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Drag-law Effects in the Goddard Problem*

P. TSIOTRAS† and H. J. KELLEY‡

For a drag-law witnessing a sharp increase in the transonic region, a more complex switching structure than the classical full-singular-coasting sequence may occur during the optimal burning program for the vertical climb of a rocket.

Key Words—Aerospace trajectories; bang-bang control; optimal control; optimization; singular arcs; singular control.

Abstract—Presently studied is the problem of maximizing the altitude of a rocket in vertical flight in a resisting medium, when the amount of propellant is specified, known as the Goddard problem. The case is studied in which the drag coefficient is a function of the Mach number, witnessing a sharp increase in the transonic region. Analysis shows the possibility of a more complex switching structure than the classical full-singular-coast sequence, with the appearance of a second full-thrust subarc in the transition from the subsonic to the supersonic region. Necessary conditions such as the Legendre–Clebsch condition for singular subarcs and the McDanell–Powers condition for joining singular and non-singular subarcs were checked, and were found to be satisfied. It is shown that the results obtained depend heavily on the assumed form of the drag law, and on the magnitude of the upper bound on the thrust.

1. INTRODUCTION

THE PROBLEM of optimum thrust programming for maximizing the altitude of a rocket in vertical flight, for a given amount of propellant, has been extensively analyzed over the past sixty years. Briefly, we can refer to the pioneering work of Goddard (1919), Hamel (1927), Tsien and Evans (1951) and Leitmann (1956, 1957). However, as Leitmann (1963) first pointed out, the problem's solution continued to be far from complete, mainly due to difficulty arising from the requirement that the mass be monotonically non-increasing. In this work, the possibility of a more complex sequence of subarcs for the case of sharp transonic drag-rise was suggested.

Solutions that meet this requirement have been obtained only in a few special cases, typified by the work of Miele and Cavoti (1958), and Miele (1955), who treated the cases of flight in vacuum and flight with a power law for drag, and later by Bryson and Ross (1958). Miele (1962), using a totally different approach, also proved the *sufficiency* of the optimal solution established by his predecessors, i.e. that the optimal burning program involves a rapid boost at the beginning of the flight, usually followed by a period of continuous burning (sustain phase) and ending with a zero-thrust period. Miele and Cicala (1956) were also the first to suggest the possibility of a more complex sequence of subarcs for the case of a general drag model.

One of the most complete works on Goddard's problem is perhaps the extensive treatment by Garfinkel (1963), who proved that with impulsive boosts in the velocity admitted, and for the case of a general drag model, the solution contains a finite number of such boosts in the transonic velocity region, and contains no coasting arcs except the terminal one.

As already has been established by previous researchers, the drag plays a significant role in the switching structure of the problem. In particular, it has been shown by Tsien and Evans (1951) and later by Miele (1955), that for the special case when drag is ideally zero, the variable-thrust subarc disappears from the extremal solution, which consequently reduces to subarcs flown with maximum engine output and coasting subarcs. Moreover, the approximation $C_D \approx$ const. may be of use at low altitudes, when the speed of optimum climb still belongs to

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† Formerly with the Department of Aerospace and Ocean Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A.; currently with the School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN 47906, U.S.A.

‡ Department of Aerospace and Ocean Engineering, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, U.S.A. Dr H. J. Kelley passed away 9 February 1988.

the region of quasi-incompressible flow. As the altitude increases, both the velocity and the Mach number increase with such a rapidity that the hypothesis $\partial C_D/\partial M = 0$ is soon no longer satisfied, and a more accurate drag model should be used. Constant drag coefficient C_D is then replaced by a Mach-dependent drag coefficient featuring a sharp increase in the transonic region. When such a model is used, two optimal solutions for the singular surface arise: one in the subsonic-transonic region, and the other in the supersonic region of the velocity. For the case of level flight of a rocket-powered aircraft Miele and Cicala (1956) showed that another full-thrust subarc may occur during transition from the subsonic to supersonic region.

In the current work, Goddard's problem is examined with relaxed restrictions on the assumed drag characteristics of the rocket. The relaxed assumptions allow for switching structures that were previously not considered for the case of vertical climb. Invaluable insight to the problem was obtained via a transformation to a state-space of reduced dimension, where the problem becomes more tractable. The methodology can be applied to other singular optimal control problems too, in order to determine the possible optimal solution structure, i.e. the number and the relative position of singular and bang-bang subarcs.

2. PROBLEM FORMULATION

The vertical flight of a rocket obeys the following system of differential equations:

$$\begin{aligned} \dot{\tilde{h}} &= \tilde{v} \\ \dot{\tilde{v}} &= \frac{\tilde{T}-\tilde{D}}{\tilde{m}} - \tilde{g} \\ \dot{\tilde{m}} &= -\frac{\tilde{T}}{\tilde{c}} \end{aligned} \tag{1}$$

where $\tilde{h}$, $\tilde{v}$, $\tilde{m}$ are the altitude, velocity and mass respectively, $\tilde{T}$ denotes the engine thrust, $\tilde{D}$ denotes the aerodynamic drag, $\tilde{c}$ represents the exhaust velocity of the gases from the rocket engine, and $\tilde{g}$ is the gravitational acceleration. The second of the above equations simply states the force equilibrium along the flight path, the first equation is the kinematic relation between the altitude and the velocity, and the last states that the fuel consumption is proportional to the thrust. If we assume spherical earth with an inverse-square gravitational field, the above system of equations can be suitably nondimensionalized using the following quantities:

$$\begin{aligned} \hat{h} &= R_e \\ \hat{t} &= G^{-1/2}\hat{h}^{3/2} \\ \hat{v} &= G^{1/2}\hat{h}^{-1/2} \\ \hat{g} &= G\hat{h}^{-2} \\ \hat{m} &= m_0. \end{aligned} \tag{2}$$

Here R_e denotes the radius of the earth, G the gravitational constant, $\hat{g}$ the acceleration due to gravity at the surface of the earth, and $\hat{m}$ the launching mass of the vehicle. Using the above nondimensionalization factors, the equations of motion in nondimensionalized form become:

$$\begin{aligned} \dot{h} &= v \\ \dot{v} &= \frac{T-D}{m} - h^{-2} \\ \dot{m} &= -\frac{T}{c}. \end{aligned} \tag{3}$$

If S is the characteristic cross-section area of the vehicle, and ρ denotes the atmospheric density, then the aerodynamic drag is given by the quadratic formula:

$$D = C_D(M)S\frac{\rho v^2}{2}. \tag{4a}$$

If in addition, we assume that the density of the atmosphere reduces exponentially with the altitude, the nondimensionalized form of the drag force becomes

$$D = C_D(M)bv^2\exp(\beta(1-h)) \tag{4b}$$

where the factor $bv^2\exp(\beta(1-h))$ is numerically equal to the product of the velocity head and the characteristic cross-section area of the aircraft, b, β are constants, and M is the Mach number defined as the ratio of the vehicle speed over the speed of sound. For simplicity it will be assumed that the speed of sound remains constant with altitude, an assumption which is actually valid only for stratospheric solutions. In (4) $C_D(M)$ is the zero-lift drag coefficient assumed to depend on the Mach number according to the following relationship:

$$C_D(M) = A_1\tan^{-1}(A_2(M-A_3)) + A_4. \tag{5}$$

This formula generates a quick transition from one value of C_D in the subsonic region to another higher value of C_D in the supersonic region. The A_1, A_2, A_3, A_4 are constants controlling when, and how fast, this transition takes place (Fig. 1).

The initial conditions are specified for the three state variables as h_0, v_0 and m_0. The final

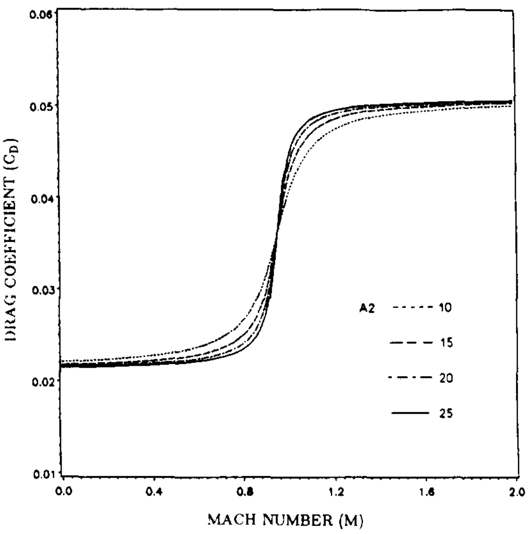

FIG. 1. Variation of the drag coefficient C_D with Mach number M.

value of the mass is also given as m_f. The problem is to determine the optimum trajectory of a rocket in vertical flight, from an assigned initial position on the surface of the earth to the final position where the altitude reaches its maximum value, i.e. we want to maximize the altitude at the terminal time. Hence, the performance index is given by

$$\mathcal{J} = h(t_f) \tag{6}$$

subject to the prescribed boundary condition

$$v(t_f) = 0 \tag{7}$$

and the dynamic equality constraints given by (3). The thrust is the control variable which is bounded according to the inequality:

$$0 \leq T \leq T_{max}. \tag{8}$$

The aerodynamic data and the vehicle's parameters, with the exception of the value for T_{max}, were taken from the work of Zlatskiy and Kiforenko (1983), and their nondimensionalized values are listed below:

$$\begin{aligned} b &= 6200 \\ \beta &= 500 \\ T_{max} &= 3.5. \end{aligned} \tag{9}$$

Furthermore, the constants in (5) are chosen as follows:

$$\begin{aligned} A_1 &= 0.0095 \\ A_2 &= 25 \\ A_3 &= 0.953467778 \\ A_4 &= 0.036. \end{aligned} \tag{10}$$

For the numerical solution it is assumed that the rocket is initially at rest at the surface of the earth, and that its fuel mass is 40% of the rocket total mass.

3. PROBLEM ANALYSIS

Define the state vector $\vec{x} = \text{col}(h, v, m)$, and the co-state vector $\vec{\lambda} = \text{col}(\lambda_h, \lambda_v, \lambda_m)$. Then the variational Hamiltonian takes the form:

$$\mathcal{H}(\vec{\lambda}, \vec{x}, T) = \lambda_j \dot{h} + \lambda_v \dot{v} + \lambda_m \dot{m} \tag{11}$$

where the propagation of the co-state vector obeys the equation

$$\dot{\vec{\lambda}} = -\frac{\partial \mathcal{H}}{\partial \vec{x}}. \tag{12}$$

Using (3) and (12), and noting that the control T appears linearly in the equations of motion, one obtains for the Hamiltonian the following form:

$$\mathcal{H} = \mathcal{H}_0 + T\mathcal{H}_1 = 0 \tag{13}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are given by:

$$\mathcal{H}_0 = \lambda_h v - \lambda_v\left(\frac{D}{m} + h^{-2}\right) \tag{14}$$

$$\mathcal{H}_1 = \frac{\lambda_v}{m} - \frac{\lambda_m}{c}. \tag{15}$$

$\mathcal{H}_1$ is the "switching function" and governs the history of the control. Using Pontryagin's Maximum Principle (Pontryagin *et al.*, 1962), three possibilities exist for an extremal control, depending on the sign of the switching function:

$$\begin{aligned} T^* = T_{max} &\quad \text{when} \quad \mathcal{H}_1 > 0 \\ 0 \leq T^* \leq T_{max} &\quad \text{when} \quad \mathcal{H}_1 = 0 \\ T^* = 0 &\quad \text{when} \quad \mathcal{H}_1 < 0. \end{aligned} \tag{16}$$

The second case indicates the possibility of an interval of singular control, i.e. an interval of finite duration over which the $\mathcal{H}_1$ vanishes identically. The following relationships must then be fulfilled simultaneously on a singular arc:

$$\mathcal{H}_1 = \dot{\mathcal{H}}_1 = \ddot{\mathcal{H}}_1 = \cdots \equiv 0. \tag{17}$$

The above equations along with (13) define a manifold $E(v, m, h) = 0$ in the three-dimensional state space of v, m, h. This manifold, often called the *singular surface*, represents the locus of all possible state trajectories, corresponding to singular control effort. Note also, that $E(v, m, h) = 0$ is also the *singular control switching boundary*, since any point of the state space which does not lie on it, must feature a bang-bang control.

The three possible types of subarcs that may appear in the solution of an optimal trajectory have already been examined; however, the composite optimal trajectory consisting of these three types of subarcs need to be determined. The analysis of the problem is complicated by the fact that the optimal solution, in general, consists of some combination of singular and non-singular subarcs, the number and sequence of which are not known *a priori*. In fact, the manner in which singular subarcs enter into composite candidates will be determined in part, by the specified two-point boundary conditions for the Euler equations. Hence, the determination of the optimal composite trajectory involves the solution of a two-point boundary value problem, frequently by means of a trial and error procedure. The next section describes a methodology that simplifies problems involving both singular and nonsingular subarcs and that can be used to determine the possible composite optimal structure.

4. TRANSFORMATION TO REDUCED STATE-SPACE

A transformation approach suggested by Kelley (1964a, b) is sometimes helpful and permits analysis of singular arcs in a state space of reduced dimension. The singular arcs become nonsingular, thus the available necessary conditions can be applied. However, this approach has the practical shortcoming that the solution of the transformation requires a closed form solution to a system of nonlinear differential equations. Fortunately, this transformation can be obtained rather easily for the present problem, allowing the structure of the problem to be studied in a reduced, two-dimensional, state-space. This is quite attractive; the complete family of singular extremals for given initial conditions can be pictured in two-dimensional space.

Omitting for brevity the theory of the transformation, Kelley *et al.* (1967) have shown, that the transformation of the original state vector $\vec{x}$ to the canonical form leads to the new state vector $\vec{z}$ with components

$$z_1 = h \quad z_2 = v \quad z_3 = me^{v/c}. \tag{18}$$

The differential equations in the new state space are derived directly from (3) and (18).

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \frac{T-D}{z_3}\exp(z_2/c) - z_1^{-2} \\ \dot{z}_3 &= -\frac{D}{c}\exp(z_2/c) - \frac{z_3}{c}z_1^{-2}. \end{aligned} \tag{19}$$

The Hamiltonian for the new system is given by

$$\mathscr{H} = \kappa_{z_1}\dot{z}_1 + \kappa_{z_2}\dot{z}_2 + \kappa_{z_3}\dot{z}_3 \tag{20}$$

where $\vec{\kappa} = \mathrm{col}(\kappa_{z_1}, \kappa_{z_2}, \kappa_{z_3})$ is the co-state vector of the new state-space, satisfying the differential equations

$$\begin{aligned} \dot{\kappa}_{z_1} &= \frac{\kappa_{z_3}}{c}\left[\exp(z_2/c)\frac{\partial D}{\partial z_1} - 2z_3 z_1^{-3}\right] \\ &\quad - \kappa_{z_2}\left[-z_3^{-1}\frac{\partial D}{\partial z_1}\exp(z_2/c) + 2z_1^{-3}\right] \\ \dot{\kappa}_{z_2} &= -\kappa_{z_1} + \frac{\kappa_{z_3}}{c}\exp(z_2/c)\left[\frac{\partial D}{\partial z_2} + \frac{D}{c}\right] \\ &\quad - \kappa_{z_2}\frac{\exp(z_2/c)}{z_3}\left[\frac{\partial D}{\partial z_2} + \frac{T-D}{c}\right] \\ \dot{\kappa}_{z_3} &= \frac{\kappa_{z_3}}{c}z_1^{-2} - \kappa_{z_2}\left[-\frac{T-D}{z_3^2}\exp(z_2/c)\right]. \end{aligned} \tag{21}$$

Notice from (19) that the control T appears only in one of the state equations, namely in the equation for $\dot{z}_2$. One can therefore discard this equation, for analysis of the singular portion of the trajector, and consider the z_2 variable as a new "control-like" variable, in the reduced state-space of variables z_1 and z_3. This change occurs through the identical vanishing of the Lagrange multiplier associated with the second equation of the state. Indeed, the switching function of the transformed problem is given as

$$\mathscr{H}_1 = \frac{\partial \mathscr{H}}{\partial T} = \frac{\kappa_{z_2}}{z_3}\exp(z_2/c) \tag{22}$$

and along a singular arc we require

$$\kappa_{z_2} \equiv 0 \tag{23}$$

because, throughout the trajectory,

$$\frac{\exp(z_2/c)}{z_3} \neq 0 \quad \text{always}. \tag{24}$$

The vanishing of κ_{z_2}, along the singular portion of the trajectory, can be verified through an analogous transformation for the co-state vector $\vec{\lambda}$ of the original state space as follows: Optimal control theory indicates that the co-state variables have a special meaning; as Breakwell (1961), and Cicala (1957) have pointed out, the value at some time t of the Lagrange multiplier λ_i, associated with the variable x_i is just $\partial \mathscr{J}/\partial x_i(t)$, where $\mathscr{J}$ is the "payoff" function with t regarded as a starting time. This interpretation of the co-states is very instructive and it will be used extensively later on. Requiring that the cost function and the Hamiltonian remain unchanged under the transformation, and using the chain rule of differentiation, the following relationship must hold along the trajectory:

$$\vec{\lambda} = \frac{\partial \mathscr{J}}{\partial \vec{x}} = \frac{\partial \vec{z}}{\partial \vec{x}}\frac{\partial \mathscr{J}}{\partial \vec{z}}. \tag{25}$$

According to the interpretation mentioned earlier,

$$\vec{\kappa} = \frac{\partial \mathscr{J}}{\partial \vec{z}} \tag{26}$$

is the co-state vector for the new state space and

$$[J] = \frac{\partial \vec{z}}{\partial \vec{x}} \tag{27}$$

is the Jacobian of the transformation, with elements

$$J_{ij} = \frac{\partial z_j}{\partial x_i} \quad i, j = 1, 2, 3. \tag{28}$$

Assuming that the transformation is nonsingular, the inverse of the Jacobian matrix exists, and the system of (25) has the unique solution

$$\vec{\kappa} = [J]^{-1}\vec{\lambda} \tag{29}$$

which can be written analytically as

$$\begin{bmatrix} \kappa_{z_1} \\ \kappa_{z_2} \\ \kappa_{z_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -m/c \\ 0 & 0 & e^{-v/c} \end{bmatrix} \begin{bmatrix} \lambda_h \\ \lambda_v \\ \lambda_m \end{bmatrix} \tag{30}$$

or in expanded form

$$\kappa_{z_1} = \lambda_h \tag{31a}$$

$$\kappa_{z_2} = \lambda_v - \frac{m}{c}\lambda_m \tag{31b}$$

$$\kappa_{z_3} = \lambda_m e^{-v/c}. \tag{31c}$$

Notice that since

$$\det [J] = e^{v/c} \neq 0 \tag{32}$$

the transformation is nonsingular everywhere. Equation (31b) offers another justification for the vanishing of κ_{z_2} along a singular arc. Comparing (15) and (31b) we see that the co-state κ_{z_2} is just the switching function of the original problem times the mass; consequently, along the singular arc κ_{z_2} should be identically zero. The co-state equations reduce to

$$\dot{\kappa}_{z_1} = -\frac{\partial \mathscr{H}}{\partial z_1} - \frac{\kappa_{z_3}}{c}\left[\exp(z_2/c)\frac{\partial D}{\partial z_1} - 2z_3 z_1^{-3}\right] \tag{33a}$$

$$\dot{\kappa}_{z_3} = -\frac{\partial \mathscr{H}}{\partial z_3} = \frac{\kappa_{z_3}}{c} z_1^{-2} \tag{33b}$$

and the Hamiltonian simplifies to

$$\mathscr{H} = \kappa_{z_1} z_2 - \frac{\kappa_{z_3}}{c}[D \exp(z_2/c) + z_3 z_1^{-2}]. \tag{34}$$

Notice that the new control z_2 does not appear linearly in the system of state and co-state equations, and the classical Legendre–Clebsch necessary conditions can be applied successfully in the reduced-state-space problem.

The extremals, then, of the transformed problem are the singular extremals of the original, and those extremals satisfying the strengthened version of the classical Legendre–Clebsch condition are maximizing, at least over short intervals. The stationary solution of the transformed problem corresponding to the singular subarc of the original problem occurs then, when

$$\frac{\partial \mathscr{H}}{\partial z_2} = \kappa_{z_1} - \frac{\kappa_{z_3}}{c}\exp(z_2/c)\left[\frac{D}{c} + \frac{\partial D}{\partial z_2}\right] = 0 \tag{35}$$

and the Legendre–Clebsch necessary condition requires, for a maximizing extremal

$$\frac{\partial^2 \mathscr{H}}{\partial z_2^2} = -\frac{\kappa_{z_3}}{c}\exp(z_2/c)\left[\frac{D}{c^2} + \frac{2}{c}\frac{\partial D}{\partial z_2} + \frac{\partial^2 D}{\partial z_2^2}\right] \leq 0. \tag{36}$$

The latter relationship assures the convexity of the Hamiltonian in the neighborhood of a solution of (35), i.e. an optimal control obtained by (35) provides *at least a local* maximum of $\mathscr{H}$.

5. NECESSARY CONDITIONS

The fact that a trajectory satisfies the Euler differential equations and the first-order necessary conditions, only guarantees its stationary character. To determine whether a maximum is attained, further investigation is in order. Thus, the Legendre–Clebsch, Weierstrass and Jacobi conditions must be checked. Each of these three conditions is a necessary condition for a maximum. All of them, suitably strengthened, in combination with the first-order necessary conditions, provide a sufficient condition. In this section, we will briefly review the available necessary conditions for the optimality of the trajectory, in the case when singular subarcs are considered as possible candidates.

Kelley condition. The mere presence of singular members of the state-Euler system solutions does not assure the appearance of such subarcs in an optimal trajectory. In fact, as Johnson and Gibson (1963) pointed out, a singular solution may not be optimal even locally. To determine local optimality a further investigation is in order. Thus, the so-called Generalized Legendre–Clebsch, or Kelley–Contensou condition must be checked; see Kelley (1964a) and Robbins (1967). This condition can be stated as follows:

$$(-1)^q \frac{\partial}{\partial T}\left[\frac{d^{2q}}{dt^{2q}}\left(\frac{\partial \mathscr{H}}{\partial T}\right)\right] \leq 0 \tag{37}$$

where q is the *order* of the singularity of the arc.

Junction conditions. An admissible control must satisfy other requirements, in addition to satisfying the given physical constraints. If the solution is totally nonsingular, or totally singular, necessary conditions for optimality testing are available in a large number of cases. The continuity of $\mathscr{H}(t)$ and the continuity of $\vec{\lambda}(t)$ across junctions between subarcs, the so-called *Weierstrass–Erdmann* corner condition, is perhaps the most important. However, the character of optimal trajectories which include both singular and nonsingular subarcs is less easily decided. The first results concerning the behavior of the optimal control at a junction between singular and nonsingular subarcs were derived by Kelley *et al.* (1967), and may be summarized as follows: If q is the order of the singular subarc, then

> If q is *odd* a jump discontinuity in control may occur at a junction between a locally minimizing singular subarc, i.e. a subarc on which the generalized Legendre–Clebsch condition is satisfied in strengthened form, with a nonsingular subarc.
> If q is *even,* jump discontinuities in control from singular subarcs satisfying the strengthened form of the generalized Legendre–Clebsch condition are ruled out.

Johnson (1967) recognized the conflict between the generalized Legendre–Clebsch condition and the junction condition for q even, and showed that analytic junctions with jumps can occur only if q is odd, but he did not identify the character of junctions between nonsingular and q even singular subarcs.

McDanell and Powers (1971), motivated by the preliminary results obtained by Kelley *et al.* (1967) and Johnson (1967), considered the problem concerning the continuity and smoothness properties of the optimal control at a junction between singular and nonsingular subarcs in more detail, and generalized the previous conclusions, with one important exception; they proved the possibility of a continuous junction for control saturation with zero slope for q odd problems, a possibility which had not been included by Kelley *et al.* and which was later ruled out for $q > 1$ by Berschchanskiy (1979). Their main result was that—for analytic junctions—the sum of the order of the singular subarc and the order of the lowest time derivative of the control which is discontinuous at the junction must be an odd integer when the strengthened generalized Legendre–Clebsch condition is satisfied.

In the McDanell and Powers results, the assumption that the control is piecewise analytic is not to be taken lightly because the junction is typically nonanalytic not only for q even, but also for q odd with $q > 1$. In fact, according to Bershchanskiy, the McDannell–Powers necessary conditions are actually of interest only for $q = 1$. As was shown in his work, for q even problems or for q odd problems with $q > 1$ the transition from a nonsingular to a singular subarc is associated with *chattering* junctions, i.e. controls that switch rapidly between the upper and the lower bound faster and faster, with a point of accumulation, and which although measurable, are nonanalytic.

Jacobi and Jacobi-like conditions. Testing of the second variation, on the other hand, such as Jacobi and Jacobi-like testing is rarely carried out for nonsingular extremal candidates, and even more rarely for candidates with isolated singular points, possibly corners, as pointed out by Kelley and Moyer (1985). Extremals corresponding to the second case, so-called broken extremals, have been studied with generality, detail and rigor by Larew (1919), Reid (1935) and Caratheodory (1967). Moyer (1965, 1970) using this idea, developed a computational technique in the case of a nonsingular extremal exhibiting corners, and used this approach successfully in an orbital transfer. However, Jacobi-like testing for composite Euler solutions including singular subarcs is still a research area, and the few attempts made in this direction, mainly due to McDanell and Powers (1970), are limited to the case of a totally singular arc. Very few methods have been also developed for the more complex case of a composite extremal, mainly by Speyer and Jacobson (1971a, b) and Moyer (1973).

All the above conditions, though only necessary, help to eliminate some of the possible subarc-sequence candidates.

6. CONTROL-LOGIC ANALYSIS

Only the free-time case was studied, but the method of solution is applicable also for any value of fixed final time. Due to the sharp increase of the partial $\partial C_D/\partial M$ near Mach 1, the singular surface witnesses also a peak in the same region (Fig. 2). Moreover, projections of the singular surface into mass-velocity and altitude-velocity planes reveal the existence of a nonadmissible portion of the variable thrust arc, since it corresponds to increasing mass (shown by a dashed line in Figs 3 and 4). Therefore, an optimal switching structure cannot include a singular arc in the transonic region, on account of the violation of the requirement the mass be monotonically nonincreasing.

The problem becomes more transparent if

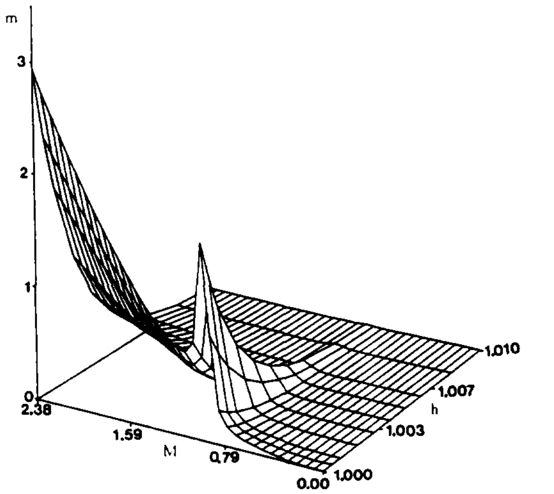

FIG. 2. Singular surface $E(v, m, h) = 0$ for Mach dependent drag coefficient.

one uses the transformation to z_1 and z_3 state-space described before. It should be noted however, that such an approach is equivalent to admitting jump discontinuities in the new control variable $z_2 = v$. Such discontinuities, occurring at corner points of the solution, imply impulsive behavior of the thrust T. Such impulsive behavior would be admissible in the absence of inequality constraints on T, but in practice, there is always a limit on the available thrust output. However, thrust impulses, while not physically possible, are convenient idealizations to very rapid burning of fuel. Thus, an optimal solution obtained in the z-space would still be of importance as an approximation to the case of a very high magnitude of the throttle setting, and in addition to this, it could provide physical insight to the problem.

The analysis can be stated briefly as follows: examine the singular arc by transforming to

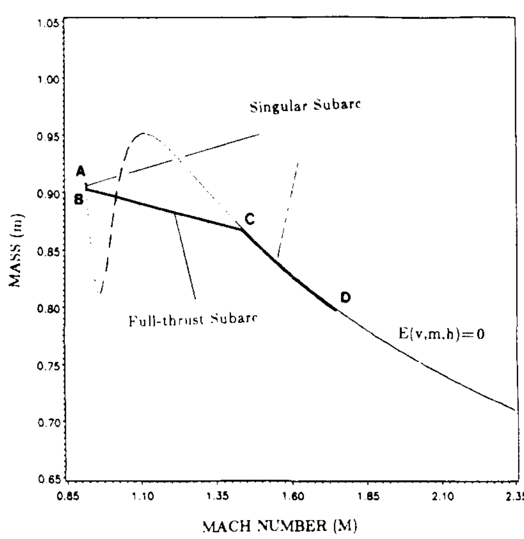

FIG. 3. Projection of the trajectory on the mass-velocity plane for $T_{max} = \infty$.

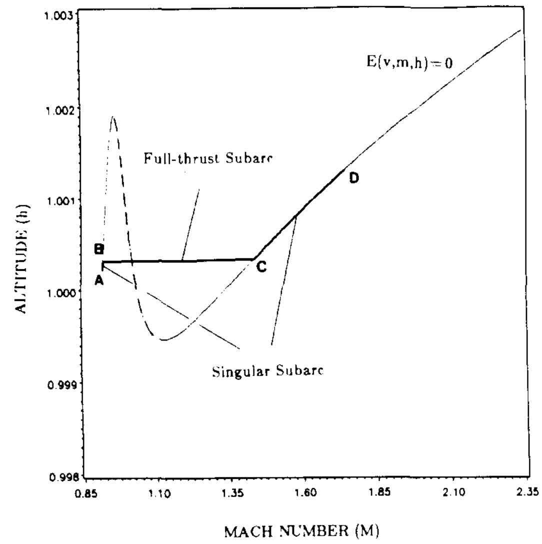

FIG. 4. Projection of the trajectory on the altitude-velocity plane for $T_{max} = \infty$.

z-space with a new control-like variable, z_2, which maximizes the new Hamiltonian. The variation of the Hamiltonian vs the velocity along the extremal, corresponding to the singular arc of the original problem, reveals that along this singular arc the Hamiltonian has three stationary values, corresponding to three solutions of the equation of the singular surface $E(v, m, h) = 0$. Two of those correspond to a maximum, and the other corresponds to a minimum value of the Hamiltonian. The first maximum corresponds to the subsonic branch, the minimum corresponds to the transonic branch, screened out, and the second maximum corresponds to the supersonic branch of the singular surface. Henceforth we shall use the terms "subsonic maximum" or "subsonic solution", and "supersonic maximum" or "supersonic solution" to distinguish between the two cases of interest. Thus, points corresponding to the transonic solution cannot be included in an optimal trajectory for a second reason, since such points provide a local minimum rather than a maximum for the Hamiltonian.

From Fig. 5 we notice that there is a point in time t_{sw} when both solutions provide the same maximum to the Hamiltonian, and the velocity then *jumps* from the subsonic to the supersonic solution. That is,

$$\mathscr{H}(v_{sub}(t_{sw})) = \mathscr{H}(v_{sup}(t_{sw})) \tag{38}$$

where the subscript "sub" denotes the subsonic solution, and the subscript "sup" denotes the supersonic solution. Hence,

$$z_2(t) = v_{sub} \quad \text{for} \quad t \leq t_{sw} \tag{39a}$$

and

$$z_2(t) = v_{sup} \quad \text{for} \quad t \geq t_{sw}. \tag{39b}$$

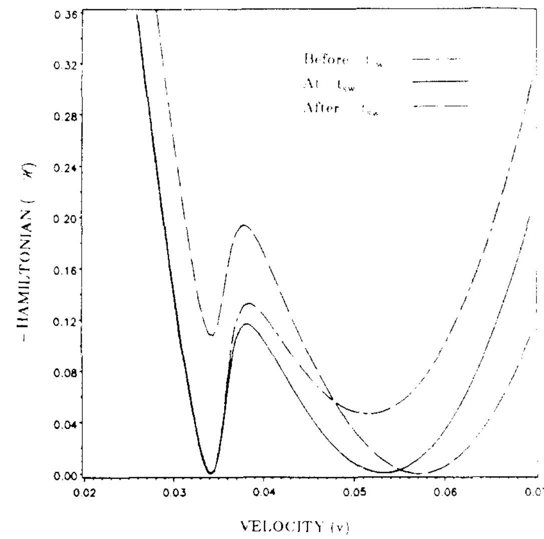

FIG. 5. Variation of $-\bar{\mathscr{H}}$ with $z_2 = v$.

That is because, although both the subsonic and the supersonic solution give a relative maximum, an optimal control should correspond to the absolute maximum of the Hamiltonian. However, the result is limited to the case in which no upper bound on thrust is imposed.

7. COMPOSITE OPTIMAL TRAJECTORY

The previous analysis indicates that an optimal trajectory should start with a full-thrust subarc until the subsonic solution of the singular surface is reached. Then a variable-thrust subarc using this solution is used up to the point when both the subsonic and the supersonic branches provide the same maximum to the Hamiltonian. A switching then to the supersonic branch occurs, and the trajectory remains on the singular surface until the time when the fuel is exhausted. Then a final coasting arc is used, until the terminal boundary conditions are satisfied.

Although this thrust history would provide the optimal switching structure for the case of $T_{max} = \infty$, this will not be necessarily true for the case of bounded thrust. In such a case discontinuities in the velocity are of course unacceptable, and the validity of the solution depends on the value of the upper bound of the thrust. Thus, the structure of the optimal trajectory is still in question. This is the topic of the following section.

8. BOUNDED-THRUST CASE

The analysis so far shows that the variational problem has a special mathematical structure, in so far as the occurrence of two optimal solutions of $E(v, m, h) = 0$ implies the existence of an infinite number of composite solutions, in the passing through the transonic region, all satisfying the same boundary conditions. The question is: which of this family extremals is to be preferred from the point of view of maximizing the altitude? For the case of unbounded thrust the answer has already been given: At a time $t = t_{sw}$, when the Hamiltonian in the reduced z-space switches from its subsonic to its supersonic maximum. Although valid only for unbounded thrust, nevertheless, this remark gives us a hint; an optimal trajectory must *accelerate* from the subsonic to the supersonic region. Since the variable-thrust case must be ruled out, our only choice is the use of full thrust between the two solutions of $E(v, m, h) = 0$. Furthermore, because for a realistic case $T_{max} < \infty$, the switching from the variable-thrust to the second full-thrust subarc must take place somewhere *before* the time t_{sw}, and such that the switching function vanishes at the points of departure and arrrival to the singular surface (points B and D in Figs 3 and 4). In addition to this, the switching function should remain positive all along the full-thrust subarc in order to satisfy the optimality condition of (16).

Thus, a trial-and-error procedure is needed to determine the points B and D. The result obtained, using the boundary-problem solver BOUNDSOL (Bulirsch, 1971), was rather disappointing; an optimal switching from the first varible-thrust subarc to the second full-thrust subarc (point B), should take place before the switching of the first full-thrust subarc to the first variable-thrust subarc (point A). Therefore, for the case of $T_{max} = 3.5$, an optimal trajectory cannot have this switching structure, but rather must have the simpler full-singular-coast sequence, with the singular subarc corresponding to the supersonic solution of $E(v, m, h) = 0$.

However, when an analogous calculation for the case $T_{max} = 6$ was performed, the new, more complex, sequence of subarcs full-singular-full-singular-coast, gave indeed a *higher* final altitude than the full-singular-coast sequence (Table 1). In Figs 6–8 is shown the history of the three components of the nondimensionalized state vector $\vec{x} = (h, v, m)$ respectively, for the optimum burning program corresponding to $T_{max} = 6$. The switching sequence for this optimum thrust program is depicted in Fig. 9. Notice the

TABLE 1. COMPARISON BETWEEN THE CLASSICAL FULL-SINGULAR-COAST SEQUENCE AND THE NEW FULL-SINGULAR-FULL-SINGULAR-COAST SEQUENCE INVOLVING A FULL THRUST SUBARC AT THE TRANSONIC REGION, FOR THE CASE OF $T_{max} = 6$

	Final time	Final altitude
F-S-C	0.197374	1.0132976
F-S-F-S-C	0.198978	1.0133038

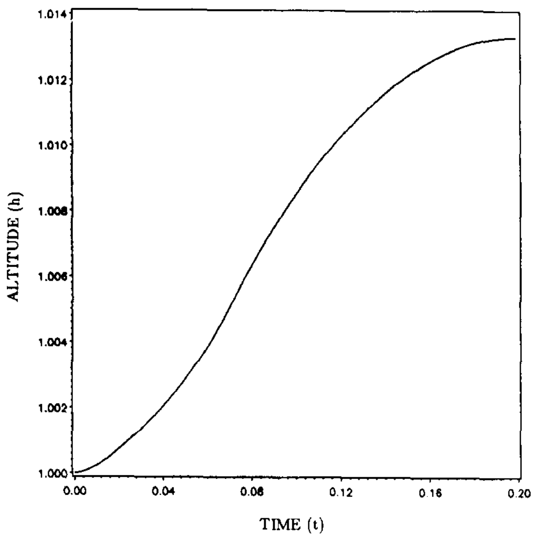

FIG. 6. Variation of altitude h with time for $T_{max} = 6$.

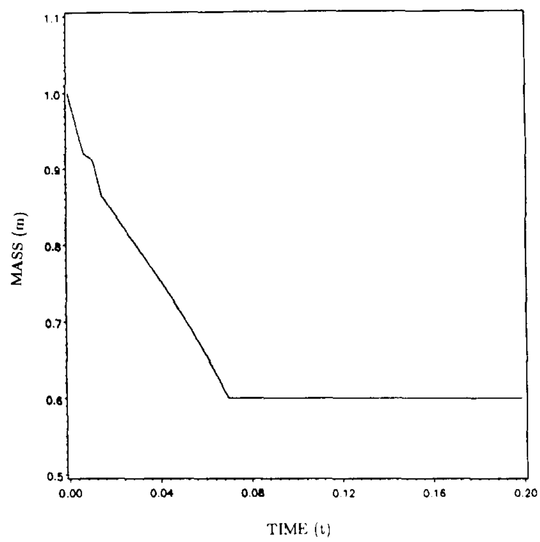

FIG. 8. Variation of mass m with time for $T_{max} = 6$.

corner points during the velocity and mass evolution, that correspond to the points of discontinuity of the thrust. This is the result of the control variable T entering directly to the right-hand side of the equations of motion for the velocity and mass, (3). On the other hand, the evolution of the state variable representing the altitude is smooth, since the thrust does not appear to the corresponding state equation.

10. CONCLUSIONS

The problem of maximizing the final altitude of a vertically ascending rocket has been analyzed for the case of bounded thrust, and quadratic drag law, with the drag coefficient as a function of the Mach number, witnessing a sharp increase in the transonic region. A more complex switching structure, with an intermediate full-thrust subarc in transition through the transonic region, was required owing to the requirement that the mass should be monotonically nonincreasing. The results are identical with those of Garfinkel, for the $T_{max} = \infty$ case, although a totally different approach was used. The solution, using a transformation to a reduced two-dimensional state space, showed that the optimality of the solution depends on the assumed upper bound on the thrust. Numerical results obtained verified the superior performance of the new thrust program, over the classical full-singular-coast sequence, at least for a sufficiently high upper bound on the thrust.

The Kelley necessary condition for singular arcs, and the McDanell and Powers condition for joining singular and nonsingular subarcs were checked, and were found to be satisfied.

A companion paper by the authors examines time-of-flight constraint effects in the problem (Tsiotras and Kelley, 1988).

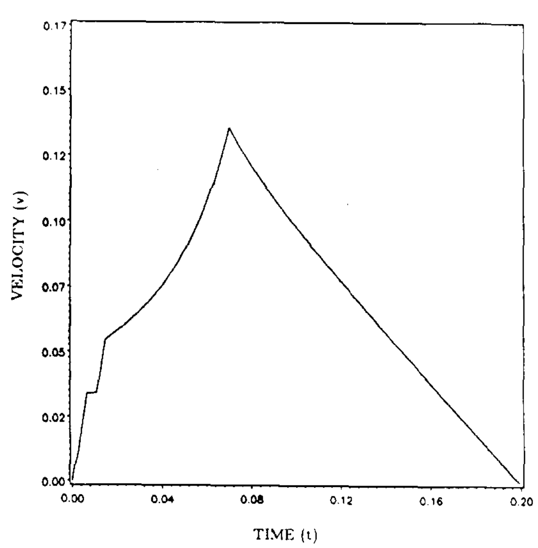

FIG. 7. Variation of velocity v with time for $T_{max} = 6$.

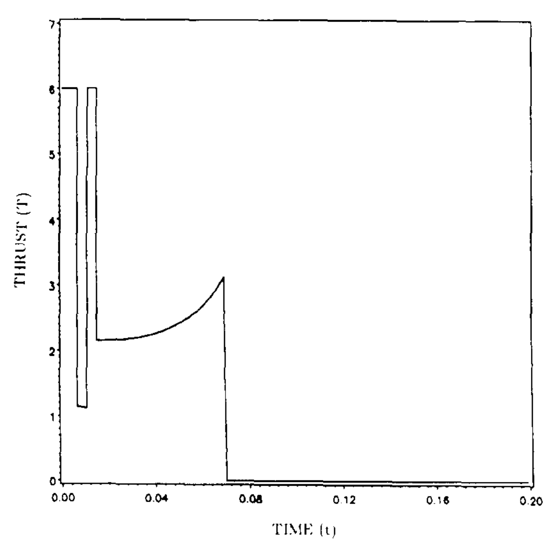

FIG. 9. Variation of thrust T with time for $T_{max} = 6$.

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