TIME-OPTIMAL CONTROL OF AXI-SYMMETRIC RIGID SPACECRAFT

Haijun Shen* and Panagiotis Tsiotras†

University of Virginia, Charlottesville, VA 22903-2442

Abstract

In this paper, we consider the minimum-time reorientation problem of an axi-symmetric rigid spacecraft with two independent control torques mounted perpendicular to the spacecraft symmetry axis. The spacecraft is allowed to spin about its symmetry axis. All possible control structures, including both singular and nonsingular arcs, are studied completely by deriving the corresponding formulae, and the necessary optimality conditions. An efficient method for solving the optimal control problem numerically, based on a cascaded computational scheme, is also presented. Numerical examples demonstrate optimal reorientation maneuvers with both nonsingular and singular subarcs.

1. Introduction

In recent years, the time-optimal reorientation problem of a rigid spacecraft has been extensively studied by many researchers. In Ref. 1, the minimum-time attitude slewing of a rigid spacecraft is considered. Quasilinearization is used to solve the Two-Point Boundary-Value Problem (TPBVP) arising from Pontryagin's Minimum Principle. An integral of a quadratic function of the control inputs is used as the performance index instead of the slewing time. The minimum slewing time is determined by sequentially shortening the final time. The corresponding fixed final time problem is solved until the solution can no longer be obtained, or until all the resulting controls are bang-bang. The Euler (eigenaxis) rotation maneuver is used as the initial guess for the numerical computation. Besides the fact that the bang-bang solutions show that the minimum-time trajectories are far from an eigenaxis rotation, numerical results also make the authors suspect that singular controls may appear for a single principle axis rotation of a symmetric body. The singular trajectory is then a rotation about a principle axis, i.e., an eigenaxis rotation. In their following work[2], the authors implemented their method to the Naval Research Laboratory's Reconfigurable Spacecraft Host for Attitude and Pointing Experiment (RESHAPE) three-axis maneuver facility. The results of these experiments are presented in Ref. 2.

In Ref. 3, Bilimoria and Wie studied the time-optimal, rest-to-rest, large-angle, principle-axis rotation of an inertially symmetric rigid body. By solving the TPBVP using a shooting method, they obtained a variety of bang-bang controls which showed that the eigenaxis rotation is not time optimal, in general. Singular controls are considered only in the sense that all three controls cannot be singular at the same time. Later, they extended their work to an axi-symmetric body (with three control torques) and they also studied the principle axis rotation[4]. The emphasis in this latter work is on examining how the gyroscopic terms in Euler's equations affect the minimum time. Comparing with the minimum final time obtained for a system with the gyroscopic terms dropped, they showed that the gyroscopic effect increases the final time for a rod-like body and decrease the final time for a disk-like body.

Seywald and Kumar[5] extended the work in this area by analyzing all the possible controls for a general minimum-time reorientation problem of an inertially symmetric rigid body. An elegant derivation of all the possible controls, including bang-bang control subarcs, finite-order singular control subarcs and infinite-order singular control subarcs was developed. It was shown that for rest-to-rest maneuvers, the eigenaxis rotation can appear as a finite-order singular arc, but it is not optimal. It follows that an eigenaxis rotation can, in fact, appear as an optimal infinite-order singular arc.

Scrivener and Thompson[6] explored the minimum-time reorientation of a rigid spacecraft numerically using a direct method via collocation and nonlinear programming. This method was first introduced by Hargraves and Paris[7]. Instead of dealing with the necessary conditions from Pontryagin's Minimum Principle, the trajectory is first discretized and the optimal trajectory is found in the finite dimensional space of the states and controls at each node using nonlinear programming. The method is shown to be robust in the sense that it does not

* Graduate Student, Department of Mechanical, Aerospace and Nuclear Engineering. Email: hs2x@virginia.edu. Student member AIAA.

†Assistant Professor, Department of Mechanical, Aerospace and Nuclear Engineering. Email: tsiotras@virginia.edu. Senior member AIAA.

require accurate initial guesses. Scrivener and Thompson applied this method to the time-optimal, rest-to-rest maneuver of a rigid spacecraft. Comparison is made between their results and the ones in Ref. 4. The results are consistent except the case when the maneuver has a reorientation angle of less than 10 deg. It turns out that in this case – although the maneuver time is the same – the switching structure is different indicating, possibly, a multiple local minimum.

A few researchers have worked with a method called Switching Time Optimization (STO). STO was originally proposed by Meier and Bryson[8] to solve the time-optimal control of a two-link manipulator. Byers[9,10] used STO to solve the time-optimal rigid body reorientation problem. More recently, Liu and Singh[11] addressed the weighted time/fuel optimal control of an inertially symmetric spacecraft performing a rest-to-rest maneuver. The authors modified the STO method to determine the switching times and total maneuver time of the bang-off-bang control profiles. The results are compared with those of Bilimoria et al. in Ref. 4. The effect of the penalty on fuel in the cost on the number of switching times is discussed and an interesting result is presented, namely, as the fuel penalty is beyond a specific value, the eigenaxis control with two switches is shown to be optimal. An apparent drawback of STO is that the switching structure, i.e., the number of switches and how the controls switch, has to be guessed or known in advance.

In relation to the work in this paper, two articles are of particular interest. First, Chowdhry and Cliff[12] considered the time-optimal reorientation of a rigid body with two control torques. The existence of singular subarcs was also studied. However, the authors only considered the time-optimal control of rigid body angular rates, i.e., they considered only the dynamics of the motion. Hermes and Hogenson[13] studied the same system as the one in this work. They applied feedback linearization to transform the system to two uncoupled linear double-integrators. The time-optimal controls can then be calculated explicitly[14]. The result is then transformed back to the original space to obtain the explicit feedback control for the original nonlinear system. However, due to the complex nonlinear relationship between the original controls and the transformed ones, the controls for the original system are not necessarily time optimal, as pointed out by the authors themselves in their conclusions. In addition, since a double-integrator system has only one switch and no singular subarcs (and the feedback linearization transformation between the original and the resulting systems is continuous), this method leads to time-optimal controls for the original nonlinear system with at most two switches and no singular subarcs. This is shown *not* to be true in this paper.

In this paper, we address the time-optimal reorientation problem for an axi-symmetric rigid body. The purpose of the control is to drive the symmetry axis from some initial orientation, with some specified angular velocity, to another final orientation, with specified angular velocity. We assume that the relative orientation of the body about the symmetry axis is irrelevant and only the location of the symmetry axis is of interest. This could be the case when the symmetry axis coincides with the boresight or line-of-sight of a camera or a gun barrel, for example. Clearly, the relative rotation of the camera or the barrel has no influence on the clarity of the photograph or the accuracy of the projectile. Spin-stabilized spacecraft also fall into this category.

For the axi-symmetric case it turns out that the objective of optimal reorientation of the symmetry axis can be achieved using only two torques about axes that span the plane perpendicular to the symmetry axis. Therefore, without loss of generality, we consider the time-optimal reorientation of an axi-symmetric rigid spacecraft with two control torques acting perpendicular to the symmetry axis and to each other. The spacecraft may be spinning about its symmetry axis with a constant angular velocity. The main effort in this paper is devoted on analyzing the formulae for the possible bang-bang and singular control subarcs and the corresponding necessary conditions.

2. Problem Formulation

Consider an axi-symmetric rigid body with two control torques as shown in Fig. 1. Without loss of generality,

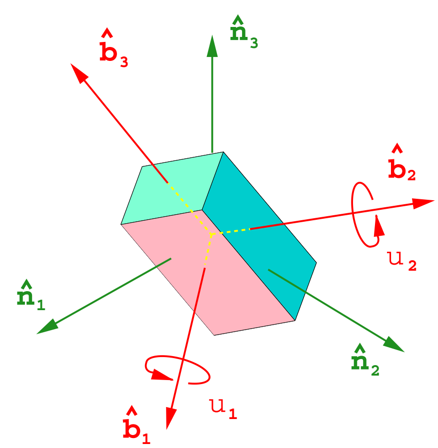

Figure 1: Axi-symmetric rigid body with two controls.

a body-fixed reference frame $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ is defined with the unit vector $\hat{b}_3$ pointing along the symmetry axis. The control system generates two control torques T_1 and T_2 along the $\hat{b}_1$ and $\hat{b}_2$ axis, respectively, as shown in Fig. 1. Let $\omega = (\omega_1, \omega_2, \omega_3)^T \in \mathcal{R}^3$ denote the angular velocity vector in the $\hat{\mathbf{b}}$ frame, and $I_1,\ I_2,\ I_3$ be the moments of inertia with respect to the three axes just defined. Then Euler's equations[15] with respect to this frame take the form

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3 + T_1 \quad (1a)$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1 + T_2 \quad (1b)$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 \quad (1c)$$

Since $I_1 = I_2$, if we let the initial condition $\omega_3(0) = \omega_{30}$, ω_3 will remain constant throughout the maneuver, and the equations reduce to

$$\dot{\omega}_1 = a\omega_{30}\omega_2 + u_1 \quad (2a)$$
$$\dot{\omega}_2 = -a\omega_{30}\omega_1 + u_2 \quad (2b)$$

where $a = (I_2 - I_3)/I_1$ and u_1 and u_2 are the new control inputs given by $u_i = T_i/I_i,\ i = 1,2$. Note that for physical systems we always have that $-1 < a < 1$.

As it has been shown in Tsiotras and Longuski[16], if $\hat{\mathbf{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ denotes the inertial reference frame, then the position of the $\hat{n}_3$ inertial axis in the body fixed $\hat{\mathbf{b}}$ frame can be uniquely described by two variables w_1 and w_2 which are defined as

$$w_1 = \frac{b}{1+c}, \qquad w_2 = \frac{-a}{1+c} \quad (3)$$

where a, b, and c denote the direction cosines of axis $\hat{n}_3$ with respect to $\hat{\mathbf{b}}$ frame, i.e., $\hat{n}_3 = a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3$. It can be shown[16] that w_1 and w_2 can be obtained by stereographically projecting the unit vector $\hat{n}_3$ onto the $\hat{b}_1$-$\hat{b}_2$ plane[17]. Moreover, w_1 and w_2 obey the following differential equations[16,17]

$$\dot{w}_1 = \omega_{30}w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \quad (4a)$$
$$\dot{w}_2 = -\omega_{30}w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \quad (4b)$$

Given the system of Eqs. (2) and (4) we seek to minimize the performance index

$$\min_{u\in\mathcal{U}} \mathcal{J} = \min \int_{t_0}^{t_f} 1\, dt \quad (5)$$

subject to the initial conditions

$$\omega(0) = [\omega_1(0), \omega_2(0)]^T \quad \text{given}$$
$$w(0) = [w_1(0), w_2(0)]^T \quad \text{given}$$

and final conditions

$$\Psi[\omega(t_f), w(t_f)] = 0 \quad (6)$$

where $\Psi : \mathcal{R}^4 \mapsto \mathcal{R}^k,\ k \leq 4$ is a given smooth function vector, and the control constraint set is given by

$$\mathcal{U} = \{u \ :\ |u_i| \leq u_{i\max}, \quad i = 1,2\} \quad (7)$$

with $u_{i\max} > 0, i = 1,2$.

This problem has the following physical interpretation. For an observer in the $\hat{\mathbf{b}}$ frame, the location of the inertial $\hat{n}_3$ axis is given by w_1 and w_2. The time-optimal control problem then consists of re-orienting the spacecraft to given relative locations of the $\hat{n}_3$ (expressed in the body frame). We point out here that using Eqs. (2) and (4) it is not possible to specify the absolute orientation of the spacecraft in the inertial frame. In particular, it is not possible to know the relative orientation of the spacecraft about the $\hat{n}_3$. That would require, of course, a third attitude parameter to complement w_1 and w_2[16]. Applications where such "reduced" attitude information may be sufficient is the case, for example, of reorientation of the symmetry axis of an axi-symmetric spacecraft along a given direction.

3. Optimality Conditions

For simplicity, let the state vector $[\omega_1, \omega_2, w_1, w_2]^T \in \mathcal{R}^4$ be denoted by the vector $[x_1, x_2, x_3, x_4]^T \in \mathcal{R}^4$ and let $m = \omega_{30}$. Then Eqs. (2) and (4) are re-written as

$$\dot{x}_1 = amx_2 + u_1 \quad (8a)$$
$$\dot{x}_2 = -amx_1 + u_2 \quad (8b)$$
$$\dot{x}_3 = mx_4 + x_2x_3x_4 + \frac{x_1}{2}(1 + x_3^2 - x_4^2) \quad (8c)$$
$$\dot{x}_4 = -mx_3 + x_1x_3x_4 + \frac{x_2}{2}(1 + x_4^2 - x_3^2) \quad (8d)$$

The Hamiltonian $\mathcal{H}$ for this problem is defined by

$$\begin{aligned}\mathcal{H} &= 1 + \lambda_1\dot{x}_1 + \lambda_2\dot{x}_2 + \lambda_3\dot{x}_3 + \lambda_4\dot{x}_4 \\ &= 1 + am\lambda_1 x_2 + \lambda_1 u_1 - am\lambda_2 x_1 + \lambda_2 u_2 \\ &\quad + m\lambda_3 x_4 + \lambda_3 x_2 x_3 x_4 + \lambda_3 x_1(1 + x_3^2 - x_4^2)/2 \\ &\quad - m\lambda_4 x_3 + \lambda_4 x_1 x_3 x_4 + \lambda_4 x_2(1 + x_4^2 - x_3^2)/2\end{aligned} \quad (9)$$

The co-state equations, defined by $\dot{\lambda} = -(\partial\mathcal{H}/\partial x)^T$, are

$$\dot{\lambda}_1 = am\lambda_2 - \lambda_3(1 + x_3^2 - x_4^2)/2 - \lambda_4 x_3 x_4 \quad (10a)$$
$$\dot{\lambda}_2 = -am\lambda_1 - \lambda_4(1 + x_4^2 - x_3^2)/2 - \lambda_3 x_3 x_4 \quad (10b)$$
$$\dot{\lambda}_3 = -\lambda_3 x_2 x_4 - \lambda_3 x_1 x_3 + m\lambda_4 - \lambda_4 x_1 x_4 + \lambda_4 x_2 x_3 \quad (10c)$$
$$\dot{\lambda}_4 = -\lambda_3 x_2 x_3 + \lambda_3 x_1 x_4 - m\lambda_3 - \lambda_4 x_1 x_3 - \lambda_4 x_2 x_4 \quad (10d)$$

with $\lambda(t_f)$ given by the transversality condition

$$\lambda^T(t_f) = \nu^T \frac{\partial\Psi}{\partial x(t_f)} \quad (11)$$

where $\nu \in \mathcal{R}^k$ is a constant multiplier vector.

From Pontryagin's Minimum Principle[14], the optimal control u^* is chosen such that the Hamiltonian $\mathcal{H}$ is minimized, i.e.,

$$u^*(t) = \arg \min_{u\in\mathcal{U}} \mathcal{H}(x(t),\lambda(t),u), \ \forall t \geq 0 \qquad (12)$$

Since both controls u_1 and u_2 appear only linearly in the Hamiltonian $\mathcal{H}$, the optimal control u_i^* is given by

$$u_i^* = \begin{cases} +u_{i\max} & \text{if } \lambda_i < 0 \\ -u_{i\max} & \text{if } \lambda_i > 0 \\ \text{singular} & \text{if } \lambda_i \equiv 0 \end{cases} \qquad i = 1,2 \qquad (13)$$

The transversality condition associated with the final time t_f is given by

$$\mathcal{H}(t_f) = 0 \qquad (14)$$

Equation (9) shows that the Hamiltonian $\mathcal{H}$ is not an explicit function of time t, hence $\mathcal{H}(t) \equiv 0$, for $t \in [t_0,\ t_f]$.

4. Singular Control Analysis

Let $S_i,\ i = 1,\ 2$ be the switching functions, defined by

$$S_i = \frac{\partial\mathcal{H}}{\partial u_i}, \qquad i = 1,2 \qquad (15)$$

Here $S_i = \lambda_i,\ i = 1,\ 2$. From Eq. (13), u_i^* is singular whenever $S_i \equiv 0$ during a nonzero interval $[t_1,\ t_2] \subset [t_0,\ t_f]$. In this case the control component u_i^* is determined implicitly by the condition $S_i \equiv 0$. Indeed, u_i^* can be obtained by differentiating $S_i \equiv 0$ until the control component u_i appears explicitly[18]. For u_i^* to be optimal, it is required that S_i be differentiated an *even* number of times until the control u_i appears[19]. Therefore, u_i^* can be determined by solving

$$u_i^* = \arg\left\{ \left(\frac{d^{2k_i}S_i}{dt^{2k_i}}\right) = 0\right\}, \qquad i = 1,2 \qquad (16)$$

where $2k_i$ is the least number of differentiations of S_i that are required so that the corresponding u_i shows up. It is also evident that the switching functions and their time derivatives up to $(2k_i-1)$th order are zero along the singular subarc $[t_1,\ t_2]$, i.e., $S_i = \dot{S}_i = \ldots = S_i^{(2k_i-1)} = 0,\ t \in [t_1,\ t_2]$. In addition, Kelley's optimality condition[20,21] (also known as the Generalized Legendre-Clebsch condition)

$$(-1)^{k_i}\frac{\partial}{\partial u_i}\left[\frac{d^{2k_i}S_i}{dt^{2k_i}}\right] \geq 0 \qquad (17)$$

has to be satisfied along an optimal singular subarc.

A complete analysis of all possible singular control cases, i.e., with two and only one control being singular, is presented in the following two subsections. Before proceeding with this analysis, recall that $\dot{S}_1$ and $\dot{S}_2$ are given by Eqs. (10a) and (10b), respectively, and that

$$\ddot{S}_1 = am^2\lambda_1 + m(1+a)\dot{\lambda}_2 + (\lambda_3x_4 - \lambda_4x_3)x_2 \quad (18a)$$
$$\ddot{S}_2 = am^2\lambda_2 - m(1+a)\dot{\lambda}_1 - (\lambda_3x_4 - \lambda_4x_3)x_1 \quad (18b)$$

Henceforth it will be assumed that $m \neq 0$ and $a \neq 0$, namely, the rigid body is *not inertially* symmetric and it is spinning about its symmetry axis with a nonzero angular velocity $\omega_3 = m$. The cases $a = 0$ and/or $m = 0$ are treated separately in Section 5.

4.1. Case I: both u_1 and u_2 are singular

From Eq. (13), when both u_1 and u_2 are singular during $t \in [t_1,\ t_2] \subset [t_0,\ t_f]$, we have

$$S_1 = \lambda_1 \equiv 0, \qquad S_2 = \lambda_2 \equiv 0 \qquad (19)$$

which imply that

$$\dot{S}_1 = \dot{\lambda}_1 \equiv 0, \qquad \dot{S}_2 = \dot{\lambda}_2 \equiv 0 \qquad (20)$$

Substitution of Eqs. (19) and (20) into Eq. (9) yields

$$\mathcal{H} = 1 + m(\lambda_3x_4 - x_3\lambda_4) \qquad (21)$$

and substitution of Eqs. (19) and (20) into Eq. (18) yields

$$\ddot{S}_1 = (\lambda_3x_4 - \lambda_4x_3)x_2, \quad \ddot{S}_2 = -(\lambda_3x_4 - \lambda_4x_3)x_1 \quad (22)$$

From Eqs. (3) and (21), we have that $\mathcal{H} \equiv 0$ implies that $(\lambda_3x_4 - x_3\lambda_4) \neq 0$. Thus, during the singular arc $[t_1,\ t_2]$ Eqs. (22) imply $x_2 = 0$ and $x_1 = 0$. Taking the third time derivative of S_1 and S_2 and using Eqs. (8a) and (8b), we get explicit expressions for controls u_1 and u_2. In particular, $S_1^{(3)} = 0$ implies that $u_2 = 0$ and $S_2^{(3)} = 0$ implies that $u_1 = 0$. The control u_2 appears in the third time derivative of S_1 and u_1 appears in the third time derivative of S_2. Since the controls appears after differentiating the switching functions an odd number of times, the controls are not optimal[22,19].

The previous analysis has shown that the case when both u_1 and u_2 are singular is ruled out. Nevertheless, a more careful look reveals that this case may be very close to optimal for large values of the spin-rate m. To this end, recall that along a singular arc, the conditions $\lambda_1 = \lambda_2 = \dot{\lambda}_1 = \dot{\lambda}_2 = 0$ should be satisfied. Thus, Eqs. (10a) and (10b) imply that

$$\begin{bmatrix} -(1+x_3^2-x_4^2)/2 & -x_3x_4 \\ -x_3x_4 & -(1+x_4^2-x_3^2)/2 \end{bmatrix} \begin{bmatrix} \lambda_3 \\ \lambda_4 \end{bmatrix} = 0 \qquad (23)$$

Since $(\lambda_3,\lambda_4) \neq (0,0)$ the previous coefficient matrix must be singular. Calculating the determinant of this matrix, one obtains that a singular arc with both $u_1 = u_2 = 0$ can occur if and only if $x_3^2 + x_4^2 = 1$.

Using the fact that $x_1 = x_2 = 0$ we have that along a singular arc the system equations are given by

$$\dot{x}_3 = mx_4 \quad (24a)$$
$$\dot{x}_4 = -mx_3 \quad (24b)$$

The solution of the previous system is given by

$$\begin{bmatrix} x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \cos m & \sin m \\ -\sin m & \cos m \end{bmatrix} \begin{bmatrix} x_3(0) \\ x_4(0) \end{bmatrix} \quad (25)$$

Note that if $x_3^2(0)+x_4^2(0) = 1$ then $x_3^2(t)+x_4^2(t) = 1$ for all $t \geq 0$. This implies that singular subarcs indeed exist for specific boundary conditions. From the definition of the state vector, $x_3^2 + x_4^2 = 1$ corresponds to the case when the inertial $\hat{n}_3$ and the body $\hat{b}_3$ axes are perpendicular to each other. For an observer in the spacecraft frame, the inertial $\hat{n}_3$ axis rotates at a constant rate $\omega_3 = m$ rad/sec in the $\hat{b}_1 - \hat{b}_2$ plane. Both initial and final conditions correspond to different locations of the $\hat{n}_3$ axis in the $\hat{b}_1 - \hat{b}_2$ plane. The singular solution suggests letting the body coast from the initial to the final position by a pure rotation about the $\hat{b}_3$ axis. This situation is shown in Fig. 2. The time to complete the maneuver can

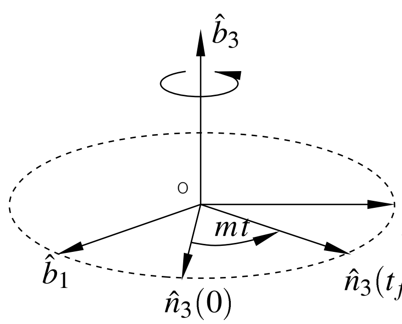

Figure 2: Singular control maneuver ($u_1 = u_2 = 0$).

be calculated explicitly from Eq. (25). For example, in case $(x_3(0),x_4(0)) = (1,0)$ and $(x_3(t_f),x_4(t_f)) = (0,1)$ one obtains that $t_f = \frac{\pi}{2m}$. The optimality conditions in this section have shown that this maneuver is not optimal. Indeed, simulations using the numerical scheme described in Section 6 showed that a bang-bang solution consisting of one switch for each u_1 and u_2 gives a better (smaller) final time. Table 1 gives the numerical results for $u_{i\max} = 1$, $i = 1,2$ and for initial and final conditions as above. The results in Table 1 show that the difference of the final time between the coasting maneuver in Eq. (25) and the bang-bang solution becomes smaller and smaller as the spin-rate m increases. This agrees with our intuition. For this example, the two solutions give essentially the same value of t_f for $m = 10$ rad/sec. For values above $m = 10$ rad/sec numerical issues prevented accurate calculation of the optimal trajectory using EZopt (see Section 6 for a discussion on

Table 1: Comparison between singular and bang-bang solutions

m (rad/sec)	t_f(singular)	t_f(bang-bang)
0.5	3.141592	2.328614
1.0	1.570796	1.507866
1.5	1.047197	1.038082
2.0	0.785398	0.783575
4.0	0.392699	0.392674
6.0	0.261799	0.261798
10.0	0.157080	0.157080
∞	0	0

the numerical scheme used in this paper to calculate the optimal trajectories). The results in Table 1 correspond to an inertia parameter $a = 0.5$, but similar results were obtained for other values of a. The conclusions are therefore generic regardless of whether the body is prolate or oblate.

4.2. Case II: only u_1 is singular

In the previous subsection, we have ruled out the possibility that both controls u_1 and u_2 become singular at the same time. This observation is in accordance with similar results for the inertially symmetric case with three controls[5]. In this subsection we will assume that only u_1 is singular during some interval $[t_1,\ t_2] \subset [t_0,\ t_f]$ while u_2 is bang-bang. From Eq. (13) we therefore have $S_1 = \lambda_1 = 0$ which implies that along the singular arc, $\dot{S}_1 = \dot{\lambda}_1 = 0$. Substitution of these two equations into Eq. (18a) yields $\ddot{S}_1 = m(1+a)\dot{\lambda}_2 + (\lambda_3 x_4 - \lambda_4 x_3)x_2$. Because the control u_1 does not appear in the equation of $\ddot{S}_1$, we have to take the third derivative of S_1,

$$S_1^{(3)} = am^3(1+a)\lambda_2 - m(1+2a)(\lambda_3 x_4 - \lambda_4 x_3)x_1 + x_1 x_2 \dot{\lambda}_2 + amx_2^2\lambda_2 + u_2(\lambda_3 x_4 - \lambda_4 x_3) \quad (26)$$

where we have made use of the fact that $\lambda_1 = \dot{\lambda}_1 = 0$ and

$$\frac{d}{dt}(\lambda_3 x_4 - \lambda_4 x_3) = (am\lambda_1 + \dot{\lambda}_2)x_1 + (am\lambda_2 - \dot{\lambda}_1)x_2 \quad (27)$$

Again, the control u_1 does not appear in the equation of $S_1^{(3)}$, so we shall take the fourth derivative of S_1. The control u_1 appears in $S_1^{(4)}$ explicitly, i.e.,

$$S_1^{(4)} = A_1 + B_1 u_1 \quad (28)$$

where A_1 and B_1 are the coefficients which are given by

$$A_1 = [2a(a+1)^2 m^3 - 2amx_1^2 + 2amx_2^2 + 2x_1 u_2]\dot{\lambda}_2 - (4a^2m^2x_1 - 3amu_2)x_2\lambda_2 \quad (29a)$$
$$B_1 = x_2\dot{\lambda}_2 - m(1+2a)(\lambda_3 x_4 - \lambda_4 x_3) \quad (29b)$$

and $\dot{\lambda}_2$ is given by Eq. (10b). From the discussion in Section 4, the optimal singular control u_1 is of second order and is given by

$$u_1 = -\frac{A_1}{B_1} \tag{30}$$

Kelley's necessary condition for optimality requires that

$$B_1 \geq 0 \tag{31}$$

Since the singular control is of second order, one can explicitly calculate the *singular surface*, i.e., the manifold in the state space where all trajectories with control (30) have to live. To construct the singular surface $\mathcal{S}_1$, recall that the following conditions have to be satisfied along a singular arc,

$$\lambda_1 = 0 \tag{32a}$$

$$\dot{\lambda}_1 = am\lambda_2 - \frac{1+x_3^2-x_4^2}{2}\lambda_3 - x_3x_4\lambda_4 = 0 \tag{32b}$$

$$\ddot{\lambda}_1 = [-m(1+a)x_3x_4 + x_2x_4]\lambda_3 - [m(1+a)\frac{1+x_4^2-x_3^2}{2} + x_2x_3]\lambda_4 = 0 \tag{32c}$$

$$\dddot{\lambda}_1 = [am^3(1+a) + amx_2^2]\lambda_2 - [m(1+2a)x_1x_4 - u_2x_4 + x_1x_2x_3x_4]\lambda_3 + [m(1+2a)x_1x_3 - u_2x_3 - x_1x_2\frac{1+x_4^2-x_3^2}{2}]\lambda_4 = 0 \tag{32d}$$

These four equations are homogeneous in terms of the four co-states λ_i, $(i = 1,...,4)$. Setting the determinant of the coefficient matrix to zero, one obtains the singular surface as $\mathcal{S}_1 = \{x \in \mathcal{R}^4 : \det M_1(x) = 0\}$ where $M_1(x)$ is the coefficient matrix from Eq. (32), i.e, $M_1(x)\lambda = 0$. The equation $\det M_1(x) = 0$ can be used in practice to locate the beginning of a singular arc, without keeping track of the co-state λ_1. This is a significant simplification of the control implementation, since the optimal control law can be implemented in a *feedback-form*. Assuming that the optimal switching structure is known, one need not integrate the co-state equations in Eqs. (10). Instead, it is enough to keep track whether the equation $\det M_1(x) = 0$ is satisfied. Notice, however, that care must be exercised when following this approach, since the switching surface $\mathcal{S}_1$ has two branches due to the appearance of the control $u_2 = \pm u_{2\max}$ in Eq. (32d).

4.3. Case III: only u_2 is singular

The same analysis as for the case when only u_1 is singular can be repeated for u_2 if only u_2 is singular while u_1 is bang-bang. In general, the optimal singular control is of second order for u_2 and is given by

$$u_2 = -\frac{A_2}{B_2} \tag{33}$$

where

$$A_2 = [-2a(a+1)^2m^3 - 2amx_1^2 + 2amx_2^2 + 2x_2u_1]\dot{\lambda}_1 - (4a^2m^2x_2 + 3amu_1)x_1\lambda_1 \tag{34a}$$

$$B_2 = x_1\dot{\lambda}_1 - m(1+2a)(\lambda_3x_4 - \lambda_4x_3) \tag{34b}$$

Kelly's necessary condition for optimality requires

$$B_2 \geq 0 \tag{35}$$

The calculation of the singular surface $\mathcal{S}_2$ can be done from the equations

$$\lambda_2 = 0 \tag{36a}$$

$$\dot{\lambda}_2 = -am\lambda_1 - x_3x_4\lambda_3 - \frac{(1+x_4^2-x_3^2)}{2}\lambda_4 = 0 \tag{36b}$$

$$\ddot{\lambda}_2 = [m(1+a)\frac{1+x_3^2-x_4^2}{2} - x_1x_4]\lambda_3 + [m(1+a)x_3x_4 + x_1x_3]\lambda_4 = 0 \tag{36c}$$

$$\dddot{\lambda}_2 = [amx_1^2 - am^3(1+a)]\lambda_1 - [m(1+2a)x_2x_4 + u_1x_4 + x_1x_2\frac{1+x_3^2-x_4^2}{2}]\lambda_3 + [m(1+2a)x_2x_3 + u_1x_3 - x_1x_2x_3x_4]\lambda_4 = 0 \tag{36d}$$

Setting the determinant of the coefficient matrix to zero, one obtains the equation of the singular surface as $\mathcal{S}_2 = \{x \in \mathcal{R}^4 : \det M_2(x) = 0\}$ where $M_2(x)$ is the coefficient matrix from Eq. (36), i.e, $M_2(x)\lambda = 0$. Again, the singular control can be implemented in feedback form, but care must be taken when following this approach since $\mathcal{S}_2$ has two branches due to the appearance of the bang-bang control $u_1 = \pm u_{1\max}$ in Eq. (36d).

5. Special Cases

In the discussion in the previous section, it was assumed that $m \neq 0$ and $a \neq 0$. In this section, we will consider two special cases when $a = 0$ and $m = 0$, respectively. These two cases correspond to an *inertially* symmetric rigid body and a *nonspinning axi-symmetric* rigid body, respectively. For these cases, the equations are simplified significantly and a better insight is gained about the optimal solution.

5.1. Inertially Symmetric Rigid Body $(a = 0)$

For an inertially symmetric rigid body, $a = 0$, and the dynamics are simply

$$\dot{x}_1 = u_1 \tag{37a}$$

$$\dot{x}_2 = u_2 \tag{37b}$$

while the kinematics remain the same as given by Eqs. (8c) and (8d). In this section, we assume that $m \neq 0$, i.e., the rigid body has a nonzero angular velocity component about the $\hat{b}_3$ axis. As before, we examine the three different cases separately.

5.1.1. *Case* I*: both u_1 and u_2 are singular*

In this case, as in Section 4.1, since both u_1 and u_2 are singular, we have $\lambda_1 = \dot{\lambda}_1 = \lambda_2 = \dot{\lambda}_2 = 0$. The Hamiltonian then becomes $\mathcal{H} = 1 + m(\lambda_3 x_4 - x_3\lambda_4)$. From Eq. (18) the second derivatives of S_1 and S_2 yield $\ddot{S}_1 = (\lambda_3 x_4 - \lambda_4 x_3)x_2$ and $\ddot{S}_2 = -(\lambda_3 x_4 - \lambda_4 x_3)x_1$. Since the Hamiltonian has to be zero along the whole trajectory, and since $m \neq 0$, one obtains $(\lambda_3 x_4 - x_3\lambda_4) \neq 0$. Now letting the second derivatives of S_1 and S_2 be zero yields $x_2 = 0$ and $x_1 = 0$. Taking the third time derivative of S_1 and S_2 and letting them be zero we get $u_2 = 0$ and $u_1 = 0$. As in Section 4.1, since the controls u_1 and u_2 appear in the third time derivative of S_1 and S_2, these controls are not optimal[22].

5.1.2. *Case* II*: only u_1 is singular (u_2 is bang-bang).*

In this case, since u_2 is bang-bang, we have $x_2 \neq 0$, except possibly at some isolated points. The control u_1 is assumed singular, so we have $\lambda_1 = \dot{\lambda}_1 = 0$, and following the same approach as in Section 4.2 one obtains that the singular control is of second order and given by

$$u_1 = -\frac{2x_1 u_2 \dot{\lambda}_2}{x_2\dot{\lambda}_2 - m(\lambda_3 x_4 - \lambda_4 x_3)} \tag{38}$$

where

$$\dot{\lambda}_2 = -\lambda_4 \frac{1 + x_4^2 - x_3^2}{2} - \lambda_3 x_3 x_4 \tag{39}$$

Kelley's optimality condition requires $x_2\dot{\lambda}_2 - m(\lambda_3 x_4 - \lambda_4 x_3) \geq 0$. Since in this case the Hamiltonian takes a simple form

$$\mathcal{H} = 1 + \lambda_2 u_2 - \dot{\lambda}_2 x_2 + m(\lambda_3 x_4 - \lambda_4 x_3) \tag{40}$$

and since $u_2 = -u_{2\max}\mathrm{sgn}(\lambda_2)$ we have

$$\dot{\lambda}_2 x_2 - m(\lambda_3 x_4 - \lambda_4 x_3) = 1 + \lambda_2 u_2 = 1 - u_{2\max}|\lambda_2| \tag{41}$$

so the Kelley's optimality condition is equivalent to $u_{2\max}|\lambda_2| \leq 1$. From Eqs. (38) and (41), we can see that the optimal control u_1 is only defined when $|\lambda_2| \neq 1/u_{2\max}$ except at some isolated points. In this case $\dot{\lambda}_2 \neq 0$, and since $x_2 \neq 0$, from Eq. (18a) $\ddot{S}_1 = 0$ implies that $\lambda_3 x_4 - \lambda_4 x_3 \neq 0$. Thus λ_3 and λ_4 cannot be zero at the same time. To construct the singular surface, recall the following necessary conditions along the singular arc

$$\dot{\lambda}_1 = -\frac{1 + x_3^2 - x_4^2}{2}\lambda_3 - x_3 x_4 \lambda_4 = 0 \tag{42a}$$

$$\ddot{\lambda}_1 = [-m x_3 x_4 + x_2 x_4]\lambda_3 - [m\frac{1 + x_4^2 - x_3^2}{2} + x_2 x_3]\lambda_4 = 0 \tag{42b}$$

$$\dddot{\lambda}_1 = -[m x_1 x_4 - u_2 x_4 + x_1 x_2 x_3 x_4]\lambda_3 + [m x_1 x_3 - u_2 x_3 - x_1 x_2 \frac{1 + x_4^2 - x_3^2}{2}]\lambda_4 = 0 \tag{42c}$$

Since $(\lambda_3, \lambda_4) \neq (0,0)$, one gets the singular surface as $\mathcal{S}_1 = \{x \in \mathcal{R}^4 : \det[M_1(x)^T M_1(x)] = 0\}$ where $M_1(x)$ is the coefficient matrix from Eq. (42), i.e., $M_1(x)(\lambda_3, \lambda_4)^T = 0$. It is possible that $|\lambda_2| = 1/u_{2\max}$ during the singular arc and u_1 is no longer defined in Eq. (38). In this case, since λ_2 is continuous[14], either $\lambda_2 = 1/u_{2\max}$ or $\lambda_2 = -1/u_{2\max}$ holds. Therefore, $\dot{\lambda}_2 = 0$, and $\ddot{S}_1 = 0$ implies $\lambda_3 x_4 - \lambda_4 x_3 = 0$ since $x_2 \neq 0$. From Eq. (18a), we can see that $S_1^{(3)}$ automatically equals to zero, and all higher order derivatives of S_1 will be zero identically as well. Therefore, the optimal control u_1 is an infinite order singular control and it can be chosen arbitrarily as long as the boundary conditions are satisfied[5]. In this case, substitution of $\lambda_3 x_4 - \lambda_4 x_3 = 0$ into $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = 0$ yields $\lambda_3 = \lambda_4 = 0$ during the infinite order singular arc.

5.1.3. *Case* III*: only u_2 is singular (u_1 is bang-bang).*

This case is similar to Case II above. If $|\lambda_1| \neq 1/u_{1\max}$ except possible at some isolated points, the optimal singular control u_2 is second order and is written as

$$u_2 = -\frac{2x_2 u_1 \dot{\lambda}_1}{x_1\dot{\lambda}_1 - m(\lambda_3 x_4 - \lambda_4 x_3)} \tag{43}$$

where

$$\dot{\lambda}_1 = -\lambda_3 \frac{1 + x_3^2 - x_4^2}{2} - \lambda_4 x_3 x_4 \tag{44}$$

Kelley's optimality condition requires $|\lambda_1| \leq 1/u_{1\max}$. The singular surface can be calculated from the following equations

$$\dot{\lambda}_2 = -x_3 x_4 \lambda_3 - \frac{(1 + x_4^2 - x_3^2)}{2}\lambda_4 = 0 \tag{45a}$$

$$\ddot{\lambda}_2 = [m\frac{1 + x_3^2 - x_4^2}{2} - x_1 x_4]\lambda_3 + [m x_3 x_4 + x_1 x_3]\lambda_4 = 0 \tag{45b}$$

$$\dddot{\lambda}_2 = -[m x_2 x_4 + u_1 x_4 + x_1 x_2 \frac{1 + x_3^2 - x_4^2}{2}]\lambda_3 + [m x_2 x_3 + u_1 x_3 - x_1 x_2 x_3 x_4]\lambda_4 = 0 \tag{45c}$$

The singular surface is obtained as $\mathcal{S}_2 = \{x \in \mathcal{R}^4 : \det[M_2(x)^T M_2(x)] = 0\}$ where $M_2(x)$ is the coefficient matrix from Eq. (45), i.e., $M_2(x)(\lambda_3, \lambda_4)^T = 0$. If $|\lambda_1| = 1/u_{1\max}$ along the singular trajectory, the optimal singular control u_2 is of infinite order and can be chosen arbitrarily as long as all the boundary conditions are satisfied. In this case necessarily $\lambda_3 = \lambda_4 = 0$ during the singular arc.

5.2. Nonspinning Axi-Symmetric Rigid Body ($m = 0$)

If the body is not spinning, i.e., $m = 0$, then the system equations are simplified as

$$\dot{x}_1 = u_1 \tag{46a}$$
$$\dot{x}_2 = u_2 \tag{46b}$$
$$\dot{x}_3 = x_2x_3x_4 + x_1(1 + x_3^2 - x_4^2)/2 \tag{46c}$$
$$\dot{x}_4 = x_1x_3x_4 + x_2(1 + x_4^2 - x_3^2)/2 \tag{46d}$$

These equations describe the dynamics and kinematics of both an axi-symmetric and an inertially symmetric body. Again in the analysis, we will discuss the possibility of both controls and only one control being singular.

5.2.1. *Case* I*: both u_1 and u_2 are singular*

In this case, as in Section 4.1, since both u_1 and u_2 are singular during $[t_1,\, t_2] \subset [t_0,\, t_f]$, we get $\lambda_1 = \dot{\lambda}_1 = \lambda_2 = \dot{\lambda}_2 = 0$. Substituting these equations into Eq. (9), it follows immediately that $\mathcal{H} = 1$. This contradicts the necessary condition which states that the Hamiltonian has to be zero along the whole trajectory. Therefore both controls being singular is impossible for a nonspinning symmetric body.

5.2.2. *Case* II*: only u_1 is singular (u_2 is bang-bang)*

In this case, since $u_2 = -u_{2\max}\,\mathrm{sgn}(\lambda_2)$, we have $x_2 \neq 0$ except possibly at some isolated points. The control u_1 is assumed to be singular, so by taking successive derivatives of λ_1 one obtains $\dot{\lambda}_1 = -\lambda_3(1 + x_3^2 - x_4^2)/2 - \lambda_4x_3x_4 = 0$ and $\ddot{\lambda}_1 = (\lambda_3x_4 - \lambda_4x_3)x_2 = 0$. Since $x_2 \neq 0$, necessarily $\lambda_3x_4 - \lambda_4x_3 = 0$. From Eq. (26) we have for the third derivative of S_1,

$$S_1^{(3)} = x_1x_2\dot{\lambda}_2 \tag{47}$$

If $\dot{\lambda}_2 \neq 0$, then letting $S_1^{(3)} = 0$, we have $x_1 = 0$. Taking the fourth derivative of S_1, we have that u_1 explicitly appears in $S_1^{(4)}$. Letting $S_1^{(4)} = 0$, we get the explicit expression for the second order optimal singular control u_1 as

$$u_1 = 0 \tag{48}$$

Kelley's optimality condition requires that $x_2\dot{\lambda}_2 \geq 0$. Since in this case the Hamiltonian takes the simple form $\mathcal{H} = 1 + \lambda_2u_2 - \dot{\lambda}_2x_2$ we have

$$\dot{\lambda}_2x_2 = 1 + \lambda_2u_2 = 1 - u_{2\max}|\lambda_2| \tag{49}$$

and Kelley's optimality condition is equivalent to $u_{2\max}|\lambda_2| \leq 1$. Substituting $\lambda_3x_4 - \lambda_4x_3 = 0$ into $\dot{\lambda}_1 = 0$, we get $\lambda_3 = 0$ necessarily during the singular arc, so $\dot{\lambda}_2 \neq 0$ implies $\lambda_4 \neq 0$. Therefore, $\lambda_3x_4 - \lambda_4x_3 = 0$ implies that $x_3 = 0$. Thus the singular surface in this case is defined by $\mathcal{S}_1 = \{x \in \mathcal{R}^4 : x_1 = x_3 = 0\}$. If $\dot{\lambda}_2 = 0$ during the singular arc, from Eq. (47), we can see that $S_1^{(3)}$ automatically equals to zero, and all higher order derivatives of S_1 will be zero identically as well. Therefore, the optimal control u_1 is an infinite order singular control and it can be chosen arbitrarily as long as the boundary conditions are satisfied[5]. From Eq. (49), $\dot{\lambda}_2 = 0$ implies $\lambda_2 = \pm 1/u_{2\max}$. Substitution of $\lambda_3x_4 - \lambda_4x_3 = 0$ into $\dot{\lambda}_1 = 0$ and $\dot{\lambda}_2 = 0$ yields that $\lambda_3 = \lambda_4 = 0$ during an infinite order singular arc.

5.2.3. *Case* III*: only u_2 is singular (u_1 is bang-bang).*

Consider the case only u_2 is singular and u_1 is bang-bang, then the same analysis as in Case II yields the following results. If $\dot{\lambda}_1 \neq 0$, except possible at some isolated points, the optimal singular control u_2 is of second order and given by

$$u_2 = 0 \tag{50}$$

Kelley's optimality condition requires that $u_{1\max}|\lambda_1| \leq 1$. The singular surface is defined as $\mathcal{S}_2 = \{x \in \mathcal{R}^4 : x_2 = x_4 = 0\}$. As another necessary condition, during the singular arc, $\lambda_4 = 0$. If $\dot{\lambda}_1 = 0$ along the singular arc, the optimal singular control u_2 is of infinite order and can be chosen arbitrarily as long as all the boundary conditions are satisfied. In this case, $\lambda_1 = \pm 1/u_{1\max}$ and $\lambda_3 = \lambda_4 = 0$.

Remark: From Eqs. (48) and (50), it can be seen that the second order singular arc for a nonspinning body is an eigenaxis rotation.

6. A Numerical Approach for Computing Optimal Solutions

The optimal solutions are obtained numerically using a cascaded computational scheme. Both a direct method and an indirect method are used in this scheme. A direct method is applied first to get initial guesses for the indirect method, which is then solved to obtain accurate optimal solutions. The idea of combining direct and indirect methods for solving optimal control problems was introduced by Stryk and Bulirsch[23] and later by Seywald and Kumar[24] to take advantage of both the good convergence properties of the direct methods and and accuracy of the indirect methods.

Three programs are used in this numerical approach: EZopt, COSCAL and BNDSCO. BNDSCO solves the problem using an indirect method, i.e., the optimal solutions are determined by solving the Multipoint Boundary Value Problem arising from Pontryagin's Minimum Principle using a multiple shooting method[5,25]. It converges quickly to the optimal solution, and the solution obtained is of very high accuracy. However, the radius of convergence of this method is rather small since it requires very good initial guesses for the states, controls, co-states, Lagrange multipliers and switching structure[23,24]. A major difficulty lies in the fact that in most cases we do not know the optimal switching structure in advance. Also, initial guesses for the co-states and the Lagrange multipliers are nontrivial because these do not, in general, have any intuitive physical interpretation.

EZopt[26] solves the problem using a direct method, namely, the optimal solution is determined by directly minimizing the cost criterion through collocation and nonlinear programming. The radius of convergence of the direct method is usually much larger than that of the indirect method[23,24]. The program converges upon much less accurate initial guesses. The speed of convergence is, however, much lower compared with BNDSCO. Since this method does not involve co-states, one needs to provide only initial guesses for the states and controls. In addition, the switching structure does not have to be known in advance. A disadvantage of this method is that the solutions obtained may not be as accurate as those obtained from a indirect method[6,23]. This is especially true around switching points and when singular subarcs appear as part of the overall optimal solution. The accuracy of the solution depends on the discretization scheme and the number of the discrete nodes. However, these solutions are good enough to roughly determine the trajectories, states, controls, switching structure and, if there exist, singular subarcs. Thus, they provide good initial guesses for a direct optimization software such as BNDSCO.

Based upon the foregoing discussion, we have developed a software package which combines the two programs (EZopt and BNDSCO) together to overcome the drawbacks of each method. That is, we use the results from EZopt as an initial guess for BNDSCO. With this initial guess, BNDSCO typically converges very fast and gives accurate and reliable results. In addition, the optimality of the solution can be readily checked from the time history of the corresponding switching functions. One major obstacle with this approach is that BNDSCO needs the initial guesses for the co-states (in addition to the states and the correct switching structure) which EZopt does not provide. Thus, the program COSCAL was developed to calculate the co-states at each node from the Kuhn-Tucker multipliers associated with the nonlinear programming. In the following section, we will briefly discuss the methodology used in COSCAL for estimating the co-states and Lagrangian multipliers. For a complete discussion of this approach, one may consult Ref. 24.

7. Numerical Examples

EZopt, COSCAL and BNDSCO together form a cascaded computational scheme which is very effective in carrying out the optimal control computations. It has been used extensively by the authors to solve several optimal control problems. In this section we present two numerical examples for the min-time reorientation problem, one demonstrating a bang-bang control and the other one demonstrating a solution with a singular subarc.

7.1. Bang-bang control example

For the problem at hand, bang-bang control is obtained in most situations, including both rest-to-rest and non rest-to-rest maneuvers. The optimal control is given by Eq. (13). As an example of a bang-bang maneuver, consider the following initial and final conditions $x(0) = [0,0,1.5,-0.5]$ and $x(t_f) = [0,0,0,0]$. The parameters a and m are chosen to be $a = 0.5$ and $m = -0.5$ rad/sec. The control inputs are assumed to be bounded by $u_{1\max} = u_{2\max} = 1.0$.

This example represents a "rest-to-rest" maneuver with respect to the two control axes since the body is spinning about its symmetry axis at a constant rate. The initial boundary condition corresponds to a relative position such that the initial angle between the $\hat{b}_3$ axis and $\hat{n}_3$ axis is 115.38 deg. An optimal control is to be found to re-orient the body until $\hat{b}_3$ and $\hat{n}_3$ axes are aligned in a rest situation. The optimal control for this example was found to be bang-bang with the first control having one switch and the second control having two switches. The minimum time to complete this maneuver is 2.61 sec.

Figures 3 and 4 show the control inputs and the corresponding switching functions. Figure 5 shows the history of the angular velocities ω_1 and ω_2, and Fig. 6 shows the time history of w_1 and w_2. Recall that w_1 and w_2 represent the relative position of the inertial axis $\hat{n}_3$ with respect to the body fixed frame $\hat{\mathbf{b}}$. In these figures, the solid lines stand for the optimal results obtained from BNDSCO (states, controls and co-states), while the circles show the initial guesses obtained from EZopt (states and controls) or COSCAL (co-states). From these plots, one can see that the solution obtained from EZopt almost captures the properties of the optimal solution, although some discrepancy exists at the switching points. The plots also show that COSCAL provides very accurate guesses for the co-states.

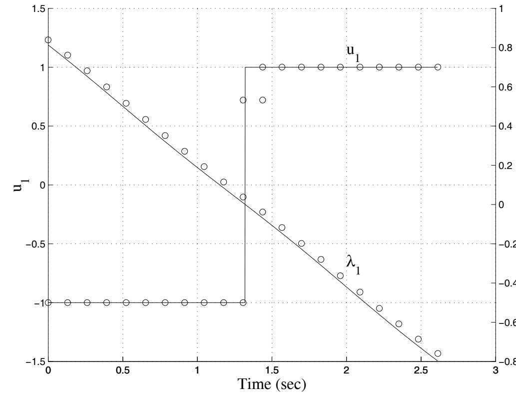

Figure 3: Control u_1 and co-state λ_1 for bang-bang maneuver.

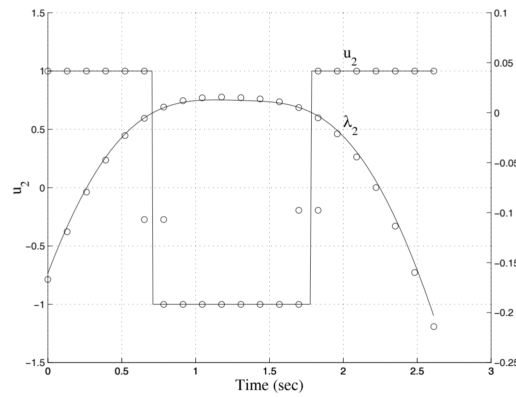

Figure 4: Control u_2 and co-state λ_2 for bang-bang maneuver.

For the calculations shown in Figs. 3-6, 21 nodes were used for EZopt. The program converged in 2 minutes on a SPARCstation 5. BNDSCO converged in about 2 seconds.

7.2. Finite order singular control example

Finite order singular subarcs can be part of an optimal solution only in some particular situations. As an example, a second order singular control is observed for the boundary conditions $x(0) = [-0.45, -1.1, 0.1, -0.1]$ and $x(t_f) = [0, 0, 0, 0]$. The parameters for this case are given by $a = 0.5$ and $m = 0$, and the control inputs are bounded by $u_{1\max} = u_{2\max} = 1.0$. Initially, the angle between $\hat{b}_3$ axis and $\hat{n}_3$ axis is 16.1 deg for these values of w_1 and w_2. The control is required to re-orient the body until the symmetry axis $\hat{b}_3$ aligns with the inertial axis $\hat{n}_3$ in a rest situation. It turns out that the first control has a singular subarc and the second control is bang-bang. The optimal singular control is given by Eq. (48). The minimum time is 2.88 sec. The results of the numerical

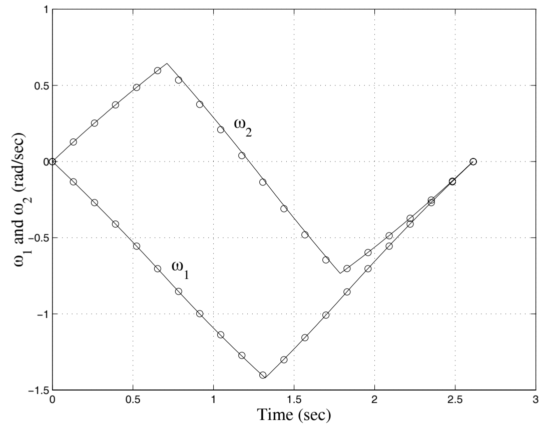

Figure 5: Angular velocities ω_1 and ω_2 for bang-bang maneuver.

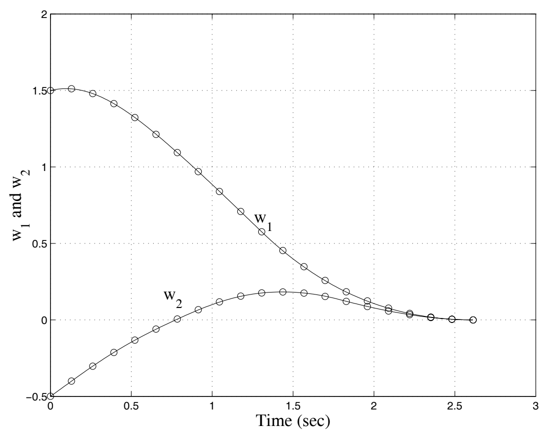

Figure 6: Time histories of w_1 and w_2 for bang-bang maneuver.

simulations are shown in Figs. 7-11. It should be noted here that although the analysis in Section 3 does not preclude the existence of singular subarcs in the case when $m \neq 0$, we were unable to find optimal trajectories with singular subarcs for the spinning case. Figures 7 and 8 show the control inputs and the corresponding switching functions. Figure 9 shows the time history of the angular velocities ω_1 and ω_2 and Fig. 10 shows the time history of w_1 and w_2. Figure 11 shows the same trajectory on the w_1-w_2 plane. From Fig. 7 we can see that the control u_1 is singular after $t = 1.904$ sec. In the w_1-w_2 plane, if the body is not spinning about its symmetry axis, then an eigenaxis rotation is represented by a straight line. Figure 11 indicates that during the bang-bang subarc, the time-optimal trajectory is not an eigenaxis rotation and during the singular subarc the time optimal trajectory is an eigenaxis rotation.

In these figures, the solid lines show the optimal solution obtained from the solution of the TPBVP and the stars indicate the solution of the direct method, which was used as an initial guess for the TPBVP solver. From these plots it is seen that the bang-bang subarc obtained from EZopt captures the optimal bang-bang subarc very well. The singular control subarc obtained from EZopt is not as accurate, but the output from EZopt gives a good understanding about the existence and location of a singular subarc. Again, COSCAL captures the time history of the co-states pretty well.

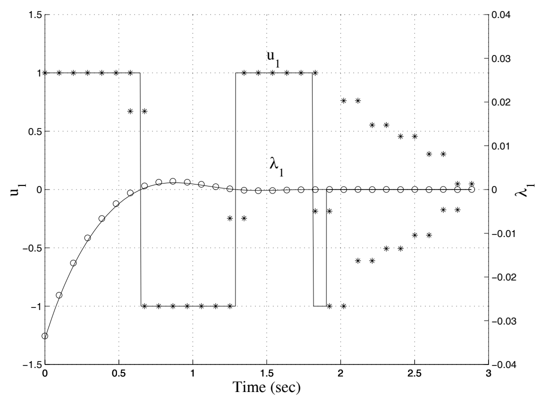

Figure 7: Control u_1 and co-state λ_1 for singular subarc maneuver.

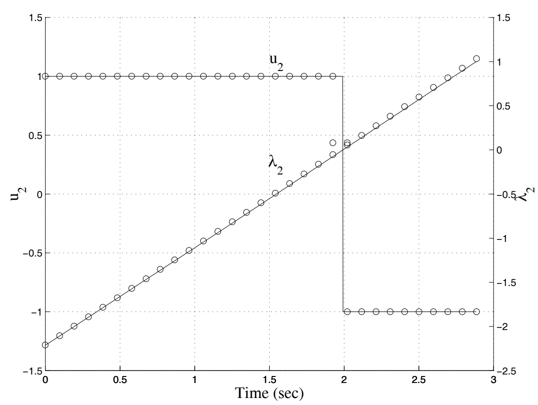

Figure 8: Control u_2 and co-state λ_2 for singular subarc maneuver.

In this example, the calculations in EZopt were performed using 31 nodes, which required EZopt about 5 minutes to converge from an all-zero initial guess. On the other hand, using the output from EZopt/COSCAL, BNDSCO converged in about 2 seconds.

The appearance of a singular subarc in the optimal trajectory deserves special mention. Kelley's necessary condition (which was found to be satisfied for this example) alone does *not* guarantee that the singular subarc will indeed be part of the composite optimal trajectory. The boundary conditions will determine if this is true

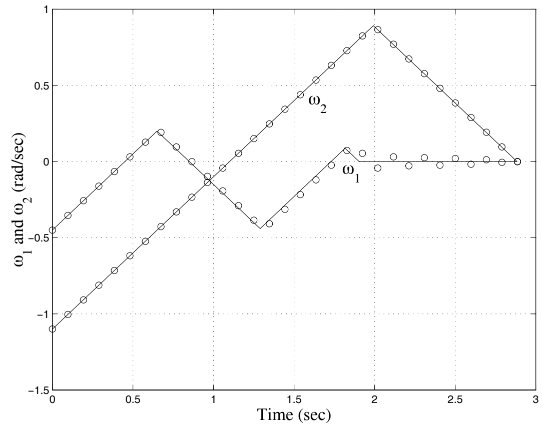

Figure 9: Angular velocities ω_1 and ω_2 for singular subarc maneuver.

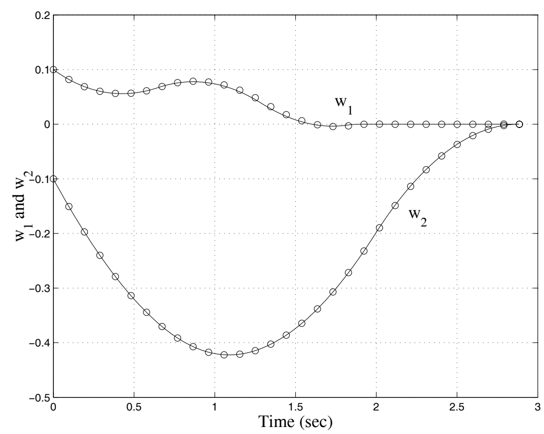

Figure 10: Time histories of w_1 and w_2 for singular subarc maneuver.

or not. Even in the case when a trajectory composed of bang-bang and singular subarcs satisfies the boundary conditions and the first order necessary conditions (including Kelley's condition on the singular subarc) it is still not guaranteed that this solution is optimal. In particular, the joining between bang-bang and singular subarcs has to satisfy certain corner conditions. Such optimality conditions are given in Refs. 22 and 27. For a second order singular arc (as the one in Fig. 7) the main condition in Ref. 27 states that the optimal control should be continuous at each junction, i.e., a jump discontinuity when joining a non-singular and a singular control is not allowed. At first glance, this seems to contradict the result in Fig. 7. However, the results in Ref. 27 assume that the control is piecewise analytic. This is shown not to be the case for singular arcs of even order (and in fact even for arcs of odd order greater than one) in Ref. 28; see also Ref. 29. Thus, for even-order singular subarcs (and odd-order singular arc of order greater than one)

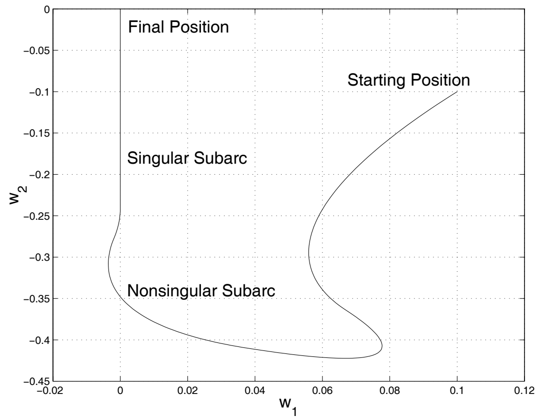

Figure 11: Optimal trajectory with singular subarc in w_1-w_2 plane.

the junction between singular and non-singular arcs is not analytic, i.e., the control consists of a sequence of an infinite number of switchings between $u = u_{\min}$ and $u = u_{\max}$ with the time between switchings rapidly decreasing. More relevant to our case is the fact that singular controls may manifest themselves as the cumulative effect of the infinite number of bang-bang control actions (chattering). If this is the case, the solution of the the differential equations have to be interpreted in the Filippov sense[30], and the singular control is then the "equivalent" control action associated with the chattering control[29].

The previous discussion reinforces our observations for the singular control in Fig. 7. The solution from EZopt shows that the optimal control switches rapidly after $t = 1.904$. The subarc after that point is identified as a singular subarc, and the solid line stands for the optimal solution (given from BNDSCO) which uses the "equivalent" singular control $u_1 = 0$, obtained using the necessary conditions. This singular control has the equivalent effect of a bang-bang control with infinite number of switchings. It must be pointed out that the substitution of a chattering bang-bang control with its "equivalent" singular form is more than a mathematical convenience. In practice, it is often preferable to use the "equivalent" singular control action instead of switching between the upper and lower bounds infinitely fast. At any rate, in both cases, the optimal state trajectory is the same.

7.3. Infinite order singular control example

As a demonstration of the infinite order singular control, the boundary conditions $x(0) = [0,0,0,0]$ and $x(t_f) = [1.0, 2.0, \text{free}, \text{free}]$ are considered. $a = 0$, $m = -0.3$ and $u_{1\max} = u_{2\max} = 1$ are also assumed in this example. The infinite order singular control corresponding to these parameters are discussed in Section 5.1.2. From the boundary conditions we can see that the purpose of this maneuver is to accelerate the angular velocity components ω_1 and ω_2 from zero to 1.0 rad/sec and 2.0 rad/sec, respectively. The final position is not important in this case. Since $\dot{x}_1 = u_1$ and $\dot{x}_2 = u_2$, the minimum time that x_2 reaches 2.0 rad/sec is 2 seconds, during which, x_1 can obtain its final value $x_1(t_f) = 1$ rad/sec in many ways. Therefore, an infinite order singular control is a possible solution for this problem. The numerical results are shown in Figs. 12 and 13. Two possible solutions for u_1 are presented. Similar results for infinite order singular arcs can be obtained when $m = 0$, with this case discussed in section 5.2.2.

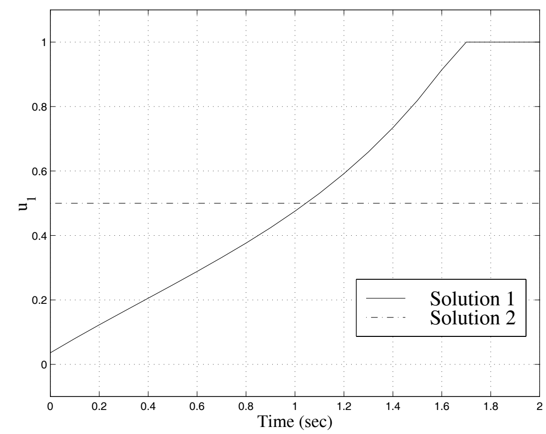

Figure 12: Two possible solutions for u_1 for infinite order singular arc maneuver.

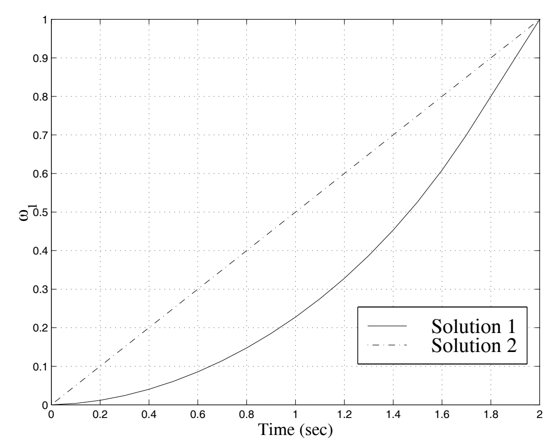

Figure 13: Two possible solutions for ω_1 for infinite order singular arc maneuver.

8. Conclusions

The time-optimal reorientation control problem of an axi-symmetric rigid spacecraft with two control torques has been studied in detail. It is assumed that no control torque is available along the symmetry axis. The spacecraft may be spinning about its symmetric axis. The relative rotation about the symmetry axis is therefore indeterminate. A complete analysis of all the possible time-optimal control structures is presented, including cases with singular and nonsingular subarcs. It is shown that second order singular arcs and infinite order singular arcs can appear as part of the optimal trajectory for specific boundary conditions. A cascaded computational scheme is developed and used for the numerical computation of the optimal trajectories. The method does not require any a priori knowledge of the optimal switching structure. Examples show that this is a very effective approach to compute optimal trajectories numerically.

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