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Brief paper

Relay pursuit of a maneuvering target using dynamic Voronoi diagrams☆

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ABSTRACT

This paper addresses the problem of the pursuit of a maneuvering target by a group of pursuers distributed in the plane. This pursuit problem is solved by associating it with a Voronoi-like partitioning problem that characterizes the set of initial positions from which the target can be intercepted by a given pursuer faster than any other pursuer from the same group. In the formulation of this partitioning problem, the target does not necessarily travel along prescribed trajectories, as it is typically assumed in the literature, but, instead, it can apply an "evading" strategy in an effort to delay or, if possible, escape capture. We characterize an approximate solution to this problem by associating it with a standard Voronoi partitioning problem. Subsequently, we propose a relay pursuit strategy, that is, a special group pursuit scheme such that, at each instant of time, only one pursuer is assigned the task of capturing the maneuvering target. During the course of the relay pursuit, the pursuer–target assignment changes dynamically with time based on the (time varying) proximity relations between the pursuers and the target. This proximity information is encoded in the solution of the Voronoi-like partitioning problem. Simulation results are presented to highlight the theoretical developments.

1. Introduction

We present a pursuit strategy for the capture of a maneuvering target by a group of pursuers distributed in the plane. Typically, problems of group pursuit of a moving target (or an evader) are dealt with by employing cooperative or non-cooperative pursuit strategies, which are based on local or global information (Blagodatskikh, 2008, 2009; Bopardikar, Bullo, & Hespanha, 2009; Bopardikar, Smith, & Bullo, 2011; Guo, Yan, & Lin, 2010; Kim & Sugie, 2007; Petrov & Shuravina, 2009; Pittsyk & Chikrii, 1982; Rappoport & Chikrii, 1997; Wang, Cruz, Chen, Pham, & Blasch, 2007). One common theme in all these approaches is that more than one pursuer is actively participating in the process of simultaneously capturing the target. In many applications, however, a more "frugal" assignment of tasks within the pursuers' group may constitute a more prudent strategy. For example, in the problem of pursuit of a moving target by a group of agents guarding a certain area, the guards may be required to remain close to their initial positions owing to fuel or power requirements, or to account for possible deceptive strategies, decoy targets, etc. In this paper, we propose a *relay pursuit* scheme, that is, a group pursuit strategy, where, at each instant of time, only one pursuer is assigned the task of capturing the moving target, whereas all other pursuers in the group remain stationary. The optimal pursuer–target assignment, at each instant of time, follows from the solution of a Voronoi-like partitioning problem with respect to a generalized, state-dependent proximity metric, namely, the minimum intercept time.

In this paper we consider the following partitioning problem: Given a team of n pursuers, who are distributed over n distinct locations in the plane, partition the plane into n "capture zones", such that each pursuer is assigned to a unique capture zone. The rule that assigns each pursuer to a capture zone is the following: a pursuer associated with a particular capture zone can intercept a target moving within the same zone, at a given instant of time, faster than any other pursuer from the given group of pursuers. The moving target is not constrained to follow a prescribed trajectory (Devillers, Golin, Kedem, & Schirra, 1996); instead, it can maneuver aiming at delaying or, if possible, avoiding capture. Henceforth, we shall refer to the previous partitioning problem as the Optimal Pursuit Dynamic Voronoi Diagram (OP–DVD) problem. The OP–DVD problem belongs to the class of dynamic Voronoi diagram problems, that is, Voronoi-like partitioning problems where the generators are moving points in the plane (Albers, Guibas, Mitchell, & Roos, 1998; Bakolas & Tsiotras, 2010a,b; Devillers et al., 1996; Okabe, Boots, Sugihara, & Chiu, 2000; Roos, 1998). Applications of dynamic Voronoi-like partitions in multi-agent problems can be found, for example, in Bakolas and Tsiotras (2010a,b), Cortés and Bullo (2005) and Cortes, Martinez, and Bullo (2005).

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In contrast to our previous treatment of similar partitioning problems (Bakolas & Tsiotras, 2010a), where each pursuer had a priori knowledge of the "evading" strategy of the target (the so-called problem of *pursuit with anticipation* Hajek (2008)), in the current framework, the pursuers have only partial knowledge of the evading strategy of the maneuvering target. In particular, it is assumed that both the pursuer and the target can only measure their respective relative position. Neither the pursuer nor the target have information about the instantaneous velocity input (i.e., the action strategies) of the other, in contrast to the information pattern that is typically assumed in pursuit–evasion games (Friedman, 2006; Hajek, 2008; Isaacs, 1999; Matsumoto, 1975). It is shown that, under some mild assumptions on the structure of the target's strategy, the globally optimal control strategy for each pursuer can be characterized in feedback form. It is further demonstrated that the solution of the OP–DVD problem can be associated with the standard Voronoi diagram generated by the initial positions of the pursuers. Finally, we introduce a relay pursuit strategy derived from the (time varying) proximity relations between the maneuvering target and the group of pursuers, which are encoded in the solution of the OP–DVD problem.

2. The optimal pursuit problem

2.1. Problem formulation

Consider a team of n pursuers located, at time $t = 0$, at n distinct points in the plane, denoted by $\mathscr{P} := \{\bar{\mathrm{x}}^i_{\mathscr{P}} \in \mathbb{R}^2, i \in \mathcal{I}\}$, where $\mathcal{I} := \{1, \ldots, n\}$. The kinematics of the ith pursuer, where $i \in \mathcal{I}$, are given by

$$\dot{\mathrm{x}}^i_{\mathscr{P}} = u^i_{\mathscr{P}}, \qquad \mathrm{x}^i_{\mathscr{P}}(0) = \bar{\mathrm{x}}^i_{\mathscr{P}}, \tag{1}$$

where $\mathrm{x}^i_{\mathscr{P}} := (x^i_{\mathscr{P}}, y^i_{\mathscr{P}}) \in \mathbb{R}^2$ and $\bar{\mathrm{x}}^i_{\mathscr{P}} := (\bar{x}^i_{\mathscr{P}}, \bar{y}^i_{\mathscr{P}}) \in \mathbb{R}^2$ denote the position vectors of the ith pursuer at time t and time $t = 0$, respectively, and $u^i_{\mathscr{P}}$ is the control input of the ith pursuer. We assume that $u^i_{\mathscr{P}} \in \mathcal{U}_{\mathscr{P}}$, where $\mathcal{U}_{\mathscr{P}}$ consists of all piecewise continuous functions taking values in the set $U_{\mathscr{P}} := \{\mathrm{z} \in \mathbb{R}^2 : |\mathrm{z}| \leq \bar{u}_{\mathscr{P}}\}$, where $\bar{u}_{\mathscr{P}}$ is a positive constant (the maximum allowable speed of the pursuers). The goal of each pursuer, located initially at a point in $\mathscr{P}$, is to capture a moving target detected in its vicinity. It is assumed that the kinematics of such a moving target are described by

$$\dot{\mathrm{x}}_{\mathcal{T}} = u_{\mathcal{T}}, \qquad \mathrm{x}_{\mathcal{T}}(0) = \bar{\mathrm{x}}_{\mathcal{T}}, \tag{2}$$

where $\mathrm{x}_{\mathcal{T}} := (x_{\mathcal{T}}, y_{\mathcal{T}}) \in \mathbb{R}^2$ and $\bar{\mathrm{x}}_{\mathcal{T}} := (\bar{x}_{\mathcal{T}}, \bar{y}_{\mathcal{T}}) \in \mathbb{R}^2$ denote the target's position vectors at time t and time $t = 0$, respectively, and $u_{\mathcal{T}}$ is the control input of the target. It is assumed that the target employs a feedback evading strategy, which depends on the relative position of the target from the ith pursuer, that is, $u_{\mathcal{T}} = u_{\mathcal{T}}(\mathrm{x}_{\mathcal{T}} - \mathrm{x}^i_{\mathscr{P}})$. Furthermore, let $\mathrm{x}_{\mathcal{T}}(\cdot; u_{\mathcal{T}}, \bar{\mathrm{x}}_{\mathcal{T}})$ and $\mathrm{x}^i_{\mathscr{P}}(\cdot; u^i_{\mathscr{P}}, \bar{\mathrm{x}}^i_{\mathscr{P}})$ denote, respectively, the trajectories of the target and the ith pursuer using $u_{\mathcal{T}}$ and $u^i_{\mathscr{P}}$ as control inputs, originating from $\bar{\mathrm{x}}_{\mathcal{T}}$ and $\bar{\mathrm{x}}^i_{\mathscr{P}}$ for the target and pursuer, respectively. The objective of each pursuer is to determine an admissible pursuit strategy that minimizes the time T_{f} such that $|\mathrm{x}_{\mathcal{T}}(t; u_{\mathcal{T}}, \bar{\mathrm{x}}_{\mathcal{T}}) - \mathrm{x}^i_{\mathscr{P}}(t; u^i_{\mathscr{P}}, \bar{\mathrm{x}}^i_{\mathscr{P}})| > \epsilon_c$ for all $t < T_{\mathrm{f}}$ (*time of first capture*), for a sufficiently small $\epsilon_c > 0$, where ϵ_c is the *capturability radius* of the pursuit problem.

Assumption 1. There exists a Lipschitz continuous function $f : [\epsilon_c, \infty) \mapsto \mathbb{R}$ such that the evading strategy $u_{\mathcal{T}}$ of the target satisfies the following condition

$$\langle u_{\mathcal{T}}, \mathrm{x}_{\mathcal{T}} - \mathrm{x}^i_{\mathscr{P}} \rangle = f(|\mathrm{x}_{\mathcal{T}} - \mathrm{x}^i_{\mathscr{P}}|), \tag{3}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$.

The interpretation of Assumption 1 is as follows: The projection of the velocity vector of the maneuvering target on its relative position vector from the ith pursuer depends only on the relative distance between the two. This is a reasonable assumption for problems of pursuit when only measurements of the relative position between the pursuer and the target are available to both of them. In addition, in this work we do not explicitly assume that the maximum allowable speed of the target is strictly less than the speed of the pursuer. Note that if we were dealing with a problem of pursuit–evasion (Friedman, 2006; Hajek, 2008; Isaacs, 1999), rather than a problem of pursuit of a moving/maneuvering target, then the assumption that the evader may travel faster than its pursuer would automatically mean that the evader can always escape capture (Hajek, 2008). Capture for the case of a faster target can occur only if the target follows a suboptimal evading strategy. In such a case, capture may (but not necessarily) still occur for some initial conditions that belong to a non-trivial subset of $\mathbb{R}^2$. In Section 3 we characterize the *winning set* of the ith pursuer, that is, the set of initial positions of the maneuvering target from which the ith pursuer can capture the target in finite time. As we shall see in more detail later in the paper, under Assumption 1 along with the following condition

$$f(\mathrm{z}) \leq \bar{f}(\mathrm{z}), \quad \text{for all } \mathrm{z} \geq \epsilon_c, \tag{4}$$

where $\bar{f} : [\epsilon_c, \infty) \mapsto \mathbb{R}$ is a continuous function that is known to all of the pursuers, we will be able to estimate the winning set of the ith pursuer. Note that the winning set against a slower target is always the whole $\mathbb{R}^2$, regardless of whether the target plays optimally or not.

2.2. Optimal feedback pursuit strategy

Let the state transformation $\mathrm{y}^i := \mathrm{x}_{\mathcal{T}} - \mathrm{x}^i_{\mathscr{P}}$. Eq. (1) can then be written in the following compact form

$$\dot{\mathrm{y}}^i = \mathrm{u}^i + u_{\mathcal{T}}(\mathrm{y}^i), \qquad \mathrm{y}^i(0) = \bar{\mathrm{y}}^i := \bar{\mathrm{x}}_{\mathcal{T}} - \bar{\mathrm{x}}^i_{\mathscr{P}}, \tag{5}$$

where $\mathrm{u}^i := -u^i_{\mathscr{P}}$. Next, we formulate the optimal pursuit problem for the ith pursuer.

Problem 1. Let the system described by equation (5), and let $u_{\mathcal{T}}$ satisfy Assumption 1. Determine the control input $\mathrm{u}^i \in \mathcal{U}_{\mathscr{P}}$ such that

(i) The trajectory $\mathrm{y}^i_* : [0, T_{\mathrm{f}}] \mapsto \mathbb{R}^2$ generated by the control u^i_* satisfies the boundary conditions $\mathrm{y}^i_*(0) = \bar{\mathrm{y}}^i$ and $|\mathrm{y}^i_*(T_{\mathrm{f}})| \leq \epsilon_c$.
(ii) The control u^i_* minimizes, along the trajectory y^i_*, the cost functional $J(\mathrm{u}^i) := T_{\mathrm{f}}(\bar{\mathrm{y}}^i)$.

Problem 1 can be interpreted as a problem of steering a single integrator from $\bar{\mathrm{y}}^i$ to a ball of radius ϵ_c centered at the origin, in the presence of a spatially-varying drift $u_{\mathcal{T}}(\mathrm{y}^i)$, which is not precisely known, in minimum time.

Proposition 1. *If Problem 1 is feasible, then its solution is unique, and it is given in feedback form as follows*

$$\mathrm{u}^i_* = -\bar{u}_{\mathscr{P}} \mathrm{y}^i_* / |\mathrm{y}^i_*|. \tag{6}$$

Proof. Let $|\mathrm{y}^i|^2 = \langle \mathrm{y}^i, \mathrm{y}^i \rangle$ and suppose that y^i is the trajectory generated using the admissible control u^i on $[0, T_{\mathrm{f}}]$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}|\mathrm{y}^i|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{y}^i, \mathrm{y}^i \rangle = 2\langle \mathrm{y}^i, \mathrm{u}^i + u^i_{\mathcal{T}}(\mathrm{y}^i) \rangle. \tag{7}$$

First, we show that $\eta^i(t) := |\mathrm{y}^i(t)| > 0$, for all $t \in [0, T_{\mathrm{f}}]$. Indeed, let us assume that $|\bar{\mathrm{y}}^i| > \epsilon_c$ (if $|\bar{\mathrm{y}}^i| \leq \epsilon_c$, then Problem 1 admits a trivial solution and $T_{\mathrm{f}} = 0$). By continuity, if $\eta^i(t_1) = 0$ for some $t_1 > 0$, then there exists $t_2 < t_1$ such that $\eta^i(t_2) = \epsilon_c$. By definition, $T_{\mathrm{f}} = \inf\{\tau : \eta^i(\tau) = \epsilon_c\}$. It follows that $T_{\mathrm{f}} \leq t_2 < t_1$ and hence $\eta^i(t) \geq \epsilon_c > 0$, for all $t \in [0, T_{\mathrm{f}}]$.

In light of Assumption 1 and equations (5) and (7), it follows that, for all $t \in [0, T_{\mathrm{f}}]$,

$$\dot{\eta}^i = f(\eta^i)/\eta^i + v^i, \qquad \eta^i(0) = \bar{\eta}^i := |\bar{\mathrm{y}}^i|, \tag{8}$$

where υ^i is a new scalar control input given by $\upsilon^i := \langle \mathrm{u}^i, \mathrm{y}^i \rangle / \eta^i$. Note that the right hand side of Eq. (8) is well-defined, and thus $\dot{\eta}^i(t)$ exists for all $t \in [0, T_f]$. In addition, by virtue of the Cauchy–Schwartz inequality, it follows that $|\upsilon^i| \leq \bar{u}_{\mathcal{P}}$. Therefore, Problem 1 reduces to the problem of determining a scalar control υ_*^i with $|\upsilon_*^i| \leq \bar{u}_{\mathcal{P}}$ that will steer the scalar system described by equation (8) to the interval $[0, \epsilon_c]$ in minimum time. In Athans and Falb (1963), it is shown that the solution to this scalar min-time problem is given by $\upsilon_*^i = -\bar{u}_{\mathcal{P}}$. Therefore,

$$\langle \mathrm{u}_*^i, \mathrm{y}_*^i \rangle = -\bar{u}_{\mathcal{P}} \eta_*^i = -\bar{u}_{\mathcal{P}} |\mathrm{y}_*^i|, \tag{9}$$

which implies that u_*^i is a vector of length $\bar{u}_{\mathcal{P}}$ parallel to the unit vector $-\mathrm{y}_*^i/|\mathrm{y}_*^i|$, thus completing the proof. □

Proposition 1 implies that the solution of the optimal control Problem 1 is independent of the evading strategy of the target $u_{\mathcal{T}}$. In particular, the optimal strategy of Problem 1 turns out to be a "pure" pursuit strategy (Nahin, 2007).

3. The winning sets of the pursuers

Next, we examine the feasibility of Problem 1 for a given $\bar{\mathrm{y}}^i \in \mathbb{R}^2$. This will allow us to characterize the winning set of the ith pursuer, that is, the set of the initial positions of the target from which it can be captured by the ith pursuer in finite time. In other words, the winning set of the ith pursuer is given by

$$\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i) := \{\mathrm{x} \in \mathbb{R}^2 : T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i) < \infty\}, \tag{10}$$

where $T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i)$ is the time of capture of the target by the ith pursuer, when $\bar{\mathrm{x}}_{\mathcal{T}} = \mathrm{x}$. First, note that if $|\bar{\mathrm{y}}^i| \leq \epsilon_c$, then capture occurs trivially at $t = 0$. Hence, the set $\{\mathrm{x} \in \mathbb{R}^2 : |\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i| \leq \epsilon_c\}$ is necessarily a subset of the winning set for each pursuer, regardless of the dynamics of the pursuer or the target. Next, we compute the winning set for the non-trivial case $|\bar{\mathrm{y}}^i| > \epsilon_c$.

Proposition 2. *Let $\epsilon_c > 0$. Then Problem 1 is feasible for the ith pursuer, for all $|\bar{\mathrm{y}}^i| > \epsilon_c$, if and only if*

$$f(\mathrm{z}) < \bar{u}_{\mathcal{P}} \mathrm{z}, \quad \textit{for all } \epsilon_c \leq \mathrm{z} \leq |\bar{\mathrm{y}}^i|. \tag{11}$$

Proof. First we show that (11) implies the feasibility of the Problem 1. It follows from the proof of Proposition 1 and, in particular, from (7) to (9), that the closed loop dynamics of (5) with (6) can be written in terms of $\eta^i = |\mathrm{y}^i|$ as follows

$$\dot{\eta}^i = f(\eta^i)/\eta^i - \bar{u}_{\mathcal{P}}, \qquad \eta^i(0) = \bar{\eta}^i. \tag{12}$$

It follows from Condition (11) that $\dot{\eta}^i = f(\eta^i)/\eta^i - \bar{u}_{\mathcal{P}} < 0$, for all $\epsilon_c \leq \eta^i \leq |\bar{\mathrm{y}}^i|$, which implies, in turn, that the set $\{\mathrm{z} : 0 < \mathrm{z} \leq \epsilon_c\}$ is an attractive (positively) invariant set for (12), for all initial conditions $\eta^i(0) > \epsilon_c$. Furthermore, $\dot{\eta}^i < 0$ for $\eta^i = \epsilon_c$. It follows that there exists $T = T(\epsilon_c)$, such that $\eta^i(t) \leq \epsilon_c$ for $t \geq T(\epsilon_c)$, thus showing feasibility of the Problem 1.

Next, we show that the feasibility of the Problem 1 implies (11). Assume, on the contrary, that there exists $\tilde{\eta}^i = |\tilde{\mathrm{y}}|$, where $\tilde{\mathrm{y}} \in \mathbb{R}^2$, such that $\epsilon_c \leq \tilde{\eta}^i \leq |\bar{\mathrm{y}}^i|$ and $f(\tilde{\eta}^i) \geq \bar{u}_{\mathcal{P}} \tilde{\eta}^i$. Notice that the set $S := \{\mathrm{z} : \mathrm{z} \geq \tilde{\eta}^i\}$ is invariant for (12) since $f(\mathrm{z})/\mathrm{z} - \bar{u}_{\mathcal{P}} \geq 0$ for all $\mathrm{z} \in \mathrm{bd}S$, where $\mathrm{bd}S$ denotes the boundary of S. Since $\eta^i(0) \in S$, it follows that $\eta^i(t) \geq \tilde{\eta}^i$, for all $t \geq 0$, which implies that the Problem 1 is not feasible for $\epsilon_c < \tilde{\eta}^i$. If, on the other hand, $\epsilon_c = \tilde{\eta}^i$, then either $f(\epsilon_c) > \bar{u}_{\mathcal{P}} \epsilon_c$ or $f(\epsilon_c) = \bar{u}_{\mathcal{P}} \epsilon_c$. In the first case, any trajectory starting from $\eta^i(0) > \epsilon_c$ can never reach the ball $\{\mathrm{z} \in \mathbb{R} : |\mathrm{z}| \leq \epsilon_c\}$. In the second case, $\eta^i = \epsilon_c$ is an equilibrium solution for (12). Since the right hand side of (12) is Lipschitz continuous at $\eta^i = \epsilon_c$, this equilibrium can only be reached asymptotically (Bhat & Bernstein, 1998). In both cases, Problem 1 is infeasible. Thus we have reached a contradiction. □

Henceforth, we refer to (11) as the *capturability condition* of Problem 1. In order to characterize the winning set of the ith pursuer, let

$$\bar{\eta}_f := \inf\{\mathrm{z} \in [\epsilon_c, \infty) : f(\mathrm{z}) \geq \bar{u}_{\mathcal{P}} \mathrm{z}\}. \tag{13}$$

Note that $\bar{\eta}_f \geq \epsilon_c$. If $f(\mathrm{z}) < \bar{u}_{\mathcal{P}} \mathrm{z}$, for all $\mathrm{z} \in [\epsilon_c, \infty)$, it follows that $\bar{\eta}_f = \infty$, and hence $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i) = \mathbb{R}^2$. If $f(\mathrm{z}) \geq \bar{u}_{\mathcal{P}} \mathrm{z}$, for all $\mathrm{z} \in [\epsilon_c, \infty)$, we take $\bar{\eta}_f := \epsilon_c$, and hence $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i) = \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| \leq \epsilon_c\}$. Finally, if $\epsilon_c < \bar{\eta}_f < \infty$, then it follows readily from (13) that $f(\mathrm{z}) < \bar{u}_{\mathcal{P}} \mathrm{z}$, for all $\epsilon_c \leq \mathrm{z} < \bar{\eta}_f$, and hence, in light of Proposition 2, $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i) := \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| < \bar{\eta}_f\}$. For all cases, the winning set of the ith pursuer can be defined compactly as $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i) := \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| < \bar{\eta}_f\} \cup \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| \leq \epsilon_c\}$. If the target starts outside the set $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$, then the relative distance between the target and the ith pursuer will increase with a rate that the ith pursuer will not be able to compensate. In this case, capture will not take place. The opposite holds true when $\bar{\mathrm{x}}_{\mathcal{T}} \in \mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$. Note, however, that the ith pursuer does not know exactly its winning set, since it has only partial knowledge of f, and consequently of $\bar{\eta}_f$ as well. As a result, each pursuer can only compute an approximation of its actual winning set. To this end, let $\bar{\eta}_{\bar{f}}$ be defined as $\bar{\eta}_f$ in (13) modulo the replacement of f by $\bar{f}$. In light of (4), it follows that $\bar{\eta}_{\bar{f}} \leq \bar{\eta}_f$. Let $\mathcal{W}_{\bar{f}}(\bar{\mathrm{x}}_{\mathcal{P}}^i) := \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| < \bar{\eta}_{\bar{f}}\} \cup \{\mathrm{x} \in \mathbb{R}^2 : |\bar{\mathrm{x}}_{\mathcal{P}}^i - \mathrm{x}| \leq \epsilon_c\}$. Clearly, $\mathcal{W}_{\bar{f}}(\bar{\mathrm{x}}_{\mathcal{P}}^i) \subseteq \mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$. Hence, $\mathcal{W}_{\bar{f}}(\bar{\mathrm{x}}_{\mathcal{P}}^i)$ is a conservative approximation of the winning set $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$. Note that, contrary to $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$, the ith pursuer has perfect knowledge of $\mathcal{W}_{\bar{f}}(\bar{\mathrm{x}}_{\mathcal{P}}^i)$. Furthermore, the closeness of the approximation of the winning set of the ith pursuer with $\mathcal{W}_{\bar{f}}(\bar{\mathrm{x}}_{\mathcal{P}}^i)$ depends on the difference $\bar{\eta}_f - \bar{\eta}_{\bar{f}}$.

4. The dynamic Voronoi partitioning problem

4.1. Problem formulation

Next, we formulate a dynamic Voronoi-like partitioning problem based on the minimum time of Problem 1, which will allow us to assign a specific pursuer starting from the set $\mathcal{P}$ to a target moving in the plane. The space we wish to partition, denoted henceforth by $\mathcal{W}$, is the union of all $\mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^i)$, where $i \in \mathcal{I}$. Note that if $\bar{\eta}_f < \infty$, then $\mathcal{W}$ is a proper subset of $\mathbb{R}^2$. The set $\mathbb{R}^2 \setminus \mathcal{W}$ consists of all the positions from which the target cannot be captured by any pursuer starting from the set $\mathcal{P}$.

Problem 2. Given a collection of n pursuers, initially located at distinct points in $\mathcal{P} := \{\bar{\mathrm{x}}_{\mathcal{P}}^i \in \mathbb{R}^2 : i \in \mathcal{I}\}$, where $\min_{i,j \in \mathcal{I}} |\bar{\mathrm{x}}_{\mathcal{P}}^i - \bar{\mathrm{x}}_{\mathcal{P}}^j| > 2\epsilon_c$, for all $j \neq i$, and the cost function $T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i)$, for $i \in \mathcal{I}$, where T_f is the minimum time from Problem 1, determine a partition $\mathcal{V} := \{\mathcal{V}^i : i \in \mathcal{I}\}$ of $\mathcal{W}$ such that

(i) $\mathcal{W} = \bigcup_{i \in \mathcal{I}} \mathcal{V}^i$,
(ii) $T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i) < \infty$, for all $\mathrm{x} \in \mathcal{V}^i$,
(iii) $T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^i) \leq T_f(\mathrm{x} - \bar{\mathrm{x}}_{\mathcal{P}}^j)$, for all $\mathrm{x} \in \mathcal{V}^i$ and $j \neq i$.

Henceforth, we shall refer to the solution of Problem 2 as the Optimal Pursuit–Dynamic Voronoi Diagram (OP–DVD). The set $\mathcal{V}^i \in \mathcal{V}$, constitutes a Voronoi cell (Dirichlet domain) of the OP–DVD. We say that the ith and jth pursuers, where $i, j \in \mathcal{I}$, are neighbors in OP–DVD if and only if the set $\mathcal{V}^i \cap \mathcal{V}^j$ is neither nonempty nor a singleton. Because the evading strategy of any moving target is not perfectly known, we can only provide approximate solutions to Problem 2. Next, we present an efficient scheme for the construction of an approximate OP–DVD derived directly from the standard Voronoi diagram generated by the set $\mathcal{P}$.

4.2. Construction of an approximate OP–DVD

In this section we show that the minimum time of Problem 1 belongs to a class of generalized metrics that can be associated with Voronoi-like partitions, for which efficient computational techniques exist in the literature (Okabe et al., 2000).

Since $\bar{\eta}_f \geq \epsilon_c$, direct integration of Eq. (12) yields

$$T_f(\bar{y}^i) := \begin{cases} 0, & \text{if } |\bar{y}^i| \leq \epsilon_c, \\ \displaystyle\int_{\epsilon_c}^{|\bar{y}^i|} \frac{\mu \, d\mu}{\bar{u}_{\mathcal{P}}\mu - f(\mu)}, & \text{if } \epsilon_c < |\bar{y}^i| < \bar{\eta}_f, \\ \infty, & \text{otherwise.} \end{cases} \tag{14}$$

Note, in particular, that if $\bar{\eta}_f = \epsilon_c$, then capture takes place only for all initial conditions $|\bar{y}^i| \leq \epsilon_c$. Moreover, in this case $T_f(\bar{y}^i) = 0$. In order to streamline the presentation, we shall henceforth restrict our discussion to the non-trivial case $\bar{\eta}_f > \epsilon_c$.

The following result will be useful in the subsequent analysis.

Proposition 3. *Let $\bar{\eta}_f > \epsilon_c$. Given two points $\xi, \psi \in \mathbb{R}^2$, with $|\xi|, |\psi| \in (\epsilon_c, \bar{\eta}_f)$, the minimum time of Problem 1 satisfies $0 < T_f(\xi) < T_f(\psi) < \infty$ if and only if $\epsilon_c < |\xi| < |\psi| < \bar{\eta}_f$, and, furthermore, $0 < T_f(\xi) = T_f(\psi) < \infty$ if and only if $\epsilon_c < |\xi| = |\psi| < \bar{\eta}_f$.*

Proof. First, notice that the minimum time of Problem 1 satisfies

$$T_f(\psi) - T_f(\xi) = \int_{|\xi|}^{|\psi|} \phi(\mu)\, d\mu, \quad \phi(\mu) := \frac{\mu}{\bar{u}_{\mathcal{P}}\mu - f(\mu)}.$$

The function $\phi : (\epsilon_c, \bar{\eta}_f) \mapsto \mathbb{R}$ is continuous and strictly positive on $(\epsilon_c, \bar{\eta}_f)$. From the mean value theorem for Riemann integrals (Bartle, 1976), it follows that there exists $\epsilon_c < |\xi| \leq \zeta \leq |\psi| < \bar{\eta}_f$ such that

$$T_f(\psi) - T_f(\xi) = \int_{|\xi|}^{|\psi|} \phi(\mu)\, d\mu = \phi(\zeta)(|\psi| - |\xi|). \tag{15}$$

Since $\phi(\zeta) > 0$, for all $\epsilon_c < \zeta < \bar{\eta}_f$, the result follows readily. □

Next, we present the solution of Problem 2.

Theorem 1. *Let $V := \{V^i, i \in \mathcal{I}\}$ be the standard Voronoi partition generated by the set $\mathcal{P}$, and assume that $\bar{\eta}_f > \epsilon_c$. The solution of Problem 2 is given by*

$$\mathcal{V}^i = V^i \cap \mathcal{W}_f(\bar{x}^i_{\mathcal{P}}), \quad i \in \mathcal{I}, \tag{16}$$

where $\mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$ is the winning set of the ith pursuer.

Proof. Let $x \in V^i \cap \mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$. In particular, $x \in V^i$ if and only if $|x - \bar{x}^i_{\mathcal{P}}| \leq |x - \bar{x}^j_{\mathcal{P}}|$, for all $j \neq i$, which implies, in light of Proposition 3, that $T_f(x-\bar{x}^i_{\mathcal{P}}) \leq T_f(x-\bar{x}^j_{\mathcal{P}})$ for all $i \neq j$. Furthermore, if $x \in \mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$ then $T_f(x - \bar{x}^i_{\mathcal{P}}) < \infty$. It follows that $x \in \mathcal{V}^i$ and hence $V^i \cap \mathcal{W}_f(\bar{x}^i_{\mathcal{P}}) \subseteq \mathcal{V}^i$, for all $i \in \mathcal{I}$.

Next, assume $x \in \mathcal{V}^i$. By the definition of $\mathcal{V}^i$, it follows that $T_f(x - \bar{x}^i_{\mathcal{P}}) < \infty$ and $T_f(x - \bar{x}^i_{\mathcal{P}}) \leq T_f(x - \bar{x}^j_{\mathcal{P}})$, for all $j \neq i$. If $0 < T_f(x - \bar{x}^i_{\mathcal{P}}) \leq T_f(x - \bar{x}^j_{\mathcal{P}}) < \infty$, it follows from Proposition 3 that $|x - \bar{x}^i_{\mathcal{P}}| \leq |x - \bar{x}^j_{\mathcal{P}}|$, for all $j \neq i$. The same is true if $T_f(x - \bar{x}^j_{\mathcal{P}}) = \infty$, since in this case $|x - \bar{x}^j_{\mathcal{P}}| \geq \bar{\eta}_f > |x - \bar{x}^i_{\mathcal{P}}|$. Additionally, if $T_f(x - \bar{x}^j_{\mathcal{P}}) = 0$, then $T_f(x - \bar{x}^i_{\mathcal{P}}) \leq T_f(x - \bar{x}^j_{\mathcal{P}}) = 0$, which implies that $T_f(x - \bar{x}^i_{\mathcal{P}}) = T_f(x - \bar{x}^j_{\mathcal{P}}) = 0$, and thus, in light of (14) and Proposition 3, it follows that $|x - \bar{x}^j_{\mathcal{P}}| \leq \epsilon_c$ and $|x - \bar{x}^i_{\mathcal{P}}| \leq \epsilon_c$. From the triangle inequality, the last statement implies that $|\bar{x}^i_{\mathcal{P}} - \bar{x}^j_{\mathcal{P}}| \leq 2\epsilon_c$, which violates one of the hypotheses of Problem 2. Thus, in all cases, $T_f(x - \bar{x}^i_{\mathcal{P}}) \leq T_f(x - \bar{x}^j_{\mathcal{P}})$ implies that $|x - \bar{x}^i_{\mathcal{P}}| \leq |x - \bar{x}^j_{\mathcal{P}}|$, for all $j \neq i$ and $x \in \mathcal{V}^i$. Thus $x \in V^i$. Furthermore, since $T_f(x - \bar{x}^i_{\mathcal{P}}) < \infty$, then $x \in \mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$. Hence $x \in V^i \cap \mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$ and $\mathcal{V}^i \subseteq V^i \cap \mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$, for $i \in \mathcal{I}$. □

Theorem 1 suggests that the ith element of the partition that solves Problem 2 is the intersection of the winning set of the ith pursuer with the cell of the standard Voronoi diagram generated by the set $\mathcal{P}$ that is associated with the generator $\bar{x}^i_{\mathcal{P}}$. Note that the OP–DVD encodes the proximity relations between a target and the pursuers with respect to the time of capture.

Theorem 1 provides an efficient way for the construction of the exact OP–DVD provided, however, that the sets $\mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$, where $i \in \mathcal{I}$, are perfectly known. However, f is assumed to be unknown, hence the sets $\mathcal{W}_f(\bar{x}^i_{\mathcal{P}})$ are also not known to the pursuers. Only a conservative approximation of each winning set is known. This approximation uses the upper bound $\bar{f}$ in lieu of f for the construction of these sets. Therefore, an approximate solution of Problem 2 is given by $\widetilde{\mathcal{V}} := \{\widetilde{\mathcal{V}}^i, i \in \mathcal{I}\}$, where $\widetilde{\mathcal{V}}^i = V^i \cap \mathcal{W}_{\bar{f}}(\bar{x}^i_{\mathcal{P}})$, $i \in \mathcal{I}$, where $V := \{V^i : i \in \mathcal{I}\}$ is the standard Voronoi partition generated by the set $\mathcal{P}$.

5. The dynamic pursuer–target assignment problem and relay-pursuit

5.1. Problem formulation

Next, we formulate the dynamic pursuer–target assignment problem. To this end, assume that $\bar{x}_{\mathcal{T}} \in \mathcal{W}$. Without loss of generality,[2] let $\bar{x}_{\mathcal{T}} \in \operatorname{int} \mathcal{V}^i$ for some $i \in \mathcal{I}$. By assigning the target to the ith pursuer and requiring that all other pursuers in the group remain stationary, capture will occur after $T_f(\bar{y}^i)$ units of time. In this *static* pursuer–target assignment scheme, the ith pursuer is the only active pursuer during the course of the pursuit.

In this section, we wish to explore the following question: "Is it possible to expedite the capture of the moving target by dynamically changing the assignment of the active pursuer?" To this end, let $\mathcal{S}$ be the family of right continuous, piecewise constant signals $\sigma : [0, \infty) \mapsto \mathcal{I}$, such that $\sigma(t) = i$ implies that the ith pursuer, at time $t \geq 0$, is the (only) active pursuer; subsequently, we write $x^i_{\mathcal{P}} \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$ to denote this fact. The dynamics of the pursuit problem can then be described by the following switched system (Liberzon, 2003)

$$\dot{y}^{\sigma(t)} = u_{\mathcal{T}}(y^{\sigma(t)}) - \bar{u}_{\mathcal{P}} y^{\sigma(t)}/|y^{\sigma(t)}|, \qquad \dot{y}^j = 0, \tag{17}$$

where $j \neq \sigma(t)$, $y^{\sigma(0)}(0) = \bar{y}^{\sigma(0)}$, $y^j(0) = \bar{y}^j$, and $\sigma(0) = \operatorname{argmin}_{i \in \mathcal{I}} T_f(\bar{y}^i)$. If, in addition, $0 < \tau_1 < \cdots < \tau_k < \cdots < \infty$ are the switching times of the signal σ, then $y^{i_k}(\tau_k) = y^{i_k}(\tau_k^-)$ where $i_k := \sigma(\tau_k) = \sigma(\tau_k^+)$.

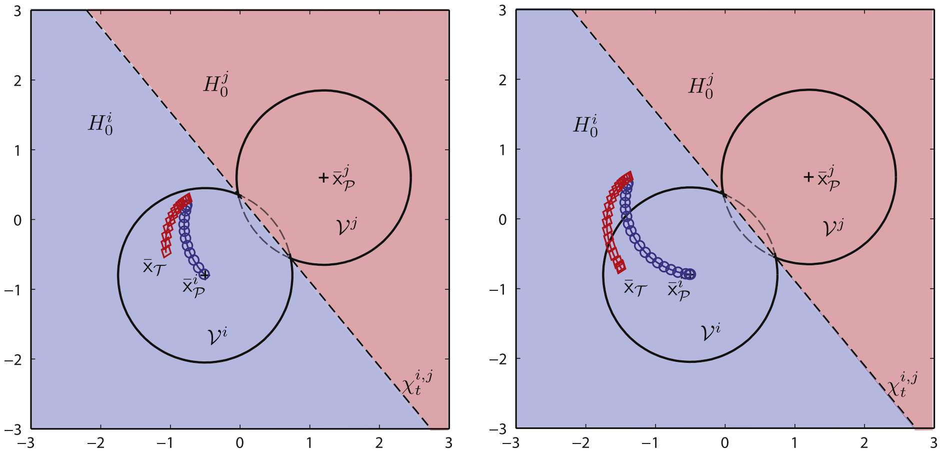
Given $\sigma \in \mathcal{S}$, let $\varphi(t; t_0, y_0, \sigma)$ denote the solution of (17) for $t \geq t_0 \geq 0$ and $y_0 = \varphi(t_0; 0, \bar{y}^{\sigma(0)}, \sigma)$. In addition, we define the minimum capture time as follows $\mathrm{T}(t_0, y_0; \sigma) := \inf\{t \geq t_0 : |\varphi(t; t_0, y_0, \sigma)| \leq \epsilon_c\}$. It follows readily that $\mathrm{T}(t, y^{\sigma(t)}(t); \sigma) = \mathrm{T}(t_0, y_0; \sigma) - (t - t_0)$, for all $t \geq t_0$, and hence, if $\sigma(t) \equiv i$, then $T_f(\bar{y}^i) = \mathrm{T}(0, \bar{y}^i; i) = \mathrm{T}(t, y^i(t); i) + t$, for all $t \geq 0$.

We will restrict the family of acceptable switching signals to a subset of $\mathcal{S}$, which includes all the signals in $\mathcal{S}$ that satisfy the following switching condition.

Switching condition. Let $\sigma \in \mathcal{S}$ and let $\tau > 0$ be a switching time, such that $i = \sigma(\tau^-)$ and $j = \sigma(\tau^+) = \sigma(\tau)$, where $j \neq i$. Then $\sigma \in \Sigma \subset \mathcal{S}$, if the following conditions hold:

(i) $x_{\mathcal{T}}(\tau) \in \operatorname{int} \mathcal{V}^j$.

[2] If $\bar{x}_{\mathcal{T}} \in \bigcap_{i \in \mathcal{J}} \mathcal{V}^i$, where $\mathcal{J} \subseteq \mathcal{I}$, we may assign as the initial pursuer any one of the elements of $\mathcal{J}$.



(a) Capture occurs in $\mathcal{V}^i$.

(b) Capture occurs in the complement of $\mathcal{V}^i \cup \mathcal{V}^j$.

Fig. 1. If $x_{\mathcal{P}}^i \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$ and $x_{\mathcal{T}}(t) \notin \mathcal{V}^j$, for all $t \geq 0$, then $T(0, \bar{y}^i; \sigma) \geq T_f(\bar{y}^i)$, for all $\sigma \in \Sigma$.

(ii) $T(\tau, y^j(\tau); \sigma) < T(\tau, y^i(\tau); \tilde{\sigma})$, *where*

$$\tilde{\sigma}(t) = \begin{cases} \sigma(t), & t \in [0, \tau), \\ i, & t \geq \tau. \end{cases}$$

The previous condition can be interpreted as follows: For any $\sigma \in \Sigma$, the assignment $x_{\mathcal{P}}^i \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$, for $t \geq 0$, is updated only if during the course of the pursuit, the target reaches a position from which, say, the jth pursuer, where $j \neq i$, can capture the target faster than the ith pursuer.

Next, we formulate the dynamic pursuer–moving target assignment problem.

Problem 3. Let $\mathcal{V} = \{\mathcal{V}^i,\ i \in \mathcal{I}\}$ denote the OP–DVD generated by the set $\mathcal{P}$ and assume that $\bar{x}_{\mathcal{T}} \in \operatorname{int} \mathcal{V}^i$, for some $i \in \mathcal{I}$. Determine a switching signal $\sigma_\star \in \Sigma$ (if one exists) such that $T(0, \bar{y}^i, \sigma_\star) < T_f(\bar{y}^i) = T(0, \bar{y}^i; i)$.

5.2. Analysis of the pursuer–target assignment problem

Before proceeding to a detailed discussion on the characterization of a solution of Problem 3, we need to introduce a few geometric concepts. In particular, let $\chi_t^{i,j} \subseteq \mathbb{R}^2$ be the moving line in the plane, where $\chi_t^{i,j} := \{x \in \mathbb{R}^2 : |x - x_{\mathcal{P}}^i(t)| = |x - x_{\mathcal{P}}^j(t)|\}$, for $t \geq 0$. At every time instant $t \geq 0$, the line $\chi_t^{i,j}$ divides $\mathbb{R}^2$ into two open half-planes, namely, $H_t^i(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) := \{x \in \mathbb{R}^2 : |x - x_{\mathcal{P}}^i(t)| < |x - x_{\mathcal{P}}^j(t)|\}$ and $H_t^j(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) := \{x \in \mathbb{R}^2 : |x - x_{\mathcal{P}}^i(t)| > |x - x_{\mathcal{P}}^j(t)|\}$.

The following proposition provides a necessary and sufficient condition for the existence of a solution to Problem 3.

Proposition 4. *Let $\mathcal{V} = \{\mathcal{V}^i,\ i \in \mathcal{I}\}$ denote the OP–DVD generated by the set $\mathcal{P}$, and assume that $\bar{x}_{\mathcal{T}} \in \operatorname{int} \mathcal{V}^i$, for some $i \in \mathcal{I}$. Then $T(0, \bar{y}^i; \sigma) \geq T_f(\bar{y}^i)$, for all $\sigma \in \Sigma$, if and only if $x_{\mathcal{T}}(t) \notin H_t^j(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) \cap \operatorname{int} \mathcal{V}^j$, for all $j \neq i$ and all $t \geq 0$.*

Proof. First we show sufficiency. Let us assume, on the contrary, that there exists a switching signal $\sigma_\star \in \Sigma$ such that $T(0, \bar{y}^i; \sigma_\star) < T_f(\bar{y}^i)$. Clearly, $\sigma_\star \not\equiv i$. If $t_1 > 0$ is the first switching time of the signal $\sigma_\star$, then, in light of the Switching Condition, there exists $j \neq i$, such that $x_{\mathcal{T}}(t_1) \in \operatorname{int} \mathcal{V}^j$ and $T(t_1, y^j(t_1); \tilde{\sigma}) < T(t_1, y^i(t_1); i)$, where $\tilde{\sigma}(t) = \sigma_\star(t) = i$, for $t \in [0, t_1)$, and $\tilde{\sigma}(t) = j$, for $t \geq t_1$. Using a similar argument as in the proof of the converse part of Theorem 1, it follows that $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| < |x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^i(t_1)|$. Hence, $x_{\mathcal{T}}(t_1) \in H_{t_1}^j(x_{\mathcal{P}}^i(t_1), x_{\mathcal{P}}^j(t_1))$, leading to a contradiction.

Conversely, given that $T(0, \bar{y}^i; \sigma) \geq T_f(\bar{y}^i)$, for all $\sigma \in \Sigma$, we wish to show that $x_{\mathcal{T}}(t) \notin H_t^j(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) \cap \operatorname{int} \mathcal{V}^j$, for all $j \neq i$ and $t \geq 0$. Let us assume, on the contrary, that there exists $j \neq i$ and $0 < t_1 < T_f(\bar{y}^i)$ such that $x_{\mathcal{T}}(t_1) \in H_{t_1}^j(x_{\mathcal{P}}^i(t_1), x_{\mathcal{P}}^j(t_1)) \cap \operatorname{int} \mathcal{V}^j$ and let the signal $\sigma_\star \in \Sigma$ be defined such that $\sigma_\star(t) = i$, for $t \in [0, t_1)$, and $\sigma_\star(t) = j$, for $t \geq t_1$. Since $x_{\mathcal{T}}(t_1) \in H_{t_1}^j(x_{\mathcal{P}}^i(t_1), x_{\mathcal{P}}^j(t_1))$, it follows that $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| < |x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^i(t_1)|$. Note that necessarily $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| > \epsilon_c$, otherwise capture would occur at $t_1 < T_f(\bar{y}^i)$, contradicting the assumption that $T(0, \bar{y}^i; \sigma) \geq T_f(\bar{y}^i)$ for all $\sigma \in \Sigma$. Furthermore, by the definition of the OP–DVD, $x_{\mathcal{T}}(t_1) \in \operatorname{int} \mathcal{V}^j$ implies that $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| < \bar{\eta}_f$. Note that, if $\epsilon_c < |x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| < \bar{\eta}_f$ and $\epsilon_c < |x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^i(t_1)| < \bar{\eta}_f$, then it follows via Proposition 3 that $T(t_1, y^j(t_1); \sigma_\star) < T(t_1, y^i(t_1); i)$. Similarly, if $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^i(t_1)| > \bar{\eta}_f$, then it follows from (14) that $T(t_1, y^i(t_1); i) = \infty$. Since $x_{\mathcal{T}}(t_1) \in \operatorname{int} \mathcal{V}^j$, it follows that $T(t_1, y^j(t_1); \sigma_\star) < \infty$. Therefore, in both cases $|x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^j(t_1)| < |x_{\mathcal{T}}(t_1) - x_{\mathcal{P}}^i(t_1)|$ implies that $T(t_1, y^j(t_1); \sigma_\star) < T(t_1, y^i(t_1); i)$ for $j \neq i$, where $x_{\mathcal{T}}(t_1) \in H_{t_1}^j(x_{\mathcal{P}}^i(t_1), x_{\mathcal{P}}^j(t_1)) \cap \operatorname{int} \mathcal{V}^j$. Therefore, the signal $\sigma_\star \in \Sigma$ satisfies $T(0, \bar{y}^i; \sigma_\star) = t_1 + T(t_1, y^j(t_1); \sigma_\star) < t_1 + T(t_1, y^i(t_1); i) = T(0, \bar{y}^i; i) = T_f(\bar{y}^i)$. Hence there exists $\sigma_\star \in \Sigma$ such that $T(0, \bar{y}^i; \sigma_\star) < T_f(\bar{y}^i)$, leading to a contradiction. □

Figs. 1 and 2 illustrate some of the cases that may appear during the pursuit of a target in the special case when $\mathcal{P} = \{\bar{x}_{\mathcal{P}}^i, \bar{x}_{\mathcal{P}}^j\}$ and $\bar{x}_{\mathcal{T}} \in \operatorname{int} \mathcal{V}^i$. In particular, Fig. 1 shows the case when, during the course of the pursuit, the target never enters the interior of $\mathcal{V}^j$. Specifically, Fig. 1(a) illustrates the scenario where the ith pursuer captures the target at some point in $\mathcal{V}^i$, whereas Fig. 1(b) illustrates the case when capture occurs at some point in the complement of $\mathcal{V}^i \cup \mathcal{V}^j$. Note that, in both cases shown in Fig. 1, the initial pursuer–target assignment does not change, since the requirements of the Switching Condition are not met. Fig. 2 illustrates the case when during the course of the pursuit, the target enters $\mathcal{V}^j$, and subsequently reaches a position within this cell from which it can be captured by the jth pursuer faster than the ith pursuer.

5.3. Implementation and analysis of the relay pursuit strategy

Next, we present a simple algorithm that will allow us to solve Problem 3 by dynamically updating the pursuer assigned to the moving target. In particular, we propose the following scheme. First, we construct the OP–DVD generated by the set $\mathcal{P}$, and determine the cell $\mathcal{V}^i$ of the OP–DVD such that $\bar{x}_{\mathcal{T}} \in \operatorname{int} \mathcal{V}^i$, and

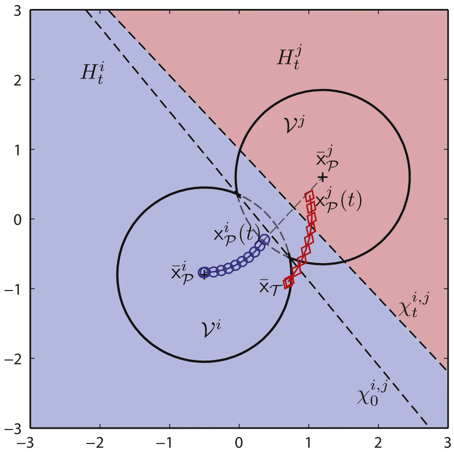


Fig. 2. If $x_{\mathcal{P}}^i \overset{0}{\rightsquigarrow} x_{\mathcal{T}}$ and there exists $t > 0$ such that $x_{\mathcal{T}}(t) \in \mathrm{int}\,\mathcal{V}^j \cap H_t^j(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t))$, where $x_{\mathcal{P}}^j(t) = \bar{x}_{\mathcal{P}}^j$, then the jth pursuer will capture the target faster than $\mathrm{T}(t, y^i(t); i)$. As a result, the pursuer–target assignment is updated at t so that $x_{\mathcal{P}}^j \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$.

let $x_{\mathcal{P}}^i \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$ for $t \in [0, T_f(\bar{y}^i)]$. If, during the course of the pursuit, the target never enters $\mathrm{int}\,\mathcal{V}^j$, for all $j \neq i$, then it follows that $\mathrm{T}(0, \bar{y}^i; \sigma) \geq T_f(\bar{y}^i)$ for all $\sigma \in \Sigma$. Hence, the pursuer target assignment is not updated. If there exists $t_1 > 0$ and $j \neq i$ such that $x_{\mathcal{T}}(t_1) \in \mathrm{int}\,\mathcal{V}^j \cap H_{t_1}^j(x_{\mathcal{P}}^i(t_1), x_{\mathcal{P}}^j(t_1))$, where $x_{\mathcal{P}}^j(t_1) = \bar{x}_{\mathcal{P}}^j$, then the signal σ with $\sigma(t) = i$ for $t \in [0, t_1)$ and $\sigma(t) = j$ for $t \geq t_1$ satisfies $\mathrm{T}(t_1, y^j(t_1); \sigma) < \mathrm{T}(t_1, y^i(t_1); i)$. Therefore, by taking $x_{\mathcal{P}}^j \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$, for $t \geq t_1$, it follows that capture can be achieved after $t_1 + \mathrm{T}(t_1, y^j(t_1); \sigma) < t_1 + \mathrm{T}(t_1, y^i(t_1); i) = T_f(\bar{y}^i)$ units of time.

The previous procedure is repeated every time the target enters a different cell of the OP–DVD during the course of its pursuit. Note that if the pursuer–target assignment is updated at some time t_1, one needs to construct the OP–DVD generated by the set comprised of the positions of the pursuers at time t_1, so that the previously described pursuer–target assignment scheme can be applied mutatis mutandis until capture occurs. In particular, one needs to compute the OP–DVD for the point-set $\mathscr{P}_{t_1} := \big(\mathscr{P} \cup \{x_{\mathcal{P}}^i(t_1)\}\big) \setminus \{\bar{x}_{\mathcal{P}}^i\}$ at time t_1. Note that the standard Voronoi diagram generated by the new set of generators can be easily constructed from the Voronoi diagram generated by the set $\mathscr{P}$ by means of well-known local/incremental algorithms (Green & Sibson, 1978; Mostafavi, Gold, & Dakowicz, 2003; Roos, 1998; Sugihara & Iri, 1992).

The previous scheme may be difficult to implement in practice due to the indeterminacy of the pursuer–target assignment scheme when the target lies on the switching line $\chi_t^{i,j}$ at some time $t \geq 0$. This is a well known problem in the theory of switched systems (Liberzon, 2003). To address it, we first redefine $\chi_t^{i,j}$ as follows $\chi_{t,\varepsilon}^{i,j} := \{x : ||x - x_{\mathcal{P}}^i(t)| - |x - x_{\mathcal{P}}^j(t)|| \leq \varepsilon\}$, where $\varepsilon > 0$ is a *hysteresis* constant. The sets H_t^i, H_t^j are then replaced, respectively, by $H_{t,\varepsilon}^i(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) := \{x : |x - x_{\mathcal{P}}^i(t)| < |x - x_{\mathcal{P}}^j(t)| - \varepsilon\}$ and $H_{t,\varepsilon}^j(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) := \{x : |x - x_{\mathcal{P}}^i(t)| > |x - x_{\mathcal{P}}^j(t)| + \varepsilon\}$. Note that after the target is assigned to, say, the ith pursuer, at time $t = 0$, based on the proximity relations encoded in the OP–DVD generated by $\mathscr{P}$, then the pursuer–target assignment cannot be updated as long as the target remains inside the set $H_{t,\varepsilon}^i(x_{\mathcal{P}}^i(t), x_{\mathcal{P}}^j(t)) \cup \chi_{t,\varepsilon}^{i,j}$, for $t > 0$ and for all $j \neq i$. In other words, if $x_{\mathcal{P}}^i \overset{t_0}{\rightsquigarrow} x_{\mathcal{T}}$ for some $t_0 \geq 0$, then the signal σ is allowed to switch at time $t_1 > t_0$ from $i = \sigma(t_0)$ to some $j \neq i$ with $j = \sigma(t_1)$ only if $\mathrm{T}(t_1, y^j(t_1); \sigma)$ is "sufficiently" smaller than $\mathrm{T}(t_1, y^i(t_1); \tilde{\sigma})$, where the signal $\tilde{\sigma}$ is defined such that $\tilde{\sigma}(t) = \sigma(t)$, for $t \in [0, t_1)$, and $\tilde{\sigma}(t) = i$, for $t \geq t_1$. The threshold difference between $\mathrm{T}(t_1, y^j(t_1); \sigma)$ and $\mathrm{T}(t_1, y^i(t_1); \tilde{\sigma})$ depends on the hysteresis constant ε.

Next, we determine a lower bound on the decrease of the capture time of the target that can be achieved by employing the previous dynamic pursuer–target assignment scheme when compared to a static pursuit scheme. In addition, we determine an upper bound on the number of switches of the signal $\sigma_\star \in \Sigma$ that solves Problem 3.

Proposition 5. *Let $\mathcal{V} = \{\mathcal{V}^i, i \in \mathcal{I}\}$ denote the OP–DVD generated by the set $\mathscr{P}$, and assume that $\bar{x}_{\mathcal{T}} \in \mathrm{int}\,\mathcal{V}^i$, for some $i \in \mathcal{I}$. In addition, let $\sigma_\star \in \Sigma$ be a solution of Problem 3 and let $N(\sigma_\star)$ denote the number of switches of $\sigma_\star$. If $\bar{\eta}_f > \epsilon_c$, then*

$$\mathrm{T}(0, \bar{y}^i; \sigma_\star) < T_f(\bar{y}^i) - N(\sigma_\star)\bar{\phi}\varepsilon, \tag{18}$$

where $\bar{\phi} := \inf_{[\epsilon_c, \bar{\eta}_f)} z/(\bar{u}_{\mathcal{P}} z - f(z))$. In particular,

$$N(\sigma_\star) < T_f(\bar{y}^i)/\varepsilon\bar{\phi}. \tag{19}$$

Proof. Let τ_k be the kth switching time of $\sigma_\star$, such that $\sigma_\star(\tau_k^-) = \ell_k$ and $\sigma_\star(\tau_k^+) = \sigma_\star(\tau_k) = \ell_{k+1}$, where $\ell_k, \ell_{k+1} \in \mathcal{I}$. Furthermore, let σ^k be the switching signal defined such that $\sigma^k(t) = \sigma_\star(t)$, for $t \in [0, t_k)$, and $\sigma^k(t) = \ell_k$, for $t \geq t_k$. Note that $i \equiv \ell_1$ and $\sigma^1 \equiv i$. By hypothesis, $x_{\mathcal{T}}(\tau_k) \in H_{\tau_k,\varepsilon}^{\ell_{k+1}}(x_{\mathcal{P}}^{\ell_k}(\tau_k), x_{\mathcal{P}}^{\ell_{k+1}}(\tau_k)) \cap \mathrm{int}\,\mathcal{V}^{\ell_{k+1}}$, which implies that $\epsilon_c < |y^{\ell_{k+1}}(\tau_k)| + \varepsilon < |y^{\ell_k}(\tau_k)| < \bar{\eta}_f$, where $y^{\ell_{k+1}}(\tau_k) := x_{\mathcal{T}}(\tau_k) - x_{\mathcal{P}}^{\ell_{k+1}}(\tau_k)$ and $y^{\ell_k}(\tau_k) := x_{\mathcal{T}}(\tau_k) - x_{\mathcal{P}}^{\ell_k}(\tau_k)$. Furthermore,

$$\mathrm{T}(\tau_k, y^{\ell_k}(\tau_k); \sigma^k) - \mathrm{T}(\tau_k, y^{\ell_{k+1}}(\tau_k); \sigma^{k+1}) = \int_{|y^{\ell_{k+1}}(\tau_k)|}^{|y^{\ell_k}(\tau_k)|} \phi(z)\,\mathrm{d}z, \tag{20}$$

where $\phi(z) := z/(\bar{u}_{\mathcal{P}} z - f(z))$. By virtue of the mean value theorem for Riemann integrals, there exists $\epsilon_c < |y^{\ell_{k+1}}(\tau_k)| \leq \zeta \leq |y^{\ell_k}(\tau_k)| < \bar{\eta}_f$, such that

$$\mathrm{T}(\tau_k, y^{\ell_k}(\tau_k); \sigma^k) - \mathrm{T}(\tau_k, y^{\ell_{k+1}}(\tau_k); \sigma^{k+1}) = \phi(\zeta)(|y^{\ell_k}(\tau_k)| - |y^{\ell_{k+1}}(\tau_k)|) > \phi(\zeta)\varepsilon. \tag{21}$$

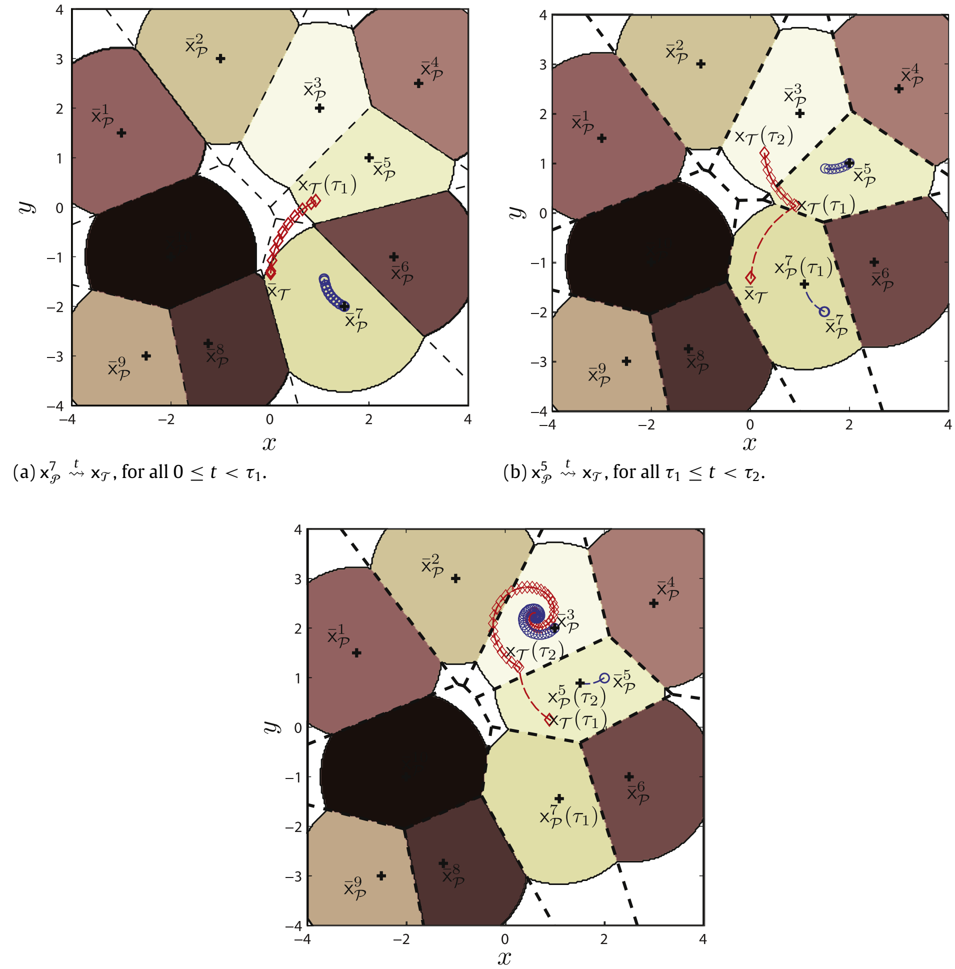
Note that ϕ is continuous and strictly positive, for all $z \in [\epsilon_c, \bar{\eta}_f)$. Furthermore, $\lim_{z \to \bar{\eta}_f} z/(\bar{u}_{\mathcal{P}} z - f(z)) = \infty$ and thus $\bar{\phi} := \inf_{[\epsilon_c, \bar{\eta}_f)} z/(\bar{u}_{\mathcal{P}} z - f(z)) > 0$. Then (21) gives $\mathrm{T}(\tau_k, y^{\ell_k}(\tau_k); \sigma^k) - \mathrm{T}(\tau_k, y^{\ell_{k+1}}(\tau_k); \sigma^{k+1}) > \bar{\phi}\varepsilon$, which, furthermore, implies that

$$\begin{aligned} T_f(\bar{y}^i) &= \tau_1 + \mathrm{T}(\tau_1, y^{\ell_1}(\tau_1); \sigma^1) \\ &> \tau_1 + \mathrm{T}(\tau_1, y^{\ell_2}(\tau_1); \sigma^2) + \bar{\phi}\varepsilon \\ &= \tau_1 + (\tau_2 - \tau_1) + \mathrm{T}(\tau_2, y^{\ell_2}(\tau_2); \sigma^2) + \bar{\phi}\varepsilon \\ &> \tau_2 + \mathrm{T}(\tau_2, y^{\ell_3}(\tau_2); \sigma^3) + 2\bar{\phi}\varepsilon \\ &\;\;\vdots \\ &> \tau_k + \mathrm{T}(\tau_k, y^{\ell_{k+1}}(\tau_k); \sigma^{k+1}) + k\bar{\phi}\varepsilon. \end{aligned} \tag{22}$$

Therefore $T_f(\bar{y}^i) > k\bar{\phi}\varepsilon$, for all $k \geq 1$, which implies that the maximum number of switches N is bounded. Furthermore, the previous inequality yields

$$\begin{aligned} T_f(\bar{y}^i) &> \tau_N + \mathrm{T}(\tau_N, y^{\ell_{N+1}}(\tau_N); \sigma_\star) + N\bar{\phi}\,\varepsilon \\ &= \mathrm{T}(0, \bar{y}^i; \sigma_\star) + N\bar{\phi}\,\varepsilon. \end{aligned}$$

Thus (18) follows readily. Finally, (19) follows immediately from the fact that $\mathrm{T}(0, \bar{y}^i; \sigma_\star) > 0$. □



(a) $x_{\mathcal{P}}^{7} \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$, for all $0 \leq t < \tau_1$.

(b) $x_{\mathcal{P}}^{5} \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$, for all $\tau_1 \leq t < \tau_2$.

(c) $x_{\mathcal{P}}^{3} \overset{t}{\rightsquigarrow} x_{\mathcal{T}}$, for all $t \geq \tau_2$.

Fig. 3. Trajectories of the active pursuers and the moving target during the course of the pursuit when the pursuer–target assignment is dynamic and is induced by the exact OP–DVD generated by $\mathcal{P}$.

6. Simulation results

In this section we present simulation results to illustrate the previous developments. We consider a scenario where the maneuvering target is faster than the ith pursuer, but the winning set of the ith pursuer is non-empty. In particular, it is assumed that the target has a constant speed and its evading strategy is given by

$$u_{\mathcal{T}}(\mathrm{y}^i) = \begin{cases} \alpha \mathrm{y}^i + \rho(\mathrm{y}^i)\mathrm{S}\mathrm{y}^i, & \text{for } \epsilon_c \leq |\mathrm{y}^i| \leq M/\alpha, \\ M\mathrm{y}^i/|\mathrm{y}^i|, & \text{for } |\mathrm{y}^i| > M/\alpha, \end{cases}$$

where M and α are some positive constants with $M > \max\{\bar{u}_{\mathcal{P}}, \alpha\}$, S is a nonzero skew symmetric matrix in $\mathbb{R}^{2\times 2}$, and $\rho(\mathrm{y}^i) := \sqrt{M^2 - \alpha^2|\mathrm{y}^i|^2}/|\mathrm{S}\mathrm{y}^i|$. It is easy to show that $f(\mathrm{y}^i) := \langle u_{\mathcal{T}}, \mathrm{y}^i\rangle$ satisfies Assumption 1. The intuition behind the evading strategy $u_{\mathcal{T}}(\mathrm{y}^i)$ is as follows: Let $\mathrm{e}_1(\mathrm{y}^i) := \mathrm{y}^i/|\mathrm{y}^i|$ be the unit vector along the line connecting the ith pursuer and the target ("line-of-sight" direction), and let $\mathrm{e}_2(\mathrm{y}^i)$ be the unit vector orthogonal to $\mathrm{e}_1(\mathrm{y}^i)$ ("tangential" direction). The strategy of the target is to allocate its velocity vector, which has a constant magnitude $M > u_{\mathcal{P}}$, along the directions $\mathrm{e}_1(\mathrm{y}^i)$ and $\mathrm{e}_2(\mathrm{y}^i)$ so that it moves with constant speed M along the line-of-sight direction when it is sufficiently far away from the pursuer, and it uses an increasingly larger tangential component as its distance from the pursuer decreases, in an effort to maneuver away or confuse its pursuer.

Assume for this example that the set $\mathcal{P}$ consists of ten locations, and let $\bar{f}$ be defined as f modulo the replacement of α with a positive scalar $\bar{\alpha}$, where $\alpha \leq \bar{\alpha} < M$. In this case, the capturability condition (11) reduces to $\eta^i(0) < \bar{u}_{\mathcal{P}}/\alpha$, which implies that $\bar{\eta}_f = \bar{u}_{\mathcal{P}}/\alpha < M/\alpha$ and $\bar{\eta}_{\bar{f}} = \bar{u}_{\mathcal{P}}/\bar{\alpha} < M/\bar{\alpha}$. Furthermore, it is easy to show that, for $\epsilon_c < |\bar{\mathrm{y}}^i| < \bar{\eta}_f$, $T_{\mathrm{f}}(\bar{\mathrm{y}}^i) = \ln((\bar{u}_{\mathcal{P}} - \alpha\epsilon_c)/(\bar{u}_{\mathcal{P}} - \alpha|\bar{\mathrm{y}}^i|))/\alpha$.

Next, we present simulation results of the relay-pursuit scheme introduced in Section 5.3. In particular, Fig. 3 illustrates the trajectories of the active pursuers and the moving target for the following data: $\mathrm{S} = \begin{bmatrix} 0 & 1.5 \\ -1.5 & 0 \end{bmatrix}$, $\varepsilon = 0.2$, $\alpha = \bar{\alpha} = 0.7$, $\bar{u}_{\mathcal{P}} = 1.2$, and $M = 3$. It is assumed that $\bar{\mathrm{x}}_{\mathcal{T}} \in \mathcal{W}_f(\bar{\mathrm{x}}_{\mathcal{P}}^7)$. Fig. 3(a) illustrates the trajectories of the target and the 7th pursuer, which is assigned to the target for $0 \leq t < \tau_1$, where τ_1 is the first switching time when the target is assigned to the 5th pursuer. Fig. 3(b) illustrates the trajectories of the target and the 5th pursuer, for $\tau_1 \leq t < \tau_2$, where τ_2 is the second switching time when the target is assigned to the 3rd pursuer. Note that $\mathrm{x}_{\mathcal{T}}(\tau_1)$ resides in the interior of the cell of the OP–DVD generated by the set $\mathcal{P}_{\tau_1} := \left(\mathcal{P} \cup \{\mathrm{x}_{\mathcal{P}}^7(\tau_1)\}\right) \setminus \{\bar{\mathrm{x}}_{\mathcal{P}}^7\}$ that is associated with the 5th pursuer. Fig. 3(c) illustrates the trajectories of the target and the 3rd pursuer for $t \geq \tau_2$. Again, we observe that, at time $t = \tau_2$, the target resides inside the cell of the OP–DVD generated by the set $\mathcal{P}_{\tau_2} := \left(\mathcal{P} \cup \{\mathrm{x}_{\mathcal{P}}^7(\tau_1)\} \cup \{\mathrm{x}_{\mathcal{P}}^5(\tau_2)\}\right) \setminus (\{\bar{\mathrm{x}}_{\mathcal{P}}^7\} \cup \{\bar{\mathrm{x}}_{\mathcal{P}}^5\})$ that is associated with the 3rd pursuer. Moreover, we observe that, although, at some time $t = \tau_3 > \tau_2$, the target enters the cell associated with the 2nd pursuer, at time $t = \tau_2$, the 3rd pursuer remains closer to the target than the 2nd pursuer, for all $t \geq \tau_3$. Consequently, the pursuer–target assignment does not change for $t \geq \tau_2$, and thus the 3rd pursuer will eventually capture the target.

7. Conclusion

We have proposed a relay pursuit scheme for the capture of a maneuvering target by a group of pursuers distributed in the plane. It is assumed that, during the course of the pursuit, only one pursuer can go after the target, whereas the rest of the pursuers remain stationary. Furthermore, it is assumed that in order to delay or, if possible, avoid capture, the target can employ a feedback "evading" strategy based on its relative position with respect to the active pursuer. The problem of assigning a pursuer from the group of pursuers to the maneuvering target is associated with the solution of a Voronoi-like partitioning problem that characterizes the sets of initial conditions of the moving target from which a particular pursuer can intercept the target faster than any other pursuer from the same group. We have presented an efficient scheme for constructing an approximate solution for this partitioning problem by associating it with a standard Voronoi partition. Based on this Voronoi partition, we have presented a scheme that dynamically assigns the task of pursuing the maneuvering target to the appropriate pursuers in the group in order to minimize capture time.

One question that has not been addressed in this paper, and is worth-pursuing in the future, is whether relay pursuit strategies can be used to capture a target which escapes capture when pursued by a single pursuer. In this case, capture will occur only if the pursuers cooperate. In that sense, relay pursuit strategies can be viewed as an intermediate option offering a simpler alternative, in lieu of attacking head-on the corresponding group pursuit game-theoretic problem involving multiple pursuers, whose solution is known to be very hard (Hajek, 2008, p. 161). Another interesting possibility for future extension is to consider scenarios where the motions of the pursuers and the target are described by more realistic kinematics, for example, those of the Isaacs–Dubins car (see for example Bakolas & Tsiotras, 2011).

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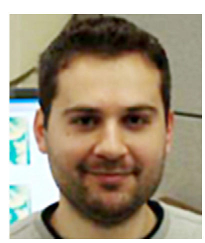

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