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REDUCED-EFFORT CONTROL LAWS FOR UNDERACTUATED RIGID SPACECRAFT*

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Abstract

Recent results show that a nonsmooth, time-invariant feedback control law can be used to stabilize an axi-symmetric rigid body using only two control torques to the zero equilibrium. This method, however, may require a significant amount of control effort, especially for initial conditions close to an equilibrium manifold. In this paper we propose a control law which reduces the control effort required. The new control law renders the equilibrium manifold unstable and drives the trajectories of the closed-loop system into a "safe" region where the original control law can be subsequently used.

1. Introduction

The problem of stabilization of a rigid body using less than three control inputs has received a lot of attention in the recent literature. Both the problems of the stabilization of the dynamics, and the stabilization of the kinematics have been treated in the literature[1–6]. The stabilization problem of the complete system, i.e., the dynamics and the kinematics, has been addressed in Refs. 7–13. The attitude stabilization of an axially symmetric rigid body using two independent control torques was studied by Krishnan, *et al.*[8,9] and Tsiotras *et al.*[10]. If the uncontrolled principal axis is not the axis of symmetry the system is strongly accessible and small time locally controllable[9]. When the uncontrolled axis coincides with the axis of symmetry, the complete system fails to be controllable or even accessible. However, the system equations are strongly accessible and small time locally controllable in the case of zero spin rate. A nonlinear control approach was developed in Ref. 8, which achieves arbitrary reorientation for this restricted case. In Refs. 14,15 the authors presented a new formulation of the attitude kinematics which was used in Ref. 10 to solve the same problem avoiding the successive switchings of Ref. 8. References 8 and 10 treat the axi-symmetric case. The non-symmetric case is dealt with in Refs. 11–13 and 16.

In this paper, we provide a modification of the control law presented in Ref. 10 for the attitude stabilization of an axi-symmetric rigid body using two independent control torques. Because the system has an equilibrium manifold which includes the origin, Brockett's necessary condition for smooth stabilizability is not satisfied and thus, any stabilizing control law is necessarily nonsmooth. (Stabilizing *time-varying* smooth control laws may still exist, however.) This nonsmoothness is evident in Ref. 10 in the form of the non-differentiability of the control law at the origin. Because of this singularity at the origin, this control law may take large values, especially for initial conditions close to the equilibrium manifold. Compared to the control law in Ref. 10 the control law proposed in this paper remedies this high control authority problem by driving the trajectories of the closed-loop system away from the singular equilibrium manifold and to a region in the state space where the "high authority" part of the control input remains small and bounded. The procedure is simple and can be easily validated from phase portrait considerations. A numerical example illustrates the control effort improvement using the new control law.

2. The Underactuated Spacecraft

The dynamics of a rigid spacecraft with two controls can be written as

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 \quad (1a)$$
$$\dot{\omega}_2 = a_2\omega_3\omega_1 + u_2 \quad (1b)$$
$$\dot{\omega}_3 = a_3\omega_1\omega_2 \quad (1c)$$

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where a_i are the inertia parameters satisfying $a_1 + a_2 + a_3 + a_1 a_2 a_3 = 0$. Here we assume a body-fixed reference frame along the principal axes of inertia.

Equations (1) describe an underactuated spacecraft with no control authority about the $3th$ principal axis. Notice that in this case ω_3 can be controlled only indirectly through judicious choice of the time histories of $\omega_1(t)$ and $\omega_2(t)$. In case of an *axi-symmetric* body (about the 3-axis), $a_3 = 0$ and $a_1 = -a_2 = a$ and Eqns. (1) reduce to

$$\begin{aligned} \dot{\omega}_1 &= a\,\omega_{30}\omega_2 + u_1 && \text{(2a)} \\ \dot{\omega}_2 &= -a\,\omega_{30}\omega_1 + u_2 && \text{(2b)} \\ \dot{\omega}_3 &= 0 && \text{(2c)} \end{aligned}$$

where $\omega_3(0) = \omega_{30}$ is constant. Introducing the complex variables $\omega = \omega_1 + i\,\omega_2$ and $u = u_1 + i\,u_2$ (with $i = \sqrt{-1}$) the previous equations can be written as

$$\dot{\omega} = -i\,a\,\omega_{30}\omega + u \tag{3}$$

3. Kinematics of the Attitude Motion

The orientation of a rigid spacecraft can be specified using various parameterizations, for example, Eulerian Angles, Euler Parameters, Cayley-Rodrigues Parameters, Cayley-Klein parameters, etc. Recently, a new parameterization using a pair of a complex and a real coordinate was introduced[14,15] which was shown to have some significant advantages for attitude analysis and control problems[10,17,18]. According to these results, the relative orientation between two given reference frames can be represented by *two rotations*, one corresponding to the real coordinate (z) and the other corresponding to the complex coordinate (w).

The kinematic equations, which provide the geometric constraints of the motion and relate the rates of the kinematic parameters z and w to the angular velocity vector, can be written as follows[10,15]

$$\begin{aligned} \dot{w} &= -i\,\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 && \text{(4a)} \\ \dot{z} &= \omega_3 + Im(\omega\bar{w}) && \text{(4b)} \end{aligned}$$

where $\omega = \omega_1 + i\,\omega_2$ and $w = w_1 + i\,w_2$. Notice that these equations can take the convenient form

$$\begin{aligned} \frac{d}{dt}|w|^2 &= (1 + |w|^2) Re(\omega\bar{w}) && \text{(5a)} \\ \dot{z} &= \omega_3 + Im(\omega\bar{w}) && \text{(5b)} \end{aligned}$$

where bar denotes complex conjugate, $Re(\cdot)$ and $Im(\cdot)$ denote real and imaginary parts of a complex number respectively, and $|\cdot|$ denotes absolute value. In Eq. (5b) only the imaginary part of the product $\omega\bar{w}$ appears, while in Eq. (5a) only the real part appears. This duality (or anti-symmetry) of Eqs. (5a) and (5b) is desirable and can be used to derive stabilizing control laws for the system of Eqs. (4). Clearly, $w = 0$ if and only if $|w| = 0$ and stabilization of the system in Eqs. (4) is equivalent to stabilization of the system in Eqs. (5).

4. Problem Statement

Consider an axi-symmetric body with the applied torque vector in the plane which is perpendicular to the symmetry axis. In such a case the system is described by Eqs. (2) and thus ω_3 remains constant. If initially $\omega_3(0) \neq 0$, no control input can bring the system to the equilibrium. The system is not controllable to the equilibrium but it is controllable to the submanifold $\omega = w = 0$ in the (ω, ω_3, z, w)-space. For a more detailed discussion on this issue, one may peruse Refs. 8–10. Therefore, for an axi-symmetric body, the stabilization to the equilibrium of the system in Eqs. (3)-(4) really makes sense only if $\omega_3 \equiv 0$. In this case, the system equations simplify to

$$\begin{aligned} \dot{\omega} &= u && \text{(6a)} \\ \dot{w} &= \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 && \text{(6b)} \\ \dot{z} &= Im(\omega\bar{w}) && \text{(6c)} \end{aligned}$$

This system can be stabilized to the origin, but any time-invariant stabilizing control law has to be necessarily *nonsmooth*, since Eqs. (6) fail Brockett's necessary condition for smooth stabilizability[19]. One is therefore compelled to use nonsmooth (albeit time-invariant) stabilizers for this system.

Equations (6) represent a system in cascade form, with the kinematics (6b)-(6c) the driven subsystem and the dynamics (6a) the driving subsystem. The methodology in Ref. 10 used this fact to derive a non-smooth control law to stabilize Eqs. (6). In essence, the controller design consists of a two-step process. In the first step only stabilization of the kinematics is addressed, with the angular velocity treated as the control input. In the second step the control torque u is chosen to shape the desired velocity profile. Since the angular velocity in the first step is (necessarily) a nonsmooth function of w and z, caution should be exercised when implementing this angular velocity in the second step. The nonsmooth controller of Ref. 10 along with its potential drawbacks is summarized in the next section.

5. A Nonsmooth Controller for the Kinematics

In Ref. 10 a nonsmooth control law was proposed for the kinematic system described by

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \quad (7a)$$
$$\dot{z} = Im(\omega \bar{w}) \quad (7b)$$

and was later implemented though the integrator in Eq. (6a). The proposed control law in Ref. 10 was motivated by the decoupling of these equations with respect to the product $\omega \bar{w}$, as it is evident from the discussion following Eqs. (5). This control law is given by

$$\omega = -\kappa w - i\mu \frac{z}{\bar{w}} \quad (8)$$

where $\mu > \kappa/2 > 0$. With this control law the closed loop system in terms of $|w|$ and z is given by

$$\frac{d|w|^2}{dt} = -\kappa(1 + |w|^2)|w|^2 \quad (9a)$$
$$\dot{z} = -\mu z \quad (9b)$$

which is globally exponentially stable. As can be easily inferred by observing Eqs. (8) and (5) the first term in the control (8) has an effect only on the differential equation for w, whereas the second term in Eq. (8) has an effect only on the differential equation for z. Moreover, the second term in Eq. (8) is a nonsmooth function of z and w.

The main disadvantage of the control law in Eq. (8) is that the last term, which involves the ratio $z/\bar{w}$, may become unbounded without careful choice of the gains. The previously imposed gain condition $\mu > \kappa/2$ ensures that the rate of decay of z is at least as large as the rate of decay of w, such that their ratio remains bounded. Actually, one can easily establish from Eqs. (9) that for $\mu > \kappa/2$, along the solutions of the system, one has $z/\bar{w} \to 0$ as $t \to \infty$.

Introducing the variable $v = |w|^2$ the system in Eqs. (9) takes the form

$$\dot{v} = -\kappa(1 + v)v \quad (10a)$$
$$\dot{z} = -\mu z \quad (10b)$$

This is a system which evolves on $\mathbb{R}_+ \times \mathbb{R}$. Typical trajectories and the vector field of the closed-loop system in Eq. (10) for $\kappa = 1$ and $\mu = 2$ are shown in Fig. 1. (Since z does not change sign it suffices to plot only the $z > 0$ case.)

Although in Eq. (8) the ratio $z/\bar{w}$, and hence the control effort ω, remains bounded by proper choice of control gains, the control input ω may take large

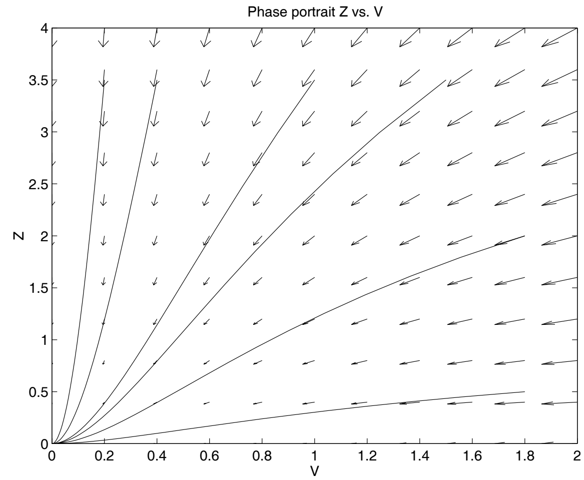

Figure 1: Phase portrait of system in Eqs. (9).

values in the region where w is small. From Eq. (9a) $|w(t)| \leq |w(0)|$ for all $t \geq 0$ and for small initial conditions $w(0)$ the control law may use a substantial amount of energy, especially in regions where $|z|$ is large. In Fig. 1, for example, the region which is close to the z axis is clearly undesirable as far as control expenditure is concerned. We wish to modify the control law in Eq. (8) such that the vector field close to the z axis points away from this axis. In short, the idea is to divide the (z, v) phase space into two regions according to the value of the ratio

$$\eta = \frac{z}{|w|^2} = \frac{z}{v} \quad (11)$$

This ratio is a direct indication of the relative magnitude between z and w. This ratio should be kept small in order to avoid high control effort. Hence, if initially the states are in an undesirable region where η attains large values, the feedback control strategy should drive the trajectories to a "safe" region in the state space where η remains relatively small. Without loss of generality, we choose as undesirable the region where $|\eta| > 1$ and as desirable the region where $|\eta| \leq 1$. These two regions, denoted by $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively, are therefore defined by

$$\mathcal{D}_1 = \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : \infty > |\eta| > 1\} \quad (12a)$$
$$\mathcal{D}_2 = \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : |\eta| \leq 1\} \quad (12b)$$

These two regions are shown in Fig. 2.

6. Main Results

The proposed modification of the control law in Eq. (8) is simple. We use positive feedback for v when the

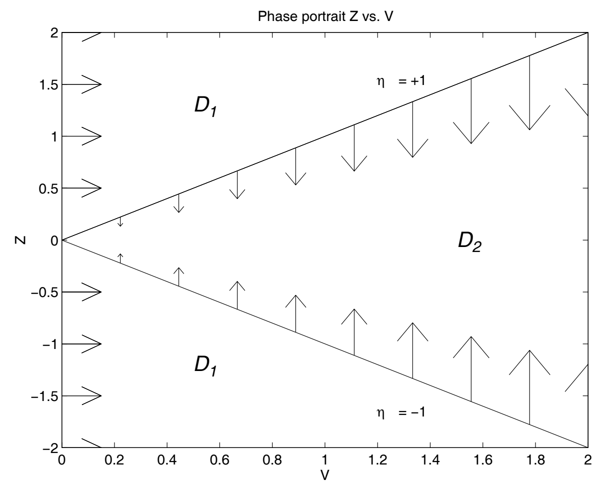

Figure 2: Regions $\mathcal{D}_1$ and $\mathcal{D}_2$ in (z, v) phase space.

trajectory is in region $\mathcal{D}_1$, while z is decreasing. This will make the manifold $v = 0$ (equivalently, $w = 0$) unstable and the trajectories will move towards the region $\mathcal{D}_2$ and subsequently stay there. The control law in region $\mathcal{D}_2$ is essentially the same as in Eq. (8). Notice that, by definition, inside the region $\mathcal{D}_2$ we have $|\eta| \leq 1$, and since $|z|/|\bar{w}| = |\eta||w|$ we can ensure that $\omega(\cdot)$ will not take excessive values as long as the trajectories remain in $\mathcal{D}_2$. These statements will be made more precise in the sequel.

6.1. Proposed Control Law for Kinematics

The proposed control law for the system in Eqs. (7) is defined by

$$\omega = -\kappa(\eta)w - i\mu(\eta)\frac{z}{\bar{w}} \tag{13}$$

where the $\kappa(\eta)$ and $\mu(\eta)$ are smooth functions satisfying

$$-\kappa_c \leq \kappa(\eta) < 0, \quad 0 \leq \mu(\eta) < \frac{\mu_c}{2}, \quad \forall (z, v) \in \mathcal{D}_1 \tag{14a}$$

$$0 \leq \kappa(\eta) \leq \kappa_c, \quad \frac{\mu_c}{2} \leq \mu(\eta) \leq \mu_c, \quad \forall (z, v) \in \mathcal{D}_2 \tag{14b}$$

and $0 < \kappa_c < \mu_c$. One possible choice is, for example,

$$\kappa(\eta) = \frac{2\kappa_c}{\pi} \arctan\left(\rho(1 - \eta^2)\right) \tag{15a}$$

$$\mu(\eta) = \frac{\mu_c}{\pi} \arctan\left(\rho(1 - \eta^2)\right) + \frac{\mu_c}{2} \tag{15b}$$

From Eqs. (15) we have immediately that

$$-\kappa_c \leq \kappa(\eta) \leq \kappa_c, \quad \text{and} \quad 0 \leq \mu(\eta) \leq \mu_c \tag{16}$$

for all $\eta \in \mathbb{R}$. Moreover, notice that $\kappa(\eta) < 2\mu(\eta)$ for all $(z, v) \in \mathcal{D}_2$.

The next theorem gives the main result of the paper.

Theorem 6.1 *Consider the system in Eqs. (7) and let the control law as in Eqs. (13)-(15) with $0 < \kappa_c < \mu_c$. Then for initial conditions $(z(0), w(0)) \in \mathbb{R} \times (\mathbb{C}\backslash\{0\})$ the following properties hold:*

(i) $w(t) \neq 0, \forall t \geq 0$.

(ii) the trajectory $(z(\cdot), w(\cdot))$ is bounded and

$$\lim_{t\to\infty} (z(t), w(t)) = 0$$

(iii) the control law $\omega(\cdot)$ is bounded and it has bounded derivative.

With the control law in Eq. (13) the closed-loop system takes the form

$$\dot{v} = -\kappa(\eta)(1 + v)v \tag{17a}$$

$$\dot{z} = -\mu(\eta)z \tag{17b}$$

where $v = |w|^2$ and η as in Eq. (11). From Eq. (16) we have that z decays monotonically for all initial conditions, whereas v increases in the region $\mathcal{D}_1$ and decreases in $\mathcal{D}_2$. The result is that the trajectories of Eqs. (17) tend to $\mathcal{D}_2$ and then to the origin, as required.

Before proving Theorem 6.1 we need to establish the following two lemmas.

Lemma 6.1 *The region $\mathcal{D}_2$ is invariant for the system in Eqs. (17).*

Proof. The boundary of the set $\mathcal{D}_2$ is given by the two lines $\eta = \pm 1$ (cf. Fig. 2). On the boundary of $\mathcal{D}_2$ we have that $\kappa(\eta) = 0$ and $\mu(\eta) = \mu_c/2$. The vector field on the boundary of $\mathcal{D}_2$ is therefore

$$\dot{v} = 0 \tag{18a}$$

$$\dot{z} = -\frac{\mu_c}{2} z \tag{18b}$$

which points into the interior of $\mathcal{D}_2$. Therefore trajectories in $\mathcal{D}_2$ cannot escape this region and thus it is invariant for the closed-loop system in Eqs. (17). ■

This lemma establishes that for initial conditions in $\mathcal{D}_2$ the trajectories of the closed-loop system remain in $\mathcal{D}_2$ for all times. Equivalently, if at some time $t' \geq 0$ the trajectory enters $\mathcal{D}_2$, it stays in $\mathcal{D}_2$ for all $t \geq t'$. Figure 2 shows the vector field on the boundary of $\mathcal{D}_2$.

Lemma 6.2 *Consider the system in Eqs. (17). For all initial conditions* $(z, v) \in \mathcal{D}_1$ *the trajectories enter the region* $\mathcal{D}_2$ *in finite time.*

Proof. As long as $(z, v) \in \mathcal{D}_1$ from Eq. (14a) we have that $0 \le \mu(\eta) < \mu_c/2$. This implies that z is bounded. Actually, $|z(t)| \le |z(0)|$ for all $t \ge 0$. Note that z does not change sign for all $t \ge 0$. Without loss of generality we can assume that $z(0) \ge 0$ (the case $z(0) \le 0$ being similar). If $(z(0), v(0)) \in \mathcal{D}_1$ then, by definition $\eta(0) > 1$. The derivative of η in $\mathcal{D}_1$ is then

$$\begin{aligned} \dot\eta &= \frac{\dot z}{v} - \frac{z}{v^2}\dot v \\ &= -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \\ &\le -\mu(\eta)\eta \le 0 \end{aligned} \tag{19}$$

since $\kappa(\eta) < 0$ and $v > 0$; hence η is bounded in $\mathcal{D}_1$. Let $cl\,\mathcal{D}_1$ denote the closure of $\mathcal{D}_1$ in $\mathbb{R}^2$, that is,

$$\begin{aligned} cl\,\mathcal{D}_1 = \mathcal{D}_1 &\cup \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : |\eta| = 1\} \\ &\cup \{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : v = 0\} \end{aligned} \tag{20}$$

Then it is an easy exercise to show that $\dot\eta \ne 0$ for all $(z, v) \in cl\,\mathcal{D}_1 \backslash \{(0, 0)\}$. Hence there exists $\epsilon > 0$ such that $\dot\eta < -\epsilon$ in $\mathcal{D}_1$ and consequently, η monotonically decreases. Thus, every trajectory starting in $\mathcal{D}_1$ will leave this set and enter $\mathcal{D}_2$ in finite time. ∎

Notice that the set $\{(z, v) \in \mathbb{R} \times \mathbb{R}_+ : v = 0 \text{ and } z \ne 0\}$ is an unstable manifold for the closed-loop system. Figure 2 shows the vector field on the boundary of $\mathcal{D}_1$. The following corollary follows directly from Lemmas 6.1 and 6.2.

Corollary 6.1 *Consider the system in Eqs. (17). For all initial conditions* $(z(0), v(0)) \in \mathbb{R} \times (\mathbb{R}_+ \backslash \{0\})$ η *is bounded for all* $t \ge 0$.

We are now ready to give the proof of Theorem 6.1.

Proof. [Theorem 6.1] From Eqs. (17a) and (16) we have that

$$\dot v \ge -\kappa_c(1+v)v \tag{21}$$

where $\kappa_c > 0$. The solution of the differential equation

$$\dot x = -\kappa_c(1+x)x, \qquad x(0) = x_0 > 0 \tag{22}$$

is given by

$$x(t) = \frac{1}{c_0 e^{\kappa_c t} - 1} \tag{23}$$

where $c_0 = (x_0 + 1)/x_0$. Clearly, $x(t) \ne 0$ for all $t \ge 0$ and $\lim_{t\to\infty} x(t) = 0$. Therefore $v(\cdot)$ is bounded below by the solutions of the differential equation (22) subject to initial condition $x_0 = v(0)$. Hence, $|w(t)| \ne 0$ for all $t \ge 0$ and $w(\cdot)$ approaches the origin asymptotically.

We now show that $\lim_{t\to\infty}(z(t), v(t)) = 0$. If $(z(0), v(0)) \in \mathcal{D}_2$ then according to Lemma 6.1 we have that $(z(t), v(t)) \in \mathcal{D}_2$ for all $t \ge 0$ and $\mathcal{D}_2$ is an invariant set for the closed-loop system. Consider now the positive definite, radially unbounded function $V : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$V(z, v) = \tfrac{1}{2}v^2 + \tfrac{1}{2}z^2, \qquad \forall\,(z, v) \in \mathcal{D}_2 \tag{24}$$

The derivative of V along the trajectories of (17) is

$$\dot V = -\kappa(\eta)(1+v)v^2 - \mu(\eta)z^2 \le 0, \quad \forall (z, v) \in \mathcal{D}_2 \tag{25}$$

therefore, the trajectories are bounded in $\mathcal{D}_2$. Moreover, $\dot V = 0$ if and only if $\kappa(\eta)(1+v)v^2 + \mu(\eta)z^2 = 0$. Using the definitions of $\kappa(\eta)$ and $\mu(\eta)$ in $\mathcal{D}_2$ and recalling that $v > 0$, one establishes that the last equality is not satisfied in $\mathcal{D}_2$ unless $z = v = 0$. By LaSalle's theorem, $\lim_{t\to\infty}(z(t), v(t)) = 0$, for all initial conditions in $\mathcal{D}_2$. To finish the proof, recall from Lemma 6.2 that if $(z(0), v(0)) \in \mathcal{D}_1$ then $|z|$ is bounded by $|z(0)|$ and there exist a time $t' > 0$ such that $(z(t'), v(t')) \in \mathcal{D}_2$. This implies that for all $t' \ge t \ge 0$ the trajectories in $\mathcal{D}_1$ are bounded, and are confined inside the strip $|z(t)| \le |z(0)|$. However, according to the previous discussion, the trajectory with initial condition $(z(t'), v(t'))$ satisfies $\lim_{t\to\infty}(z(t), v(t)) = 0$. Therefore, we have shown that for all $(z(0), v(0)) \in \mathbb{R} \times (\mathbb{R}_+ \backslash \{0\})$ the trajectories remain bounded and have the property that $\lim_{t\to\infty}(z(t), v(t)) = 0$. By the definition of v this implies that

$$\lim_{t\to\infty}(z(t), w(t)) = 0 \tag{26}$$

In order to show that ω is bounded, write the ratio $z/\bar w = \eta w$. From Eq. (13) one obtains that

$$|\omega| \le \kappa_c|w| + \mu_c|\eta||w| \tag{27}$$

From Corollary 6.1 we have that for all initial conditions $(z(0), w(0)) \in \mathbb{R} \times (\mathbb{C} \backslash \{0\})$ η is bounded. Since w is also bounded, from Eq. (27) it follows that ω is bounded.

From Eq. (7a) it follows immediately that $\dot w$ is also bounded. Moreover, since

$$\dot\eta = -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \tag{28}$$

and $\mu(\eta)$, $\kappa(\eta)$, v and η are all bounded, we have that $\dot\eta$ is bounded.

The derivative of ω is given by

$$\begin{aligned}\dot{\omega} &= -\dot{\kappa}(\eta)w - \kappa(\eta)\dot{w} - i\,\dot{\mu}(\eta)\eta\, w \\ &\quad -i\,\mu(\eta)\dot{\eta}w - i\,\mu(\eta)\eta\dot{w} \end{aligned} \tag{29}$$

Using Eqs. (15) one has

$$\dot{\kappa}(\eta) = -\frac{4\kappa_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta^2)^2}\,\eta\,\dot{\eta} \tag{30a}$$

$$\dot{\mu}(\eta) = \frac{2\mu_c}{\pi}\frac{\rho}{1+\rho^2(1-\eta^2)^2}\,\eta\,\dot{\eta} \tag{30b}$$

Since $\dot{\eta}$ is bounded, $\dot{\kappa}(\eta)$ and $\dot{\mu}(\eta)$ are both bounded. Finally, the boundedness of $\dot{\omega}$ follows directly from Eq. (29) and the fact all the terms in the right hand side of this equation are bounded. ■

The vector field and the corresponding trajectories of the closed-loop system with the control law in Eq. (13) is shown in Fig. 3 (compare with Fig. 1).

Remark 6.1 Theorem 6.1 shows that for all initial conditions $w(0) \neq 0$ the control law in Eq. (13) drives the system trajectories to the origin. This control law cannot be used if $w(0) = 0$ (and $z \neq 0$). Linearization of system (6) about $w = 0$, however, results in

$$\dot{w} = \frac{\omega}{2} \tag{31a}$$

$$\dot{z} = 0 \tag{31b}$$

and choosing, for example, a constant control $\omega = \omega_c \in \mathbb{C}$, one can move away from the z axis into the $\mathcal{D}_1$ region; once in $\mathcal{D}_1$, use of the control (13) drives the system to the origin.

Remark 6.2 Another choice of a feedback control for Eq. (7) is the sublinear control in terms of w,

$$\omega = -\kappa\frac{w}{1+|w|^2} - i\,\mu\frac{z}{\bar{w}} \tag{32}$$

which renders the closed-loop system

$$\dot{v} = -\kappa\, v \tag{33a}$$

$$\dot{z} = -\mu\, z \tag{33b}$$

globally exponentially stable. The previous methodology can be applied *mutatis mutandis* to this control law, as well. Moreover, several other similar modifications can be introduced to the control law in Eq. (8). It should be evident that the results in this section can be applied to these control laws with only minor modifications.

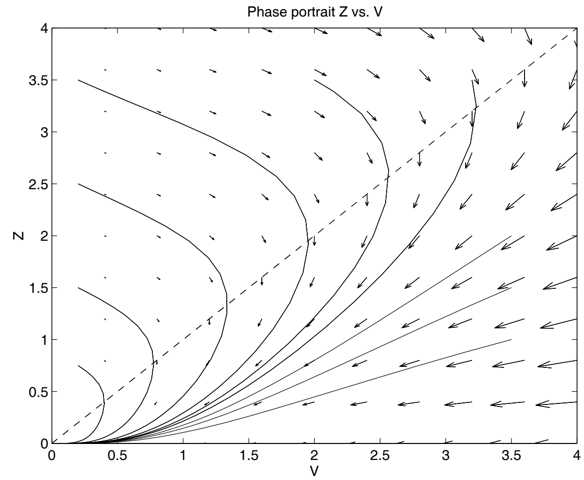

Figure 3: Phase portrait of system in Eqs. (17).

6.2. Proposed Control Law for Complete System

The control law in Eq. (13) was shown to achieve $\lim_{t\to}(z(t), w(t)) = 0$. Moreover, it is a bounded controller with bounded derivative. This allows one to implement this control through the dynamics in Eq. (6a). To this end, define the error

$$e = \omega - \omega_d \tag{34}$$

where ω_d is the *desired* angular velocity profile given in Eq. (13). Consider the following feedback control

$$u = \dot{\omega}_d - \alpha(\omega + \kappa(\eta)w + i\mu(\eta)\eta w) \tag{35}$$

where $\alpha > 0$ and where $\dot{\omega}_d$ is given in Eq. (29), along with Eqs. (30). The value of $\dot{\eta}$ is now given by

$$\begin{aligned}\dot{\eta} &= -\mu(\eta)\eta + \kappa(\eta)(1+v)\eta \\ &\quad + Im\left(\frac{e}{w}\right) - (1+v)\eta Re\left(\frac{e}{w}\right)\end{aligned} \tag{36}$$

With the control law in Eq. (35) the closed-loop system takes the form

$$\dot{e} = -\alpha\, e \tag{37a}$$

$$\dot{v} = -\kappa(\eta)(1+v)v + (1+v)Re(e\bar{w}) \tag{37b}$$

$$\dot{z} = -\mu(\eta)z + Im(e\bar{w}) \tag{37c}$$

Notice that for $e = 0$ the system reduces to the one in Eqs. (17).

For α large enough, Eq. (37a) is essentially a boundary layer subsystem to the slow system given by Eqs. (37b)-(37c). Singular perturbation theory[20] guarantees that as soon as the error becomes small enough, the (z, v) trajectories of the system will follow the ones of Eqs. (17).

Next we show that the control law in Eq. (35) is well-defined, in the sense that it remains bounded for all $t \geq 0$. We show that with α large enough $w(t) \neq 0$ for all $t \geq 0$, i.e., $w(t)$ tends to zero only asymptotically for all initial conditions inside an *a priori* given compact set.

Proposition 6.1 *Consider the system in Eqs. (37) and the compact set*

$$\mathcal{N}_\beta = \{(\omega, w, z) \in \mathcal{W} : |e| \left(\frac{1+v}{v}\right)^{\frac{1}{2}} \leq \beta\} \quad (38)$$

where $\mathcal{W} = \mathbb{C} \times (\mathbb{C}\backslash\{0\}) \times \mathbb{R}$, *and let* $\mu_c > \kappa_c > 0$ *and* $\alpha > \frac{\kappa_c+\beta}{2}$. *Then for all initial conditions in* $\mathcal{N}_\beta$, $|w|$ *is bounded below by an exponentially decaying function.*

Proof. Equation (37b) can be re-written as

$$\frac{d}{dt}|w|^2 = -(1+|w|^2)\left(\kappa(\eta)|w|^2 - Re(e\bar{w})\right) \quad (39)$$

Note that from Eq. (37a) $|e(t)| \leq |e(0)|e^{-\alpha t}$ and using Eq. (38),

$$|e(t)| \leq \beta \left(\frac{|w(0)|^2}{1+|w(0)|^2}\right)^{\frac{1}{2}} e^{-\frac{\kappa_c+\beta}{2}t}, \quad t \geq 0 \quad (40)$$

Consider now the differential equation

$$\frac{d}{dt}|\hat{w}|^2 = -(\kappa_c+\beta)(1+|\hat{w}|^2)|\hat{w}|^2 \quad (41)$$

The solution of this equation is given by

$$|\hat{w}(t)| = \frac{1}{c_0 e^{\frac{\kappa_c+\beta}{2}t} - 1} \geq c_0^{-\frac{1}{2}} e^{-\frac{\kappa_c+\beta}{2}t} \quad (42)$$

where $c_0 = (|\hat{w}(0)|^2+1)/|\hat{w}(0)|^2$. Comparison of Eqs. (40) and (42) implies that

$$|e(t)| \leq \beta\,|\hat{w}(t)|, \qquad \forall\, t \geq 0 \quad (43)$$

where $|\hat{w}|$ obeys Eq. (41) with $|\hat{w}(0)| = |w(0)|$.

Notice now that since $|Re(e\bar{w}) \leq |e||w|$, and using Eq. (43), one has from Eq. (39) that

$$\begin{aligned} \frac{d}{dt}|w|^2 &\geq -(1+|w|^2)(\kappa(\eta)|w|^2 + |e||w|) \\ &\geq -(1+|w|^2)(\kappa(\eta)|w|^2 + \beta|\hat{w}||w|) \quad (44) \end{aligned}$$

and since $-\kappa_c \leq \kappa(\eta) \leq \kappa_c$ finally,

$$\frac{d}{dt}|w|^2 \geq -(1+|w|^2)(\kappa_c|w|^2 + \beta|\hat{w}||w|) \quad (45)$$

By comparing Eqs. (41) and (45) and since $|w(0)| = |\hat{w}(0)|$, one obtains

$$\frac{d}{dt}|w(0)|^2 \geq \frac{d}{dt}|\hat{w}(0)|^2 \quad (46)$$

Therefore there exist some $t^* > 0$ such that $|w(t)| \geq |\hat{w}(t)|$ for all $0 \leq t \leq t^*$. We claim that, actually, $|w(t)| \geq |\hat{w}(t)|$ for all $t \geq 0$, and thus $|w|$ is bounded below by the exponentially decaying function $|\hat{w}|$.

Assume that at some point $t' > 0$ we have that $|w(t')| = |\hat{w}(t')|$ and $\frac{d}{dt}|w(t')| < \frac{d}{dt}|\hat{w}(t')|$; see Fig. 4. Then

$$\begin{aligned} \frac{d}{dt}|w(t')|^2 &= -(1+|w(t')|^2)(\kappa_c|w(t')|^2 + \beta|w(t')|^2) \\ &= -(1+|\hat{w}(t')|^2)(\kappa_c+\beta)|\hat{w}(t')|^2 \\ &= \frac{d}{dt}|\hat{w}(t')|^2 \quad (47) \end{aligned}$$

which leads to a contradiction. Therefore $|w(t)| \geq |\hat{w}(t)|$ and thus $w(t) \neq 0$ for all $t \geq 0$. ∎

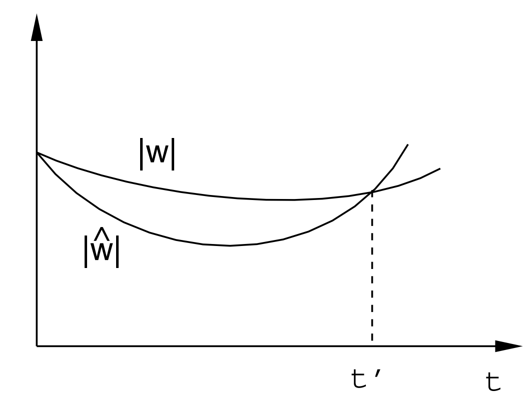

Figure 4: Time history of $|w|$ and $|\hat{w}|$.

In Ref. 10 the control law in Eq. (8) was also implemented using the same methodology. That is, the control for the complete system is given by Eq. (35), where now $\kappa = \kappa_c$, $\mu = \mu_c$ and $\dot{\kappa} = \dot{\mu} = 0$. The value of the gain α increases with β, which in turns increases as $|w|$ decreases. That is, when the initial condition is close to $w = 0$ then a faster transient for ω is required. This faster transient is achieved by talking α large enough. A potential problem in the implementation of the control in Eq. (35) is now evident. If e does not decay "fast enough" so that $\omega \to \omega_d$ sufficiently fast, then there is the danger that w will move towards the z-axis before the control law in Eq. (8) becomes effective. This is one more reason which motivated the choice of the control law in Eq. (13). Namely, it is beneficial for w

to move away from the z-axis. This can reduce the value of the gain α significantly.

In most situations it is not necessary to chose α from Proposition 6.1. Actually, as the numerical simulations in the next section show, for most practical examples it suffices to choose α to be "sufficiently larger" than the gains μ_c and κ_c. From Eq. (36) it also clear that α should be at least as large as $\kappa_c/2$.

7. Numerical Example

To illustrate the previous theoretical analysis, we have simulated the differential equations (6) with the two control laws in Eqs. (8) and (13). The gains are chosen as $\kappa_c = 0.5$ and $\mu_c = 2$. The value of the parameter $\rho = 2$. The initial conditions were taken as $w(0) = 0.3 - i\,0.25$ and $z(0) = 2.5$. The results are shown in Figs. 5 and 6. Figure 5 shows the corresponding closed-loop trajectories, and Fig. 6 shows the magnitude of the angular velocity (control input for the kinematics) $|\omega|$. Solid lines correspond to the new control law in Eq. (13) and the dashed lines correspond to the previous control law given in Eq. (8). As it is evident from these figures there is a substantial decrease in control effort by using the control law in Eq. (13), especially at the initial portion of the trajectory where z is large and $|w|$ is small.

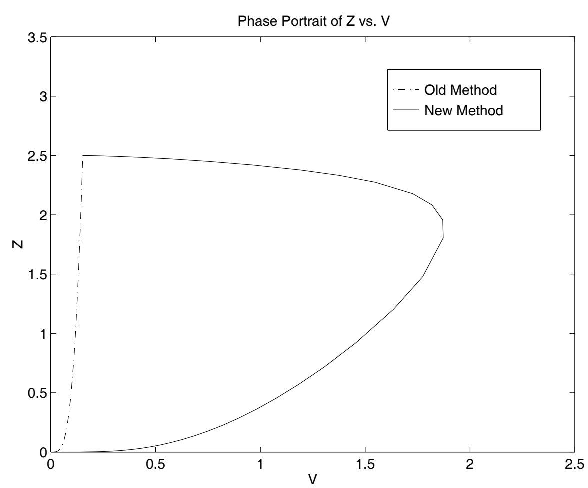

Figure 5: Closed-loop trajectories for the two methods (kinematics).

This control law was later implemented through the dynamics in Eq. (6a). A rest-to-rest maneuver was considered, thus $\omega(0) = 0$. Simulations for several values of α are shown in Figs. 7-8. The trajectories in the (z, v) space are very similar to the ones

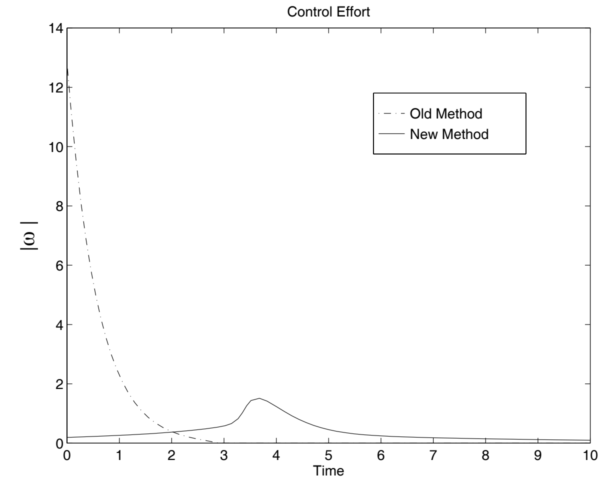

Figure 6: Control effort for the two methods (kinematics).

when ω is the control input. In fact, for $\alpha = 10$ the trajectories for the complete system are essentially identical to the ones with control law in Eq. (13). Figure (8) shows that increasing α may slightly increase the control effort, mainly because of the high-gain boundary layer part of the controller. At any rate, the corresponding control effort for the control law in Ref. 10 is several orders of magnitude higher and it is not shown here. In fact, for $\alpha = 1$ and $\alpha = 4$, the control effort for this controller is not bounded. This is due to the fact that the transient of e was not fast enough for those values of α and w drifted towards the z-axis before the control law in Eq. (8) becomes effective.

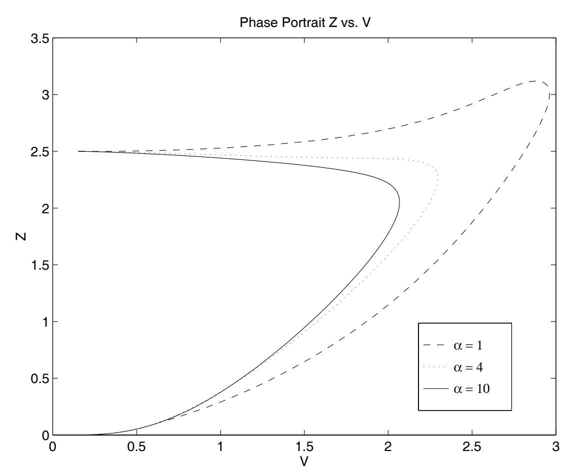

Figure 7: Closed-loop trajectories for the complete system.

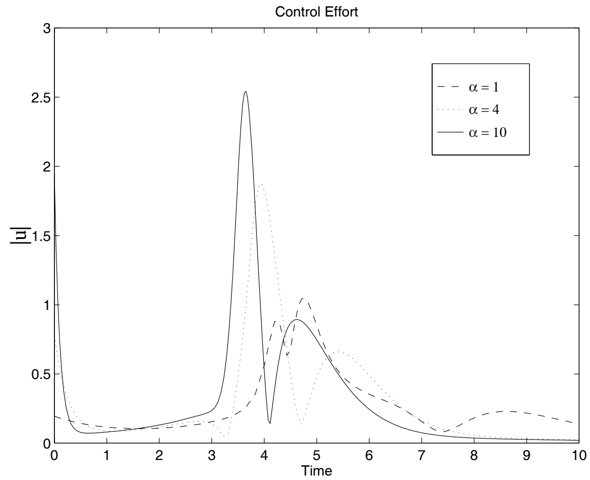

Figure 8: Control effort for the complete system.

8. Conclusions

We have constructed a nonsmooth control law which stabilizes the kinematics of an underactuated rigid spacecraft. We have shown that the proposed control law is well defined and it uses considerably less control effort than a previously derived control law. Numerical examples indicate a significant control effort reduction using the new control scheme. Because of the limited control torque on-board a spacecraft, for practical situations this may be the difference between feasibility and infeasibility of a particular reorientation maneuver. In addition, the rigid body problem subject to two control inputs is only but one example of an underactuated mechanical system. Systems of this form can be found in the class of systems subject to nonholonomic, i.e., nonintegrable constraints. Future research will be therefore directed towards extending the proposed control law to more general nonholonomic systems.

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