# SELF-SCHEDULED $H_\infty$ CONTROLLERS FOR MAGNETIC BEARINGS


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## ABSTRACT

In this paper we present a design procedure for disturbance attenuation with internal stability of a rotor supported by magnetic bearings, over the whole interval of the rotor operating speeds. The nonlinear gyroscopic equations can be simplified to a set of linear, time-varying differential equations, owing to the linear dependence of the rotor speed in the plant dynamics. For rotors operating in high speeds, even small unbalanced masses can create large synchronous disturbances. To alleviate the forces produced due to imbalance the controller automatically balances the rotor *for each operating speed.* The proposed methodology addresses both the gyroscopic compensation problem due to change of the rotor speed as well as the automatic balancing problem. The approach is based on the recently developed theory of self-scheduled $\mathcal{H}_\infty$ controllers. Owing to the linear dependence of the equations on the rotor speed, the computation of the controller can be done very efficiently using convex optimization and Linear Matrix Inequalities (LMIs).


## NOMENCLATURE

$A$ area of each pole
Co convex hull
$E_0$ nominal voltage
$\vec{F}$ external force in body frame
$F_0$ nominal force
$G_0$ nominal gap
$J_a$ axial moment of inertia
$J_r$ radial moment of inertia
$M$ mass of the rotor
$N$ number of coil turns
$\mathcal{P}$ parameter polytope
$R$ coil resistance
$\mathbb{R}^n$ $n$-dimensional vector space
$\mathcal{S}$ system matrix polytope
$\vec{T}$ external torque in body frame
$\vec{V}$ velocity of center of mass in body frame
$e_j$ voltage across the $j$th coil
$f_{ij}$ magnetic forces
$g_j$ gap length of $j$th coil
$h$ pole width
$\ell$ half the length of the shaft
$p$ rotor speed
$\Phi_0$ nominal airgap
$\gamma$ disturbance attenuation level
$\theta$ angular displacement about $Y$ axis
$\mu_0$ permeability of free space $(4\pi \times 10^{-7} H/m)$
$\phi$ angular displacement about $Z$ axis
$\phi_j$ airgap magnetic flux of $j$th coil
$\psi$ angular displacement about $X$ axis
$\omega$ angular velocity in body frame

## INTRODUCTION

Magnetic bearings (MB) increasingly become the choice for high-speed, high-performance rotating machinery because of their frictionless characteristics. They utilize a magnetic field generated by radially or axially placed electromagnets to generate the forces necessary to suspend and support a shaft without any contact with its environment. Thus, magnetic bearings are particularly useful in very high or very low temperature conditions where a lubrication-free environment is necessary. The advantages of magnetic bearings are primarily their very low power consumption (an order of magnitude lower than oil film bearings) and their very long, maintenance-free life. Some applications where magnetic bearings offer distinct advantages are high speed turbomachinery, precision milling spindles, and combined attitude control and energy storage for spacecraft and satellites.

Active magnetic bearings can support rotors without friction but they require a sophisticated control system, since the uncontrolled magnetic bearing configuration is open-loop unstable.

Although the addition of a controller is indispensable to the magnetic bearing design (complicating the overall design), it offers the ability of meeting specific performance requirements such as automatic balancing of the shaft, rejection of unwanted disturbances, and vibration isolation. Several control methodologies (both in frequency and time-domain) have been successfully used in the past for active control of magnetic bearings. Most of these techniques assume a linear time-invariant (LTI) plant. Such an assumption is valid if the rotor is to operate at a fixed speed. If, on the other hand, the speed of the rotor changes over a wide range, the linear time-invariant assumption may no longer be valid because the dynamics of the magnetic bearing change when the rotational speed varies. Control techniques from linear robust control theory (Mohamed and Busch-Vishniac, 1995; Fujita *et al.*, 1993b; Fujita *et al.*, 1993a) and adaptive control methods (Knospe *et al.*, 1995; Shafai *et al.*, 1994) have been used to attack this problem. Robust control methodologies can be, however, overly conservative for this problem, since they do not take into account the actual time variation of the rotor speed.

Another approach is a control strategy that exploits available measurements of the speed to increase the performance and robustness of the closed-loop system. In this paper we use the recently developed methodology of gain-scheduled (better, self-scheduled) $\mathcal{H}_\infty$ controllers for linear parameter varying (LPV) systems to attack this problem (Packard, 1994; Becker *et al.*, 1993; Apkarian and Gahinet, 1995; Apkarian *et al.*, 1995). The idea behind this design technique is to solve a series of standard $\mathcal{H}_\infty$ problems at a specified number of operating speeds. Using a single Lyapunov function to show stability and finite $\mathcal{L}_2$-gain at these selected points, one guarantees that these properties will also hold for all operating speeds which are linear combinations of the selected speeds. One can then interpolate between these speeds to construct the desired controller. Performance objectives (such as overshoot, disturbance rejection, robustness to unmodeled dynamics, etc.) can be effectively incorporated in this framework. The main ingredient that makes this approach computationally attractive is the linear dependence of the system matrix on the operating speed and thus, the controller can be designed by solving a set of Linear Matrix Inequalities (LMIs).

## MAGNETIC BEARING MODEL

The magnetic bearing configuration considered in this paper is shown in Fig. 1. Four pairs of electromagnets are used to suspend and control the rotor. In Fig. 1 only the two pairs of electromagnets in the $X-Z$ plane are shown. Two more pairs, lying in the $Y-Z$ plane, are not shown in the figure.

The general motion of the rotor system is one of a rotating rigid body

$$M\frac{d\vec{V}}{dt} + M\,\vec{\omega}\times\vec{V} \;=\; \vec{F} \tag{1}$$

$$\frac{d\vec{H}}{dt} + \vec{\omega}\times\vec{H} \;=\; \vec{T} \tag{2}$$

where $M$ is the total mass of the rotor, $\vec{V}$ is the velocity of the mass center, $\vec{\omega}$ is the angular velocity of the mass center, $\vec{H}$ is the angular momentum in body frame, and $\vec{F}$ and $\vec{T}$ are the total external force and moment about the mass center. The angular velocity vector in body coordinates has components $\omega_1, \omega_2, \omega_3$

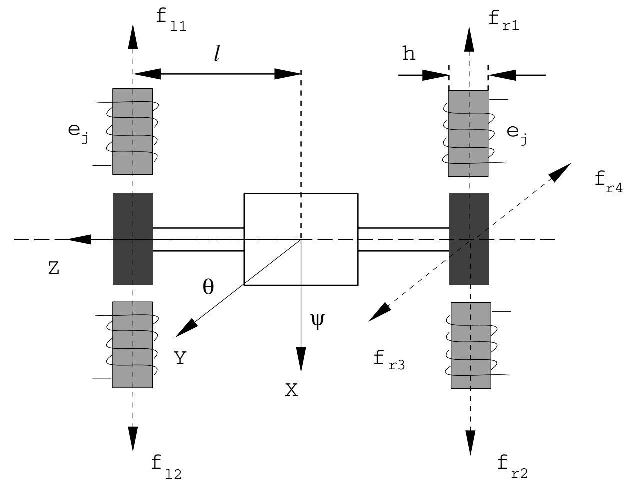


**Fig. 1: Magnetic bearing configuration.**

given by

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}\dot\theta\sin\phi + \dot\psi\cos\phi\\ \dot\theta\cos\phi - \dot\psi\sin\phi\\ \dot\phi + \dot\psi\sin\theta\end{bmatrix} \tag{3}$$

where $\phi, \theta$ and $\psi$ are angles denoting the orientation of the body-fixed frame about the axis $Z$, the $Y$ and $X$ axes respectively (3-2-1 Euler angles).

In this paper we are interested only in the rotational motion of the bearing about the $X$ and $Y$ axes. The translational motion can be treated the same way, but the equations become more complicated. The following assumptions will be made throughout the paper: the rotor is assumed to be rigid; the rotor speed can be measured in real-time; the motion of the rotor along its longitudinal axis is negligible; all electromagnets are identical.

Since the angular deviations $\phi$ and $\theta$ are typically very small, we have $\cos\theta \approx 1$ and $\sin\theta \approx 0$ and Eqs. (3) simplify to

$$\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\dot\psi\\ \dot\theta\\ \dot\phi\end{bmatrix} \tag{4}$$

Substituting $\omega_1$ and $\omega_2$ from the previous equation in Eq. (2) one obtains

$$\begin{bmatrix}\ddot\psi\\ \ddot\theta\end{bmatrix} + p\frac{J_a}{J_r}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\dot\psi\\ \dot\theta\end{bmatrix} = \frac{\ell}{J_r}\begin{bmatrix}M_\psi\\ M_\theta\end{bmatrix} \tag{5}$$

where $J_a$ is the moment of inertia of the rotor in the axial direction, $J_r$ is the moment of inertia in the radial direction, and $p = \omega_3 = \dot\phi$ is the rotor angular velocity. Let the magnetic forces produced by the magnetic bearings on the rotor be denoted by $f_{r1}, f_{r2}, f_{r3}, f_{r4}, f_{l1}, f_{l2}, f_{l3}, f_{l4}$. The body torques $M_\psi$ and $M_\theta$ are related to the inertial torques through the equation

$$\begin{bmatrix}M_\psi\\ M_\theta\end{bmatrix} = \begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}f_{r1} - f_{r2} + f_{l2} - f_{l1} + f_{d\theta}\\ f_{r3} - f_{r4} + f_{l4} - f_{l3} + f_{d\psi}\end{bmatrix} \tag{6}$$

In the previous equation $f_{d\theta}$ and $f_{d\psi}$ represent disturbance forces caused by gravity, modeling errors, imbalances, etc. Rotor imbalance, in particular, is a common source of vibration in magnetic bearing design.

The rotational motion of a magnetic bearing can finally be described by the equations (Mohamed and Busch-Vishniac, 1995)

$$\begin{aligned}\ddot{\theta} &= \frac{pJ_a}{J_r}\dot{\psi} + \frac{\ell}{J_r}(f_{r1} - f_{r2} + f_{l2} - f_{l1} + f_{d\theta}) \\ \ddot{\psi} &= -\frac{pJ_a}{J_r}\dot{\theta} + \frac{\ell}{J_r}(f_{r3} - f_{r4} + f_{l4} - f_{l3} + f_{d\psi})\end{aligned}$$

In order to complete the bearing model we also need the equations which describe the electromagnetic circuit of the coils. The voltages $e_j$ across the $j$th coil are functions of the airgap magnetic flux $\phi_j$ and the gap length $g_j$ and can be expressed as follows

$$e_j = N\frac{d\phi_j}{dt} + \frac{2R}{\mu_0 AN} g_j \phi_j, \quad j = r1, ..., r4, l1, .., l4$$

Here $N$ is the number of turns in each coil, $R$ is the coil resistance, and $A$ is the area of each pole. Note that the electromagnetic forces $f_j$ depend nonlinearly on the airgap flux and the gap length as

$$f_j = k\phi_j^2\left(1 + \frac{2g_j}{\pi h}\right), \quad j = r1, ..., r4, l1, .., l4$$

where $k$ is a constant and $h$ is the pole width. The electromagnets generate a nominal force $F_0$ which – in the absence of disturbances and modeling errors – balances the rotor. In this configuration, the nominal airgap flux, gap length and voltage are given by $\Phi_0, G_0$ and $E_0$, respectively. Assuming that $\delta f_j, \delta\phi_j, \delta g_j$ and $\delta e_j$ denote the deviation of the force, airgap flux, gap length and electromagnetic voltage from their nominal values and that the coil voltages are controlled such that $\delta e_{r2} = -\delta e_{r1}$, $\delta e_{r4} = -\delta e_{r3}$ $\delta e_{l2} = -\delta e_{l1}$, $\delta e_{l4} = -\delta e_{l3}$ we can write the linearized equations as follows

$$\begin{aligned}\ddot{\theta} &= \frac{pJ_a}{J_r}\dot{\psi} + \frac{\ell}{J_r}(-4c_2\ell\theta + 2c_1\phi_\theta + v_1) && \text{(7a)} \\ \ddot{\psi} &= -\frac{pJ_a}{J_r}\dot{\theta} + \frac{\ell}{J_r}(-4c_2\ell\psi + 2c_1\phi_\psi + v_2) && \text{(7b)} \\ N\dot{\phi}_\theta &= e_\theta + 2d_2\ell\theta - d_1\phi_\theta && \text{(7c)} \\ N\dot{\phi}_\psi &= e_\psi + 2d_2\ell\psi - d_1\phi_\psi && \text{(7d)}\end{aligned}$$

where $\phi_\theta = \delta\phi_{r1} - \delta\phi_{l1}$, $\phi_\psi = \delta\phi_{r3} - \delta\phi_{l3}$, $e_\theta = \delta e_{r1} - \delta e_{l1}$, and $e_\psi = \delta e_{r3} - \delta e_{l3}$. The constants $c_1, c_2, d_1, d_2$ depend on $\Phi_0, G_0, R, A, N, \mu_0$ and the geometry of the bearing as follows $c_1 = 2k\Phi_0\left(1 + \frac{2G_0}{\pi h}\right)$, $c_2 = \frac{2k\Phi_0^2}{\pi h}$, $d_1 = \frac{2RG_0}{\mu_0 AN}$, $d_2 = \frac{2R\Phi_0}{\mu_0 AN}$. (For a more detailed exposition the interested reader may consult Mohamed and Busch-Vishniac (1995) and the references therein).

Letting now $x_1 = \ell\theta$, $x_2 = \ell\psi$, $x_3 = \ell\dot{\theta}$, $x_4 = \ell\dot{\psi}$, $x_5 = \phi_\theta$, $x_6 = \phi_\psi$ $u_1 = e_\theta$, $u_2 = e_\phi$, $v_1 = f_{d\theta}$, $v_2 = f_{d\psi}$ the linearized equations of the magnetic bearing can be written in state-space form as follows

$$\begin{aligned}\dot{x}_1 &= x_3 && \text{(8a)} \\ \dot{x}_2 &= x_4 && \text{(8b)} \\ \dot{x}_3 &= -\frac{4c_2}{m}x_1 + p\frac{J_a}{J_r}x_4 + \frac{2c_1}{m}x_5 + \frac{1}{m}v_1 && \text{(8c)} \\ \dot{x}_4 &= -\frac{4c_2}{m}x_2 - p\frac{J_a}{J_r}x_3 + \frac{2c_1}{m}x_6 + \frac{1}{m}v_2 && \text{(8d)} \\ \dot{x}_5 &= \frac{1}{N}(u_1 + 2d_2x_1 - d_1x_5) && \text{(8e)} \\ \dot{x}_6 &= \frac{1}{N}(u_2 + 2d_2x_2 - d_1x_6) && \text{(8f)}\end{aligned}$$

where $m := J_r/\ell^2$. In these equations $u_1, u_2$ represent the control inputs (electromagnetic voltages) and $v_1, v_2$ represent undesirable disturbances.

**Remark 1** The original, nonlinear rotational equations of the rotor system including the gyroscopic terms in Eq. (2) can be written in the form

$$\dot{\omega} = A(\omega)\omega + M \quad (9)$$

where the matrix $A(\omega)$ is a linear function of the state. The interesting observation here is that $\omega_3$ is completely determined by $\dot{\phi}$ which can be measured in real-time and it is, therefore, known. Since the time history of $\omega_3$ is given, this nonlinear system becomes, in fact, a linear, parameter-varying system (Shamma and Athans, 1991; Becker *et al.*, 1993; Packard, 1994).

## PROBLEM FORMULATION

Our objective is to design a feedback controller that will achieve *automatic balancing* of the rotor over all rotor speeds. By automatic balancing we mean that no unbalance forces are transmitted from the rotor to the stator. Automatic balancing is desirable for rotor operation in vibration-free environments, such as flywheel systems onboard high-precision pointing satellites or spacecraft performing microgravity experiments. Elimination of the forces between the rotor and the stator can be achieved by allowing the rotor to rotate about its inertial axis. In Eqs. (8) $x_1$ and $x_2$ then denote deviations of the axis of rotation from the inertia axis of the rotor (see Fig. 2).

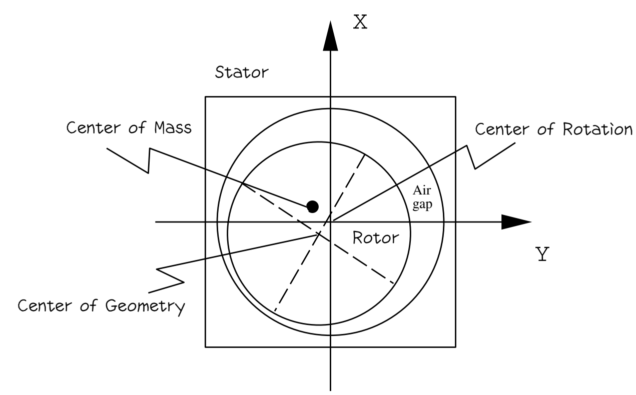


**Fig. 2: Rotor unbalance.**

The purpose of the controller is to drive the displacements $x_1$ and $x_2$ (equivalently, the angular deviations $\theta$ and $\psi$) to zero using moderate control effort. We assume that the only available measurements the controller has at its disposal are the $Y$ and $X$ displacements of the shaft $x_1 = \ell\theta$ and $x_2 = \ell\psi$, respectively. The unbalance response of the rotor is modeled as a sinusoidal signal measured by the sensors (Mohamed and Busch-Vishniac, 1995)

$$\begin{aligned}w_1 &= \tilde{d}\ cos(pt + \tau) && \text{(10a)} \\ w_2 &= \tilde{d}\ sin(pt + \tau) && \text{(10b)}\end{aligned}$$

where $\tilde{d}$ is the magnitude of the unbalance and $\tau$ corresponds to some initial phase angle. No other forces act on the system except the electromagnetic forces. Under this assumption the system equations can be rewritten compactly in the form

$$\begin{aligned} \dot{x} &= A(p)x + B_2 u &\quad (11a)\\ y &= C_2 x + w &\quad (11b) \end{aligned}$$

where $x = [x_1, x_2, x_3, x_4, x_5, x_6]^T \in \mathrm{I\!R}^6$ $w = [w_1\ w_2]^T \in \mathrm{I\!R}^2$ and $u = [u_1\ u_2]^T \in \mathrm{I\!R}^2$, and where the state matrices are given by

$$A(p) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{4c_2}{m} & 0 & 0 & \frac{pJ_a}{J_r} & \frac{2c_1}{m} & 0 \\ 0 & -\frac{4c_2}{m} & -\frac{pJ_a}{J_r} & 0 & 0 & \frac{2c_1}{m} \\ \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N} & 0 \\ 0 & \frac{2d_2}{N} & 0 & 0 & 0 & -\frac{d_1}{N} \end{bmatrix} \quad (12)$$

$$B_2 = \frac{1}{N}\begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix} \quad (13)$$

In Eq. (11) $y = [y_1\ y_2]^T \in \mathrm{I\!R}^2$ is the measured output available to the controller.

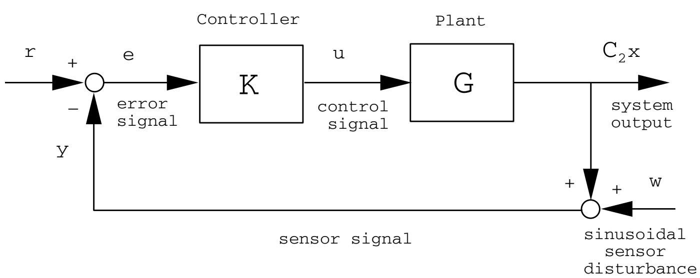


**Fig. 3: Block diagram of system interconnection.**

The regulated output is

$$z = \begin{bmatrix} C_2 x \\ u \end{bmatrix} \quad (14)$$

and includes a penalty on the actuator effort. The matrix $A(p)$ is affine in the parameter $p$ and can be written in the form

$$A(p) = A_0 + pA_p \quad (15)$$

where $A_0$ and $A_p$ are easily obtained from (12).

Given the plant Eqs. (11) and (14) our objective becomes then one of designing a controller that will reduce the map from $w$ to $z$ as much as possible, and at the same time, will guarantee asymptotic stability (in the absence of any external disturbances). To this end, notice that if the angular velocity of the rotor $p$ were fixed, then the system in Eqs. (8) is a linear time-invariant (LTI) system and the previous disturbance attenuation problem can be conveniently cast as an $\mathcal{H}_\infty$ optimization problem. For the case of a rotor with changing operating speed, $p$ is not fixed and Eqs. (8) represent a linear time-varying (LTV) system. $\mathcal{H}_\infty$ controllers for linear time-varying systems have been reported in (Ravi *et al.*, 1991). From a practical point of view, however, the implementation of such controllers is demanding. They require the solution of a system of matrix Riccati *differential* equations.

Although $p$ is, in general, time varying, it is also measurable on-line. This implies that the system (8) has the form of a linear parameter-varying (LPV) system (Shamma and Athans, 1991). Linear parameter varying systems differ from general linear time-varying systems in the sense that the time variation of the system matrix is not known a priori but the actual time variation can be determined from parameter measurements which are available in real time. Therefore, knowledge of $p$ can be used for controller design. In particular, $p$ can be used to adjust the controller gains on line, i.e, it can play the role of a gain-scheduling parameter.

## BACKGROUND THEORY

In this paper we use the methodology of Apkarian *et al.* (1995) and Apkarian and Gahinet (1995) in order to design self-scheduled LTI controllers (with respect to the parameter $p$) for the magnetic bearing system described by Eqs. (11). To this end, consider a general *polytopic* linear parameter-varying system (LPV) of the form

$$\begin{aligned} \dot{x} &= A(p(t))x + B(p(t))u &\quad (16a)\\ y &= C(p(t))x + D(p(t))u &\quad (16b) \end{aligned}$$

whose system matrices are fixed *affine* functions of some time-varying parameter vector $p(t)$ which takes values in a polytope $\mathcal{P}$ of vertices $\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_r$; that is,

$$\mathcal{P} = \mathrm{Co}\{\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_r\} = \left\{ \sum_{i=1}^{r} \alpha_i \hat{p}_i : \alpha_i \geq 0,\ \sum_{i=1}^{r} \alpha_i = 1 \right\}$$

where $\mathrm{Co}\{\cdot\}$ denotes the convex hull. Owing to the affine dependence of the system matrices $A(p), B(p), C(p)$ and $D(p)$ on the parameter $p$, these matrices range in a matrix polytope, whose vertices are the ones calculated at the vertices of the parameter polytope $\mathcal{P}$.

A useful indication of performance for LPV systems is the notion of quadratic $\mathcal{H}_\infty$ performance (Apkarian *et al.*, 1995).

**Definition 1 (Quadratic $\mathcal{H}_\infty$ performance.)** The LPV system in Eqs. (16) is said to have *quadratic $\mathcal{H}_\infty$ performance* $\gamma$ if and only if there exists a positive definite matrix $X > 0$ which satisfies the following linear matrix inequality (LMI)

$$\begin{pmatrix} A^T(p)X + XA(p) & XB(p) & C^T(p) \\ B^T(p)X & -\gamma I & D^T(p) \\ C(p) & D(p) & -\gamma I \end{pmatrix} < 0 \quad (17)$$

for all values of the parameter vector $p \in \mathcal{P}$.

Quadratic $\mathcal{H}_\infty$ performance guarantees global asymptotic stability and $\mathcal{L}_2$-gain of the map from $u$ to $y$ less than $\gamma$

$$\|y\|_2 < \gamma \|u\|_2 \quad (18)$$

for all possible parameter trajectories $p(t) \in \mathcal{P}$. Therefore, quadratic $\mathcal{H}_\infty$ performance establishes internal stability and robust performance in the sense of inequality (18).

**Remark 2** Quadratic $\mathcal{H}_\infty$ performance is a more conservative notion than standard $\mathcal{H}_\infty$ performance for each fixed $p$, since it requires the existence of a *fixed* (parameter-independent) Lyapunov function for the entire operating range. Quadratic $\mathcal{H}_\infty$ performance does incorporate, however, a constraint on the induced $\mathcal{L}_2$-gain of the system via the inequality (18).

In general, it is not an easy task to check the existence of a matrix $X$ such that the inequality (17) will be satisfied for all admissible parameter values $p$ in an arbitrary set. In the case of polytopic LPV systems, however, it is easily shown that (17) will hold for all $A(p), B(p), C(p), D(p)$ for some fixed $X$ if and only if it holds at the vertices $A_i, B_i, C_i, D_i$ for $i = 1, ..., r$ (Boyd *et al.*, 1994). Based on this observation Apkarian and Gahinet (1995), and Apkarian *et al.* (1995) developed a computationally efficient algorithm for the solution of the quadratic $\mathcal{H}_\infty$ performance problem for LPV systems.

## SELF-SCHEDULED CONTROLLERS

In this section the synthesis problem for polytopic LPV systems is reviewed. We consider (polytopic) LPV systems of the form

$$\begin{aligned} \dot{x} &= A(p)x + B_1(p)w + B_2(p)u && (19a)\\ z &= C_1(p)x + D_{11}(p)w + D_{12}(p)u && (19b)\\ y &= C_2(p)x + D_{21}(p)w + D_{22}(p)u && (19c) \end{aligned}$$

The system matrices are assumed to belong to the polytope $\mathcal{S}$ defined by

$$\mathcal{S} := \mathrm{Co}\left\{ \begin{pmatrix} A_i & B_{1i} & B_{2i} \\ C_{1i} & D_{11i} & D_{12i} \\ C_{2i} & D_{21i} & D_{22i} \end{pmatrix}, i = 1, ..., r \right\} \quad (20)$$

where $A_i, B_{1i}, ...,$ denote the values of the matrices $A(p), B_1(p), ...$ at the vertices $\hat{p}_i$ of the parameter polytope $\mathcal{P}$. Henceforth we will assume that in Eqs. (19) the system matrices $B_2(p), C_2(p), D_{12}(p)$ and $D_{21}(p)$ are parameter-independent. In addition, the disturbance does not affect the performance output and there is no feedthrough term from the input to the measured output, i.e., $D_{11}(p) = D_{22}(p) = 0$. These simplifying assumptions can be relaxed, at the expense of increased complexity in the resulting formulas (Apkarian *et al.*, 1995).

Under the natural assumption that the pairs $(A(p), B_2)$ and $A(p)^T, C_2^T)$ are quadratically stabilizable over the polytope $\mathcal{P}$ (see Corless (1993) for a definition of quadratic stability/stabilizability) we seek a controller that establishes quadratic $\mathcal{H}_\infty$ performance for the closed-loop system. In particular, we are interested in LPV controllers, that is, controllers with state space representation

$$\Omega(p) := \begin{pmatrix} A_K(p) & B_K(p) \\ C_K(p) & D_K(p) \end{pmatrix} \quad (21)$$

where $A_K$,$B_K$,$C_K$,$D_K$ are affine in $p$. The closed-loop system can be compactly written in the form

$$\begin{aligned} \dot{x}_{cl} &= A_{cl}(p)x_{cl} + B_{cl}(p)w && (22a)\\ z &= C_{cl}(p)x_{cl} + D_{cl}(p)w && (22b) \end{aligned}$$

where $A_{cl}(p) = A_0(p) + \tilde{B}\Omega(p)\tilde{C}$, $B_{cl}(p) = B_0 + \tilde{B}\Omega(p)\tilde{D}_{21}$ $C_{cl}(p) = C_0 + \tilde{D}_{12}\Omega(p)\tilde{C}$ $D_{cl}(p) = D_{11} + \tilde{D}_{12}\Omega(p)\tilde{D}_{21}$ and

$$A_0(p) = \begin{pmatrix} A(p) & 0 \\ 0 & 0_{k\times k} \end{pmatrix},\ B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},\ \tilde{D}_{21} = \begin{pmatrix} 0 \\ D_{21} \end{pmatrix}$$

$$C_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix},\quad \tilde{B} = \begin{pmatrix} 0 & B_2 \\ I_k & 0 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} 0 & I_k \\ C_2 & 0 \end{pmatrix},\quad \tilde{D}_{12} = \begin{pmatrix} 0 & D_{12} \end{pmatrix}$$

In the previous equations $k$ is the order of the controller ($k \le n$). The important observation here is that the controller is uniquely characterized by $\Omega(p)$ which enters the closed-loop equations in an affine way. Moreover, since both $\Omega(p)$ and $A_0(p)$ depend affinely on $p$, the closed-loop system (22) also depends affinely on the parameter $p$. This observation leads us to the following result (Apkarian *et al.*, 1995).

**Theorem 1** *Consider the polytopic LPV plant (19) and let $\mathcal{N}_R$ and $\mathcal{N}_s$ denote bases of the null spaces of the matrices $(B_2^T, D_{12}^T)$ and $(C_2, D_{21})$ respectively. There exists an LPV controller guaranteeing quadratic $\mathcal{H}_\infty$ performance $\gamma$ along all parameter trajectories in the parameter polytope $\mathcal{P}$ if and only if there exist two symmetric matrices $R$ and $S$ in $\mathbb{R}^{n\times n}$ satisfying the system of $2r+1$ LMIs*

$$\left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_iR + RA_i^T & RC_{1i}^T & B_{1i} \\ C_{1i}R & -\gamma I & D_{11i} \\ \hline B_{1i}^T & D_{11i}^T & -\gamma I \end{array}\right) \times \left(\begin{array}{c|c} \mathcal{N}_R & 0 \\ \hline 0 & I \end{array}\right) < 0, \quad i = 1, \ldots, r \quad (23)$$

$$\left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right)^T \left(\begin{array}{cc|c} A_i^TS + SA_i & SB_{1i} & C_{1i}^T \\ B_{1i}^TS & -\gamma I & D_{11i}^T \\ \hline C_{1i} & D_{11i} & -\gamma I \end{array}\right) \times \left(\begin{array}{c|c} \mathcal{N}_S & 0 \\ \hline 0 & I \end{array}\right) < 0 \quad i = 1, \ldots, r \quad (24)$$

$$\begin{pmatrix} R & I \\ I & S \end{pmatrix} \ge 0 \quad (25)$$

*Moreover, there exist a* k*th-order LPV controller solving the same problem if and only if $R$ and $S$ satisfy, in addition, the rank constraint*

$$rank(I - RS) \le k \quad (26)$$

Once the matrices $R$ and $S$ have been found, the Lyapunov matrix $X_{cl}$ and the vertex controllers $\Omega_i$ are obtained as follows:

### Controller Synthesis

**Step 1** Compute full-rank matrices $M, N \in \mathbb{R}^{n\times k}$ such that

$$MN^T = I - RS \quad (27)$$

**Step 2** Compute $X_{cl}$ as the unique solution to the matrix equation $\Pi_2 = X_{cl}\Pi_1$, where

$$\Pi_2 := \begin{pmatrix} S & I \\ N^T & 0 \end{pmatrix},\quad \Pi_1 := \begin{pmatrix} I & R \\ 0 & M^T \end{pmatrix} \quad (28)$$

**Step 3** Compute the vertex controllers $\Omega_i$ by solving the convex feasibility problem

$$\begin{pmatrix} A_{cl}^T(\hat{p}_i)X_{cl} + X_{cl}A_{cl}(\hat{p}_i) & X_{cl}B_{cl}(\hat{p}_i) & C_{cl}^T(\hat{p}_i) \\ B_{cl}^T(\hat{p}_i)X_{cl} & -\gamma I & D_{cl}^T(\hat{p}_i) \\ C_{cl}(\hat{p}_i) & D_{cl}(\hat{p}_i) & -\gamma I \end{pmatrix} < 0 \quad (29)$$

for $i = 1, 2, \ldots, r$, where now $X_{cl}$ is known.

If Eq. (29) holds, a possible choice of an LPV controller is the polytopic controller given by

$$\Omega(p) = \sum_{i=1}^{r} \alpha_i \Omega_i = \sum_{i=1}^{r} \alpha_i \begin{pmatrix} A_{Ki} & B_{Ki} \\ C_{Ki} & D_{Ki} \end{pmatrix} \tag{30}$$

where $(\alpha_1, \ldots, \alpha_r)$ is any solution of the convex decomposition problem $p = \sum_{i=1}^{r} \alpha_i \hat{p}_i$.

## MAGNETIC BEARING APPLICATION

In this section we apply Theorem 1 to the magnetic bearing system as described by Eqs. (11) and (12)-(13). Our first objective is to formulate the equations of the magnetic bearing/rotor system as in Eqs. (19). There is only one parameter here (the rotation speed of the shaft) which is assumed to be between $p_{\min}$ and $p_{\max}$. Therefore the polytope $\mathcal{P}$ in this case is simply the line segment $\mathcal{P} = \{p : p_{\min} \leq p \leq p_{\max}\}$.

For the automatic balancing problem, the state-space realization of the magnetic bearing/rotor system is given by Eqs. (11)-(13). A state-space realization of the disturbance in Eq. (10) is given by

$$A_w = \begin{bmatrix} 0 & -p \\ p & 0 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_w = I_2 \tag{31}$$

The noise $w$ is then given by

$$\dot{x}_w = A_w x_w + B_w \tilde{d} \tag{32}$$

$$w = C_w x_w \tag{33}$$

This model is easily incorporated into the system since the matrix $A_w$ is affine in $p$. The augmented system is formed as

$$\dot{x}_{au} = A_{au}(p) x_{au} + B_{1au} \tilde{d} + B_{2au} u \tag{34a}$$

$$z = C_{1au} x_{au} + D_{12au} u \tag{34b}$$

$$y = C_{2au} x_{au} \tag{34c}$$

with $x_{au} = \begin{bmatrix} x \\ x_w \end{bmatrix}$ and

$$A_{au}(p) = \begin{bmatrix} A(p) & 0 \\ 0 & A_w(p) \end{bmatrix}, \quad B_{2au} = \begin{bmatrix} B_2 \\ 0_{2\times 2} \end{bmatrix} \tag{35}$$

$$B_{1au} = \begin{bmatrix} 0_{6\times 1} \\ B_w \end{bmatrix}, \quad C_{1au} = \begin{bmatrix} C_2 & 0_{2\times 2} \\ 0_{2\times 6} & 0_{2\times 2} \end{bmatrix} \tag{36}$$

$$C_{2au} = [C_2 \;\; C_w], \quad D_{12au} = \begin{bmatrix} 0_{2\times 2} \\ I_2 \end{bmatrix} \tag{37}$$

The parameter $p$ contributes both in the plant system matrix due to gyroscopic effects, as well as in the synchronous unbalance disturbance model. Note that the augmented system matrix $A_{au}(p)$ is still affine in $p$. The performance measure $z$ consists of the control effort of the $X$ and $Y$ actuators and the axial displacements in the $X$ and $Y$ directions. This model can be further refined through the addition of output shaping filters as follows

$$\tilde{z} = diag(W_z(s), W_u(s)) z \tag{38}$$

The filters $W_u$ and $W_z$ can be used to place constraints on the control $u$ and on the plant output $C_2 x$. The complete system is shown in Fig. 4.

Zero steady-state error from the command input $r$, can be accommodated, if necessary, by adding an integrator at the controller input (not considered here and hence not shown in Fig. 4). Once in this format, and given a disturbance attenuation level $\gamma$, the computation of a feasible controller is determined by the solution of the five LMIs in Eqs. (23)-(25) for $i = 1, 2$, where $A_1 = A_{au}(p_{\min})$, $A_2 = A_{au}(p_{\max})$. The solution of these equations will provide two controllers $\Omega_1 = \Omega(p_{\max})$ and $\Omega_2 = \Omega(p_{\max})$ at the vertices of $\mathcal{P}$. Writing for any $p_{\min} \leq p \leq p_{\max}$ one obtains

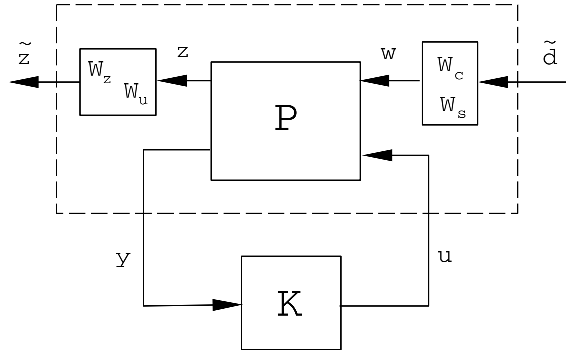


**Fig. 4: Block diagram of $\mathcal{H}_\infty$ system**

$$p = \left( \frac{p_{\max} - p}{p_{\max} - p_{\min}} \right) p_{\min} + \left( \frac{p - p_{\min}}{p_{\max} - p_{\min}} \right) p_{\max} \tag{39}$$

Thus, $\alpha_1 = \left(\frac{p_{\max} - p}{p_{\max} - p_{\min}}\right)$ and $\alpha_2 = \left(\frac{p - p_{\min}}{p_{\max} - p_{\min}}\right)$ in Eq. (30). For every operating speed the self-scheduled $\mathcal{H}_\infty$ controller is therefore computed by

$$\Omega(p) = \alpha_1 \Omega(p_{\min}) + \alpha_2 \Omega(p_{\max}) \tag{40}$$

The methodology of the previous sections establishes the feasibility of a self-scheduled controller for some prescribed performance attenuation level $\gamma$. By decreasing $\gamma$ until the feasibility problem is not solvable we can in fact solve the disturbance attenuation (sub)optimal problem: we can find the self-scheduled controller which gives the best quadratic $\mathcal{H}_\infty$ performance for the magnetic bearing. This minimization problem only slightly increases the computational burden since it involves only an additional line search on the scalar parameter $\gamma$.

## NUMERICAL SIMULATIONS

The following system parameters, shown in Table 1, were chosen for the magnetic bearing example.

**Table 1: System parameters**

| Par. | Value | Par. | Value |
|---|---|---|---|
| $A$ | $1531.79\ mm^2$ | $h$ | $40\ mm$ |
| $G_0$ | $0.55\ mm$ | $J_r$ | $0.333\ kg \cdot m^2$ |
| $J_a$ | $0.0136\ kg \cdot m^2$ | $\ell$ | $0.13\ m$ |
| $k$ | $4.6755576 \times 10^8$ | $N$ | 400 turns |
| $R$ | $14.7\ Ohm$ | $\Phi_0$ | $2.09 \times 10^{-4}\ Wb$ |

These parameters were taken from Mohamed and Busch-Vishniac (1995). The rotor is expected to operate between $p_{\min} = 315\ rad/sec$ $(3,000\ rpm)$ and $p_{\max} = 1,100\ rad/sec$ $(10,500\ rpm)$. For this system, an imbalance

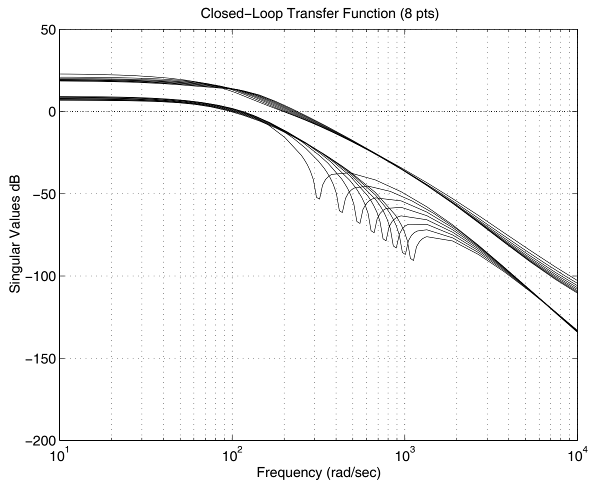


**Fig. 5: Closed loop frequency response of LPV controller.**

of $\tilde{d} = 1.3 \times 10^{-5} m$ is assumed in Eq. (10). To meet the control design requirements, two shaping filters $W_u(s)$ and $W_z(s)$ are added to the performance measure for the $\mathcal{H}_\infty$ synthesis as in Eq. (38). The following filters were chosen after some trial and error

$$W_u(s) = 100 \left( \frac{s + 1500}{s + 10000} \right)^2 I_2, \quad W_z(s) = 5000\, I_2$$

The zero at $1,500\ rad/sec$ adds an increased penalty to control effort at frequencies beyond the expected operating speed by rapidly decreasing the controller bandwidth with roll-off $-40\, dB$. The pole at $10,000\ rad/sec$ is included to make the augmented plant proper, a requirement for the synthesis software. Given the system data $A_{au}$, $B_{1au}$, $B_{2au}$, $C_{1au}$, $D_{12au}$, $C_{2au}$ as in Eqs. (34) a controller was computed using Eqs. (27), (28), (29) and (30). The actual calculations were performed using the LMI TOOLBOX of MATLAB (Gahinet *et al.*, 1993). The results are plotted in Figures 5-6.

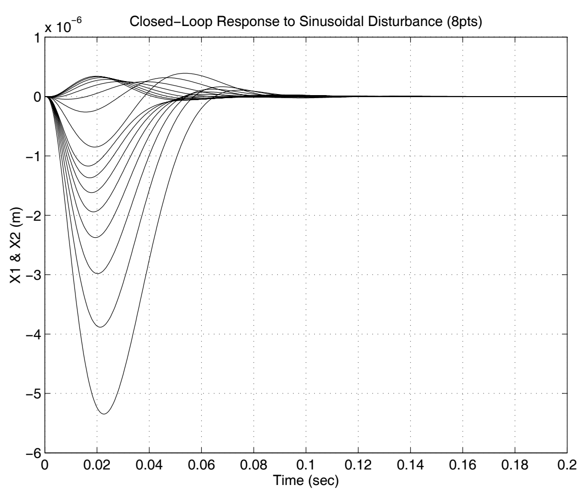


**Fig. 6: Output response of LPV controller.**

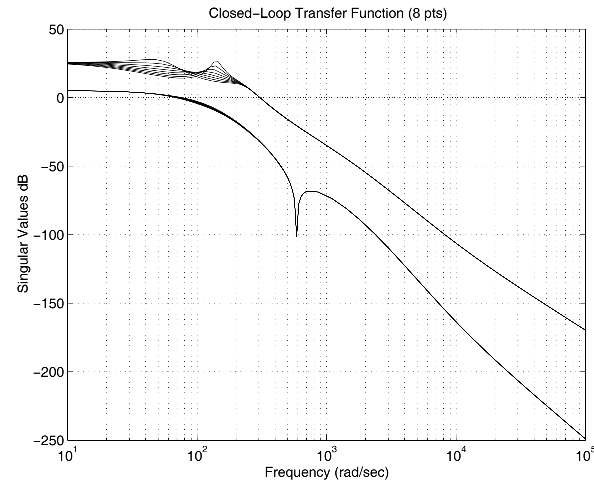


**Fig. 7: Closed loop frequency response of $\mathcal{H}_\infty$ controller.**

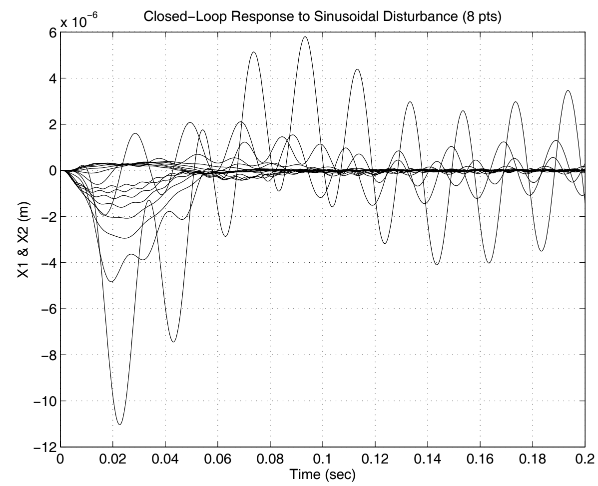


**Fig. 8: Output response of $\mathcal{H}_\infty$ controller.**

Figure 5 shows the closed-loop singular value plots from the disturbance to the plant output for eight different operating speeds (315, 426, 538, 650, 763, 875, 987 and $1100\, rad/sec$). Figure 6 shows the output response to sinusoidal synchronous disturbance. The controller successfully rejects the sinusoidal disturbance over all speeds.

For comparison, we have have also calculated a standard (not gain-scheduled) $\mathcal{H}_\infty$ robust controller for a constant speed $\bar{p} = 587\, r/s$. (This is the geometric mean of $p_{\min}$ and $p_{\max}$.) The corresponding results are shown in Figures 7 and 8. The fixed-speed controller cannot handle very well operating points away from the design speed $\bar{p}$.

## CONCLUSIONS

In this paper we apply the recently methodology of self self-scheduled $\mathcal{H}_\infty$ controllers to a magnetic bearing supporting a rotating rigid shaft. The controller is naturally gain-scheduled on the angular velocity of the rotor and provides (sub)optimal disturbance attenuation to external forces and to alleviate the effect of gyroscopics over all operating speeds. Moreover, the resulting controller is guaranteed to be stabilizing over the whole range of speeds. The main advantage of this method is the ability for casting the controller synthesis problem in terms of linear matrix inequalities which can be solved very efficiently using existing convex optimization packages. Finally, in light of Remark 1 the self-scheduled procedure can also be used for plants with *nonlinear* dynamics of the form $\dot{x} = A(x)x + Bu$ as long as some a priori knowledge of the system state response is available. If, for instance, one knows that the state trajectories of the system do not leave a certain polytope $\mathcal{P}$, one can treat them as parameters and apply the results of (Apkarian and Gahinet, 1995) and (Apkarian *et al.*, 1995). This requires, of course, that the states entering the $A$ matrix are measurable on-line.

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