to its rest position $\phi = \theta = \psi = 0$ and keep it there. The data of the problem correspond to an initial condition not on the manifold $\mathcal{S}$ (see Fig. 3). The control law (27) therefore first drives the trajectory to $\mathcal{S}$ (approximately at 4 s in Fig. 5) and then—while on $\mathcal{S}$—to the origin. The feedback gains are chosen as $\kappa = 0.5$, $\mu = 1.25$ and $\alpha = 2$, and the results are shown in Figs 4 and 5.

6. CONCLUDING REMARKS

We have presented a novel formulation of the attitude kinematics of a rotating rigid body, which promises to be very useful for control

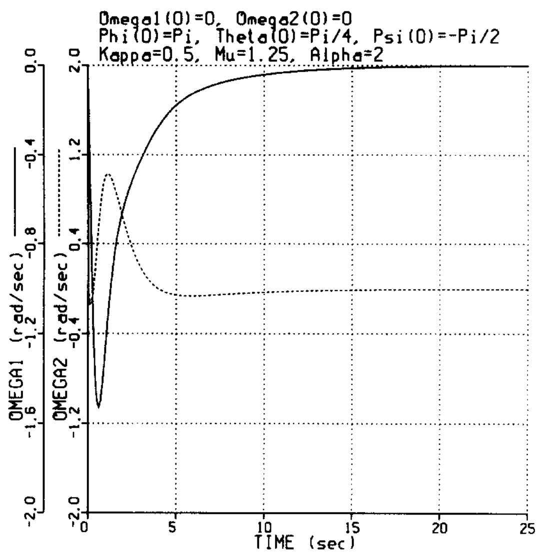

Fig. 4. Angular velocity components ω_1 and ω_2.

purposes. We have demonstrated this by deriving a simple control strategy that achieves arbitrary reorientation of a rotating axisymmetric spacecraft, when the two available control torques span the two-dimensional plane perpendicular to the axis of symmetry and the initial spin rate is zero. A two-dimensional manifold has been constructed, which is used to derive a feedback control that drives the complete system to the origin from arbitrary initial conditions. The control law asymptotically stabilizes the complete attitude equations from all initial conditions inside an a priori arbitrary large compact set. Controllers that satisfy this property have been proposed and derived by Byrnes and Isidori (1991b), where the term *stabilization on compacta* for this stabilization property was also introduced. This is an important extension of the notion of global

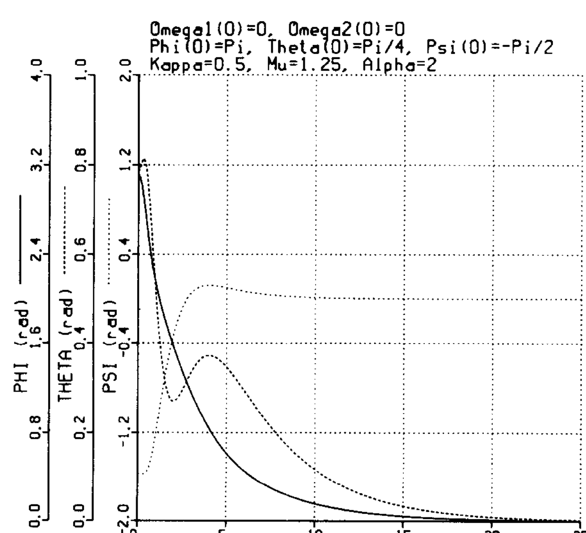

Fig. 5. Eulerian angles ϕ, θ and ψ.

stabilizability, since controllers achieving stabilization on compacta essentially provide asymptotic stabilization from all 'practical' initial conditions, assuming that the feedback gain is sufficiently large. Unfortunately, in the general case there is no explicit formula for a lower bound on the control gain (e.g. α in (32a)) that achieves stabilization on compacta, or an explicit characterization of the corresponding compact set of initial conditions. In this paper, however, we have provided such explicit formulas for both the lower bound on the control gain and the associated compact set of initial conditions for the problem addressed.

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