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# ON THE OPTIMAL REGULATION OF AN AXI-SYMMETRIC RIGID BODY WITH TWO CONTROLS

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**Abstract**

We present a partial solution to the problem of optimal feedback reorientation of the symmetry axis of an axially-symmetric rigid body. The performance index is quadratic in the state and the control variable and the optimal reorientation maneuver requires the use of only two control torques. Because of the passivity characteristics and the cascade structure of the system we first state two optimal regulation problems for the dynamics and the kinematics subsystems, separately. In this case one is able to find explicit solutions to the associated Hamilton-Jacobi equations. For the complete system the optimal regulation problem is not solvable in general. We present solutions for two partial cases. The first case is when there is no penalty on the control input. In this case, one can asymptotically recover the cost for the kinematics by making the dynamics sufficiently fast. The second case investigates restrictions imposed by optimality considerations on the aforementioned control law to avoid high gain.

## 1. Introduction

Optimal control of a rigid body has a long history stemming mainly from the interest of aerospace engineers in the control of rigid spacecraft. Several performance indices have been used in the formulation of the optimal control problem. The earliest results are perhaps those reported by Athans *et al.*[1] and Windeknecht[2]. These references address the problem of optimal fuel- and energy- regulation of the angular velocity of a rotating body. Similar results have been derived by Dabbous and Ahmed[3] where the authors develop optimal controls to regulate the angular momentum of a satellite subject to both reaction jets and flywheels, and by Dixon *et al.*[4] where the fuel-optimal reorientation problem is addressed. Most of these references either address the optimal control problem of the angular velocity equations only (without any reference to the kinematics), or they solve the open-loop optimal control problem. Pontryagin's Maximum Principle allows the formulation of the latter as a Two-Point-Boundary-Value Problem which is solved using numerical techniques[4−7]. The *synthesis* problem (i.e., optimal feedback problem) on the other hand, has been mainly addressed in the context of *time-optimal* maneuvers[8,9]. The survey paper by Scrivener and Thomson[10] gives a comprehensive treatment of the time-optimal problem. LQR-type formulations for feedback control results have been reported in the literature[11]. More recently, Carrington and Junkins[12] have used a polynomial expansion approach in order to approximate the solution to the Hamilton-Jacobi-Bellman equation. Similar results were reported by Dwyer[13] and Dwyer and Sena[14]. Finally, the book by Junkins and Turner[15] provides a comprehensive compilation of most of the existing results on the rigid body optimal control problem.

The work of Dwyer[13,14,16] has perhaps the closest connection to the results of this paper. He also seeks closed form solutions to the feedback optimal control problem via the Hamilton-Jacobi equation method. The main difference with our approach is that Dwyer applies a linearizing feedback transformation to the equations, resulting to a linear system in double integrator form. The quadratic regulator problem can then be easily solved either over a finite or an infinite time horizon. In the present work we address the nonlinear problem directly. No linearizing transformation is necessary. We rely on the special structure and the passivity properties of the equations in order to find closed-form solutions to the Hamilton-Jacobi-Bellman equation associated with the optimization problem.

In this paper we seek solutions to the optimal *feedback* regulation problem of a rigid body where *both* the angular velocity *and* the orientation of the body are regulated. We consider the case of an (inertially) axi-symmetric rigid body. Therefore, the purpose of the stabilizing optimal control is to drive the system to its final rest position which is along a *specified* direction of the symmetry axis (better, align the

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body symmetry axis with an inertially fixed axis). We assume that the relative orientation of the body about the symmetry axis is irrelevant; only the location of the symmetry axis is of interest. This could be the case when the symmetry axis coincides with the boresight or line-of-sight of a camera or a gun barrel, for example. Clearly, the relative rotation of the camera or the barrel has no influence on the clarity of the photograph or the accuracy of the projectile. Most importantly, spin-stabilized spacecraft also fall into this category.

For the axi-symmetric case it turns out that the objective of optimal regulation of the symmetry axis can be achieved using only two torques about axes that span the plane perpendicular to the symmetry axis. Therefore, without loss of generality, we restrict ourselves to the two control input case. This configuration does not allow any freedom to change the angular velocity along the symmetry axis. The angular velocity along this axis is fixed to its initial value. These statements will become more clear in the sequel. We note in passing that the case of optimal regulation of a general (non-symmetric) rigid body using three control torques has been addressed elsewhere[17].

Taking into consideration the cascade interconnection of the system equations and the passivity properties of the system, we first state the optimal regulation problem for the kinematics of the attitude motion when the angular velocity acts as a control input. The cost includes a penalty on the orientation parameters and the angular velocity. The actual control input is, of course, the acting torque entering the system through Euler's equations (the dynamics). The optimal regulation when the dynamics is included in the problem, and for general performance indices is not yet solved — as far as the author knows. However, the optimization problem for the kinematics provides a lower bound on the achievable performance for the whole system for the same cost functional. Actually, we show that if the dynamics is fast (or can be made fast enough through the appropriate choice of the control input) one is able to recover this performance asymptotically. We show how such a controller can be constructed — and thus achieve the optimal performance — under the assumption that there is no penalty on the control effort. This controller will include, in general, a high gain portion. Motivated by the optimal characteristics of this controller we derive an optimal controller which will penalize its high gain portion. A numerical example illustrates the theoretical developments.

## 2. Dynamics and Kinematics

We consider a rigid body with an axis of symmetry and *two control torques* about axes spanning the two-dimensional plane perpendicular to this axis. Without loss of generality we take a body-fixed reference frame $\boldsymbol{\hat{b}} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ with the unit vector $\hat{b}_3$ along the symmetry axis. Euler's equations with respect to this frame then take the form

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + b_{11}M_1 + b_{12}M_2 \quad (1a)$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + b_{21}M_1 + b_{22}M_2 \quad (1b)$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 \quad (1c)$$

where $b_{ij}$ denote the direction cosines of the torque vector with respect to the body axes $\hat{b}_1$ and $\hat{b}_2$ (Fig. 1).

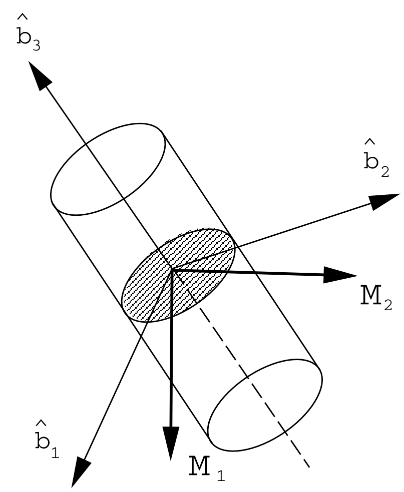


Figure 1: Axi-symmetric rigid body.

For $I_1 = I_2$ and letting the initial condition $\omega_3(0) = \omega_{30}$ we can rewrite the previous equations as

$$\dot{\omega}_1 = a\omega_{30}\omega_2 + u_1 \quad (2a)$$

$$\dot{\omega}_2 = -a\omega_{30}\omega_1 + u_2 \quad (2b)$$

where $a = (I_2 - I_3)/I_1$ and where $u_1, u_2$ are the new control torques given by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$$

The matrix in the previous matrix is assumed to be invertible, so that the two independent torques $M_1$ and $M_2$ correspond to two independent control inputs $u_1$ and $u_2$. If $\boldsymbol{\hat{n}} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$ denotes the inertial reference frame then, as it was shown in

Ref. [18], the position of the $\hat{n}_3$ inertial axis in the $\hat{\boldsymbol{b}}$ frame can be uniquely described by two variables $w_1$ and $w_2$ which obey the differential equations

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2) \quad (3a)$$
$$\dot{w}_2 = -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2) \quad (3b)$$

The variables $w_1$ and $w_2$ can be actually combined together to form a single complex variable $w_c = w_1 + i\, w_2$ and the system (3) then reduces to the following single complex differential equation[18]

$$\dot{w}_c = -i\,\omega_3 w_c + \frac{\omega_c}{2} + \frac{\bar{\omega}_c}{2} w_c^2 \quad (4)$$

where $\omega_c = \omega_1 + i\,\omega_2$ and bar denotes complex conjugate. The complex variable $w_c$ completely determines the angle $\theta$ between the body 3-axis $\hat{b}_3$ and the inertial 3-axis $\hat{n}_3$ from

$$\theta = \arccos\left(\frac{1 - |w_c|^2}{1 + |w_c|^2}\right)$$

where $|w_c|^2 = w_c \bar{w}_c$ denotes absolute value. Moreover, the plane spanned by $\hat{b}_3$ and $\hat{n}_3$ is perpendicular to the unit vector

$$\hat{h} = \left(\frac{w_c + \bar{w}_c}{2|w_c|}, \frac{i(\bar{w}_c - w_c)}{2|w_c|}, 0\right)$$

One can then go from the $\hat{n}_3$ axis to the $\hat{b}_3$ axis by rotating about $\hat{h}$ at an angle $\theta$. Figure 2 depicts this situation. Notice that $w_c = 0$ if and only if the body and inertial 3-axes are aligned. The complete description of the kinematics requires one more parameter ($z$) which denotes an initial rotation about the axis $\hat{n}_3$. In Fig. 2 the axes $\hat{n}_1'$ and $\hat{n}_2'$ are intermediate axes obtained from the inertial frame through the initial rotation $z$ about the $\hat{n}_3$ axis. The differential equation for $z$ is given by[18]

$$\dot{z} = \omega_3 + Im(\omega \bar{w}) = \omega_3 - \omega_1 w_2 + \omega_2 w_1 \quad (5)$$

where $Im(\cdot)$ denotes the imaginary part of a complex number. Therefore, equation (4) along with equation (5) can be used as an alternative to the standard kinematic descriptions in terms of the Eulerian angles, Euler-Rodrigues parameters, quaternions, etc.

For a more detailed discussion on the derivation and properties of the kinematic parameters $w_c$, and $z$, as well as their ramifications on attitude analysis and control problems, the interested reader may peruse Refs. [18, 19].

The $(w, z)$ kinematic parameterization is especially suitable for attitude description and control

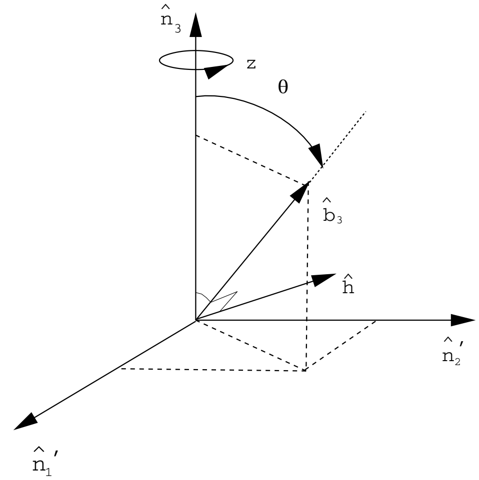


Figure 2: Attitude description using $w_c$ and $z$ coordinates.

of axi-symmetric bodies, where typically only the location of the symmetry axis is of interest. The location of this axis can be determined by $w_c$, equivalently $w_1$ and $w_2$. Since $z$ does not affect Eq. (4) one can then use only this equation to keep track the deviation of the symmetry axis from the $\hat{n}_3$ inertial axis.

In this paper we choose to work with the real set of differential equations (3) instead of the complex equation (4) mainly for clarity of exposition. The equations (2) and (3) can also be written in a vector form as

$$\dot{\omega} = aS(\omega_{30})\omega + u \quad (6a)$$
$$\dot{w} = S(\omega_{30})w + F(w)\omega \quad (6b)$$

where $\omega = [\omega_1\ \omega_2]^T$, $w = [w_1\ w_2]^T$, where $F : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is the symmetric, matrix-valued function defined by

$$F(w) = \tfrac{1}{2}\left((1 - w^T w)I + 2ww^T\right)$$

and where $S(\omega_{30})$ is the $2\times 2$ skew-symmetric matrix

$$S(\omega_{30}) = \begin{bmatrix} 0 & \omega_{30} \\ -\omega_{30} & 0 \end{bmatrix}$$

Equation (6a) is the *dynamics* of the attitude motion, whereas Eq. (6b) is the *kinematics*. Given Eqs. (6), the main objective of this paper is to derive feedback control laws $u = u(\omega, w)$ that will drive $w$ and $\omega$ to zero in some optimal fashion. According to the previous discussion, this amounts to optimally reorienting the symmetry axis to a desired position (assumed to be the inertial axis $\hat{n}_3$).

## 3. Equation Structure and Passivity

Equations (6) have the nice structure of a system in cascade form (see Fig. 3). That is, w does not enter into the dynamics in Eq. (6a) and $u$ does not affect the kinematics in Eq. (6b). In fact, the kinematics can only be manipulated through appropriate choice of the angular velocity profile. This motivates the decomposition of the complete system into a dynamics and a kinematics subsystem. The control input for the dynamics is $u$ and the output is $\omega$; the input to the kinematics is $\omega$ and the output is w.

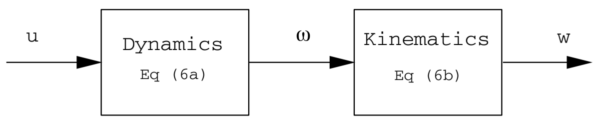


Figure 3: Cascade connection of dynamics and kinematics.

Another important property of the system (6) is that it represents a cascade interconnection of two *passive* systems. This allows for *linear*, globally asymptotically stabilizing control laws for the system (6). Recall that a system with input $u \in \mathbb{R}^m$ and output $y \in \mathbb{R}^m$ is *passive* (with storage function $V$) if there exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that[20]

$$\int_0^T y^T(t)u(t)\,dt \geq V(x(T)) - V(x(0)) \qquad (7)$$

where $x \in \mathbb{R}^n$ is the state of the system. It is *strictly passive* (with storage function $V$ and dissipation rate $\psi$) if there exists positive definite functions $V : \mathbb{R}^n \to \mathbb{R}_+$ and $\psi : \mathbb{R}^n \to \mathbb{R}_+$ such that[20]

$$\int_0^T y^T(t)u(t)\,dt \geq V(x(T)) - V(x(0)) + \int_0^T \psi(x(t))\,dt \qquad (8)$$

Passivity is invariant under feedback interconnection but cascade interconnection of two passive systems is not necessarily passive. Nevertheless, as we will show in this section the cascade interconnection of two passive systems can always be globally asymptotically stabilized by *linear* feedback of the subsystem outputs. We will state and prove this result for the system interconnection (6a)-(6b). This result can easily be extended, however, to the case of a cascade interconnection of any two (nonlinear) passive systems. Below $\|\cdot\|$ denotes the 2-norm, that is, $x^T x = \|x\|^2$ for any $x \in \mathbb{R}^n$.

**Proposition 3.1** *(i) Consider the system (6a) with input $u$ and output $\omega$. This system is passive with storage function*

$$V_1(\omega) = \tfrac{1}{2}\|\omega\|^2 \qquad (9)$$

*(ii) Consider the system (6b) with input $\omega$ and output w. This system is passive with storage function*

$$V_2(\mathrm{w}) = \ln(1 + \|\mathrm{w}\|^2) \qquad (10)$$

*Proof.* (i) In order to show that the dynamics subsystem (6a) is passive notice that the derivative of $V_1$ in Eq. (9) along the trajectories of (6a) is

$$\frac{dV_1}{dt} = \omega^T u \qquad (11)$$

Integrating both sides of the previous equation form 0 to $T$, we arrive at Eq. (7).

(ii) In order to show that the kinematics subsystem (6b) is passive notice that the derivative of $V_2$ in Eq. (10) along the trajectories of (6b) is

$$\frac{dV_2}{dt} = \mathrm{w}^T \omega \qquad (12)$$

Integrating both sides we arrive at Eq. (7).

∎

This proposition shows that the system in Eqs. (6) is a cascade interconnection of two passive systems. We now show that the cascade interconnection of the two passive systems in Eqs. (6a) and (6b) can be globally asymptotically stabilized using *linear* feedback in terms of the subsystem outputs. Hence the following lemma.

**Lemma 3.1** *The control law*

$$u = -k_1\omega + \nu \qquad (13)$$

*with $k_1 > 0$ renders the subsystem (6a) strictly passive from $\nu$ to $\omega$ with storage function $V_1$ and dissipation rate $\psi(\omega) = k_1\|\omega\|^2$.*

*Proof.* Letting $V_1$ as in Eq. (9) and using Eqs. (11) and (13) we get that

$$\frac{dV_1}{dt} = -k_1\|\omega\|^2 + \omega^T \nu$$

Integrating both sides of the previous equation one obtains

$$\int_0^T \omega^T \nu\,dt = V_1(\omega(T)) - V_1(\omega(0)) + k_1 \int_0^T \|\omega\|^2\,dt$$

which, according to Eq. (8) implies that the system from $\nu$ to $\omega$ is strictly passive. ∎

This lemma shows that we have a cascade interconnection of a strictly passive system (from $\nu$ to $\omega$) with a passive system (from $\omega$ to $w$). Let us now choose a negative feedback from $w$ to $\nu$ (say, $\nu = -k_2 w$). The resulting closed-loop system is then a *feedback* interconnection of a passive with a strictly passive system and global asymptotic stability can be easily shown under an observability assumption – which in our case is satisfied. The following theorem formalizes this observation and shows that the cascade interconnection of the two passive systems (6a) and (6b) is globally asymptotically stabilized using *linear* feedback in terms of the subsystem outputs.

**Theorem 3.1** *Consider the cascade interconnection (6a)-(6b). The linear control*

$$u = -k_1\omega - k_2 w \tag{14}$$

*where $k_1, k_2 > 0$ globally asymptotically stabilizes this system.*

*Proof.* Choosing the negative feedback $\nu = -k_2 w$ one obtains a feedback interconnection of a strictly passive system with a passive system. Therefore, by the Passivity Theorem[20], the closed-loop system is globally asymptotically stable. To see this, let the positive definite, radially unbounded function

$$V(\omega, w) = V_1(\omega) + k_2 V_2(w) = \tfrac{1}{2}\|\omega\|^2 + k_2 \ln(1+\|w\|^2)$$

Taking the derivative of $V$ along the trajectories of Eqs. (6)-(14) one obtains

$$\begin{aligned}
\dot V &= \omega^T\dot\omega + \frac{2k_2}{1+\|w\|^2} w^T \dot w \\
&= -k_1\|\omega\|^2 - k_2\omega^T w \\
&\quad + \frac{k_2 w^T}{1+\|w\|^2}\left(S(\omega_{30})w + F(w)\omega\right) \\
&= -k_1\|\omega\|^2
\end{aligned}$$

and the system is stable. Asymptotic stability follows using a standard LaSalle-type argument. ■

### 3.1. The three input case

The linear control law in Eq. (14) is a result of the passivity property of system (3). The choice of kinematic coordinates makes the kinematics subsystem in Eq. (6b) passive. Does the same property holds for the complete system (3)-(5)? That is, is the map from $(\omega_1, \omega_2, \omega_3)$ to $(w_1, w_2, z)$ passive? First, in

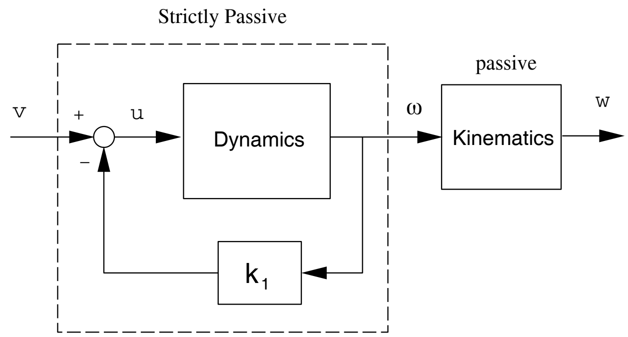


Figure 4: Passive interconnection with control $u = -k_1\omega + \nu$.

light of the discussion in the previous section, observe that if this map is passive then the linear feedback

$$\omega_1 = -w_1, \quad \omega_2 = -w_2, \quad \omega_3 = -z \tag{15}$$

would trivially render the kinematic subsystem (3)-(5) globally asymptotically stable. In fact, the following theorem states exactly this fact.

**Theorem 3.2** *Consider the kinematic subsystem (3)-(5) with the linear control law (15). Then the resulting closed-loop system is globally asymptotically stable.*

*Proof.* Consider the following positive definite, radially unbounded function

$$V(w, z) = \ln\left((1+\|w\|^2)(1+\tan\frac{z}{2})\right)$$

Taking the derivative of $V$ along (3)-(5), one obtains

$$\begin{aligned}
\dot V &= w_2\omega_2 + w_1\omega_1 + \tan\left(\frac{z}{2}\right) w_1\omega_2 \\
&- \tan\left(\frac{z}{2}\right) w_2\omega_1 + \tan\left(\frac{z}{2}\right)\omega_3
\end{aligned}$$

Using (15) one finally obtains

$$\dot V = -\|w\|^2 - \tan\left(\frac{z}{2}\right) z \le 0 \tag{16}$$

The last inequality is strict for $w \neq 0$ and $z \neq 0$. Hence, the closed-loop system is globally asymptotically stable. ■

The last term in Eq. (16) is shown in Fig. (5), along with the quadratic function $z^2$, for comparison. Clearly, this term is positive definite for all $-\pi \le z \le \pi$.

Theorem 3.2 indicates that the kinematics (3)-(5) *may be* passive. As shown next, however, this

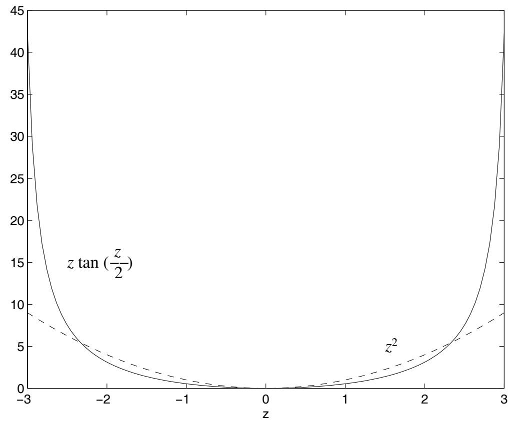


Figure 5: Plot of the function $z\tan(z/2)$.

conjecture may not be easily provable. To this end, let $q = [w_1 \ w_2 \ z]^T$ and notice that

$$\int_0^T q^T \omega \, dt = \int_0^T q^T H(q)\dot{q}\, dt = \int_{q(0)}^{q(T)} q^T H(q)\, dq$$

where

$$H(q) = \frac{1}{1+\|w\|^2}\begin{bmatrix} 2 & 0 & -2w_2 \\ 0 & 2 & 2w_1 \\ 2w_2 & -2w_1 & 1-\|w\|^2 \end{bmatrix}$$

**Proposition 3.2** *The integral*

$$\int_{q(0)}^{q(T)} q^T H(q)\, dq \qquad (17)$$

*depends on the path from $q(0)$ to $q(T)$.*

*Proof.* The integral in Eq. (17) is written as

$$\int_{q(0)}^{q(T)} \phi_1(q)\, dq_1 + \phi_2(q)\, dq_2 + \phi_3(q)\, dq_3$$

where

$$\phi_1(q) = 2\left(\frac{w_1+zw_2}{1+\|w\|^2}\right), \quad \phi_2(q) = 2\left(\frac{w_2-zw_1}{1+\|w\|^2}\right),$$
$$\phi_3(q) = z\left(\frac{1-\|w\|^2}{1+\|w\|^2}\right)$$

An easy calculation shows that $curl(\phi_1, \phi_2, \phi_3) \neq 0$ According to Stokes theorem[21] the integral (17) depends on the path from $q(0)$ to $q(T)$. ■

This result states that it may be difficult to prove the passivity of the system (3)-(5) since a storage function does not exist. This is in contrast to the Cayley-Rodrigues and the Modified Rodrigues kinematic parameters case, where the storage functions can be easily computed[22,23].

We now return to the question of *optimal* regulation of the system (6). Motivated by the results of this section we concentrate on the kinematics subsystem first.

## 4. Optimal Regulation

### 4.1. The Kinematics Subsystem

Given the kinematics system in Eq. (6b) — where $\omega$ is treated as a control-like variable — we introduce the following performance index

$$\mathcal{J}_1(w,\omega) = \tfrac{1}{2}\int_0^\infty \{r_1\|w(t)\|^2 + r_2\|\omega(t)\|^2\}\, dt \quad (18)$$

where $r_1$ and $r_2$ are some positive constants. Notice that this functional is a true performance index in the sense that it penalizes the state ($w$) and the control input ($\omega$). Although for nonlinear systems the choice of quadratic performance criteria is questionable, we nevertheless choose such a cost functional because it has a physical interpretation in terms of the system energy. Problems with nonlinear dynamics and nonquadratic cost functionals have been addressed in Ref. [24].

According to Hamilton-Jacobi theory, the optimal feedback control $\omega^*$ for the previous problem is given by

$$\begin{aligned} 0 &= \min_{\omega}\Big\{\frac{r_1}{2}\|w\|^2 + \frac{r_2}{2}\|\omega\|^2 \\ &+ \frac{\partial V}{\partial w}\left(S(\omega_{30})w + F(w)\omega\right)\Big\} \end{aligned}$$

where $\frac{\partial V}{\partial w}$ denotes the gradient of $V$ (row vector). Therefore, the Hamilton-Jacobi Equation (HJE) associated with the optimal control problem (6b)-(18) is given by

$$\frac{r_1}{2}\|w\|^2 - \frac{1}{2r_2}\|F(w)\frac{\partial^T V}{\partial w}\|^2 + \frac{\partial V}{\partial w}S(\omega_{30})w = 0 \quad (19)$$

The optimal control is given by

$$\omega^*(w) = -\frac{1}{r_2}F(w)\frac{\partial^T V}{\partial w} \quad (20)$$

We claim that the positive definite function $V : \mathbb{R}^3 \to \mathbb{R}_+$ defined by

$$V(w) = \sqrt{r_1 r_2}\ln(1+\|w\|^2) \quad (21)$$

solves the Eq. (19). Indeed, noticing that

$$\frac{\partial V}{\partial w} = \frac{2\sqrt{r_1 r_2}}{1+\|w\|^2}\, w^T$$

and that

$$F(w)\frac{\partial^T V}{\partial w} = \sqrt{r_1 r_2}\, w$$

substituting in (19), and using the fact that

$$w^T S(\omega_{30}) w = 0$$

we obtain the desired result. The optimal control is given by Eq. (20) and takes the very simple form

$$\omega^*(w) = -\sqrt{\frac{r_1}{r_2}}\, w \tag{22}$$

Note that the optimal control in Eq. (22) is unique. Moreover, using $V$ from Eq. (21) as a Lyapunov function for the closed-loop system, it is not difficult to show that the optimal control is exponentially stabilizing. The minimum value of the cost (18) is given by

$$\mathcal{J}_1^*(w(0)) = \sqrt{r_1 r_2}\ln(1 + \|w(0)\|^2) = V(w(0)) \tag{23}$$

It is interesting to note that the optimal control for the previous optimization problems is linear, although the solution to the HJE (which is also a Lyapunov function for the closed-loop system) is not quadratic. This is of course due to the fact that the system is not linear.

### 4.2. The Dynamics Subsystem

So far, we have only considered the kinematics subsystem or the attitude equations, i.e., Eq. (6b), with $\omega$ acting as a control variable. The optimal regulation problem for the Eq. (6a) has been addressed and solved elsewhere[1]. We only state the result for completeness, without proof.

To this end, consider the system (6a) where $u$ is the control input and let the quadratic performance index

$$\mathcal{J}_2(\omega, u) = \tfrac{1}{2}\int_0^\infty \{q_1\|\omega(t)\|^2 + q_2\|u(t)\|^2\}\, dt \tag{24}$$

where $q_1$ and $q_2$ are some positive constants. Then the control law

$$u^*(\omega) = -\sqrt{\frac{q_1}{q_2}}\, \omega \tag{25}$$

renders the closed-loop system globally exponentially stable at the origin and minimizes (24). Moreover, the minimum value of the cost is

$$\mathcal{J}_2^*(\omega(0)) = \tfrac{1}{2}\sqrt{q_1 q_2}\, \|\omega(0)\| \tag{26}$$

### 4.3. The Complete System

So far, we have considered the kinematics and the dynamics subsystems of the attitude equations separately. The natural question is of course "What conclusions can be drawn about the complete system interconnection ?" Previous attempts include approximate solutions using truncated Taylor series expansions of the Hamilton-Jacobi Equation[12], or exact solutions of a feedback linearized version of the problem[16]. The feedback linearization technique is especially appealing but has the drawback that the optimization is performed in the transformed variables (which may not be directly amenable to a physical interpretation) and that the penalty on the control does not include the feedback linearizing portion.

Our approach is based on the observation that we have already an exact solution of the optimal regulation problem for the kinematics. We wish to use this knowledge from the kinematics problem instead of formulating an entirely new problem for the complete system. This approach limits our freedom in choosing the performance index, but allows the analytic derivation of optimal feedback controllers in *closed form*. The fact that the derivation of optimal feedback solutions is possible for the attitude problem is related to the Lie group structure of the configuration space[25].

If the dynamics subsystem is sufficiently fast then the results of section 4.1 suffice. In these cases, the optimal angular velocity profile can be implemented through the dynamics without significant degradation in performance. Actually, one can always recover the cost in Eq. (23) asymptotically, using the control input

$$u_{as} = -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w - \lambda(\omega + kw) \tag{27}$$

where $k = \sqrt{\frac{r_1}{r_2}}$. That is, by choosing $\lambda$ large enough, the cost

$$\int_0^\infty \{r_1\|w\|^2 + r_2\|\omega\|^2\}\, dt \longrightarrow \sqrt{r_1 r_2}\, \ln(1 + \|w(0)\|^2) \tag{28}$$

and it can be made arbitrarily close to $\mathcal{J}_1^*(w(0))$.

This result can be shown by introducing the new variable

$$z = \omega + kw \tag{29}$$

and rewriting the system (6) with the control (27) in the form

$$\dot{z} = -\lambda z \tag{30a}$$

$$\dot{w} = S(\omega_{30})w - kF(w)w + F(w)z \tag{30b}$$

Notice from Eq. (29) that since $z \to 0$ then $\omega \to \omega^*$. We can explicitly calculate the value of the cost $\mathcal{J}_1(w, \omega)$ along the trajectories of (30) using the positive definite function

$$\begin{aligned} V(w,\omega) &= 2\sqrt{r_1 r_2}\ln(1+\|w\|^2) + \frac{r_2}{2\lambda}\|\omega + kw\|^2 \\ &= 2\sqrt{r_1 r_2}\ln(1+\|w\|^2) + \frac{r_2}{2\lambda}\|z\|^2 \end{aligned}$$

Then

$$\begin{aligned} \frac{dV}{dt} &= \frac{4\sqrt{r_1 r_2}}{1+\|w\|^2} w^T(-kF(w)w + F(w)z) - r_2\|z\|^2 \\ &= -2\sqrt{r_1 r_2}k\|w\|^2 + 2\sqrt{r_1 r_2}w^T z - r_2\|z\|^2 \\ &= -r_1\|w\|^2 - r_2\|z - kw\|^2 \\ &= -r_1\|w\|^2 - r_2\|\omega\|^2 \le 0 \end{aligned}$$

Since $\dot{V}$ is negative definite, the control law (27) is asymptotically stabilizing. Thus $\lim_{T\to\infty} V(T) = 0$. Integrating both sides and taking limits as $T \to \infty$ one obtains

$$V(T) - V(0) = -\int_0^T \{r_1\|w\|^2 + r_2\|\omega\|^2\}\, dt$$

Finally, from the previous equation

$$V(w(0), \omega(0)) = \int_0^\infty \{r_1\|w\|^2 + r_2\|\omega\|^2\}\, dt$$

Since

$$\begin{aligned} V(w(0),\omega(0)) &= 2\sqrt{r_1 r_2}\ln(1+\|w(0)\|^2) \\ &+ \frac{r_2}{2\lambda}\|\omega(0) + kw(0)\|^2 \qquad (31) \end{aligned}$$

then Eq. (28) follows by letting $\lambda \to \infty$. A simple singular perturbation analysis shows that the effect of large $\lambda$ is that of making the dynamics in Eq. (6a) sufficiently fast.

The optimal cost in Eq. (18) provides a lower bound on the achievable performance when the actual control input is the body fixed torque $u$. The disadvantage of the control law in Eq. (27) is that it may require high gain. This may not be acceptable if there are bounds on the available control effort. A more realistic performance index should incorporate a penalty on the control effort $u$ as well. Unfortunately, the optimization problem for a performance index which is quadratic in the state and the control effort is rather formidable. Motivated by the control law (27), we use an alternative approach. We investigate the optimality properties of (27) and, in particular, we modify this control law such that its high-gain portion its penalized.

The procedure in this section is similar in spirit to the results of Ref. [26], where the authors examine the optimality properties of a class of feedback control laws for relative degree one minimum phase systems and the results in Ref. [17] where the optimal regulation problem for a general (i.e., nonsymmetric body) is addressed. Close examination of the control law in Eq. (27) shows that the first three terms are used for canceling the nonlinearities. The only possible high gain portion of the control law (27) is the last term. We therefore consider a modified control law of the form

$$u = -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w + v \qquad (32)$$

Recalling now the desirable properties of the relationship $\omega = -kw$ for the kinematic subsystem we again introduce the variable $z = \omega + kw$ and develop control laws which will make $z \to 0$. The performance index should therefore include a penalty on $z$ as well as a penalty on the control effort $v$.

Using Eqs. (32) and (29) the system in Eqs. (6) is written in the form

$$\begin{aligned} \dot{z} &= v & (33a) \\ \dot{w} &= S(\omega_{30})w + F(w)(z - kw) & (33b) \end{aligned}$$

**Theorem 4.1** *Consider the system in Eqs. (33) and the control law*

$$v^*(w,z) = -\frac{w}{\lambda} - \lambda z \qquad (34)$$

*Then this control law makes the system (33) exponentially stable and minimizes the cost*

$$\mathcal{J}_3(w,z,v) = \tfrac{1}{2}\int_0^\infty \{\|v + \frac{w}{\lambda}\|^2 + 2k\|w\|^2 + \lambda^2\|z\|^2\}\, dt \qquad (35)$$

*Moreover, the minimum value of the cost is*

$$\mathcal{J}_3^*(w(0), z(0)) = \ln(1+\|w(0)\|^2) + \frac{\lambda}{2}\|z(0)\|^2$$

*Proof.* First, notice that the HJE associated to the previous optimal control problem is given by

$$\begin{aligned} &\tfrac{1}{2}\ \|\frac{\partial V}{\partial z}\|^2 + k\|w\|^2 + \frac{\lambda^2}{2}\|z\|^2 - \frac{\partial V}{\partial z}\left(\frac{w}{\lambda} + \lambda z\right) \\ &- \frac{\partial V}{\partial w}S(\omega_{30})w + \frac{\partial V}{\partial w}F(w)(z - kw) = 0 \qquad (36) \end{aligned}$$

and the optimal control is given by

$$v^*(w,z) = -\frac{w}{\lambda} - \frac{\partial^T V}{\partial z} \qquad (37)$$

Then notice that the positive definite function $V_3 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ defined by

$$V_3(w,z) = \ln(1+\|w\|^2) + \frac{\lambda}{2}\|z\|^2 \qquad (38)$$

satisfies the Hamilton-Jacobi Equation (36). The exponential stabilizability of the control (34) is easily verified by using (38) as a Lyapunov function for the closed-loop system. The minimum value of the cost is given by $\mathcal{J}_3^*(w(0), z(0)) = V_3(w(0), z(0))$.

■

From Eq. (32) and Eq. (37) we have that the optimal control is

$$\begin{aligned} u^*(\omega, w) &= -aS(\omega_{30})\omega - kF(w)\omega - kS(\omega_{30})w \\ &- \lambda(\omega + kw) - \frac{w}{\lambda} \end{aligned} \quad (39)$$

Moreover, $u^* = u_{as} - \frac{w}{\lambda}$.

Comparison of Eqs. (34) and (35) shows that the control law

$$\tilde{v}^*(z) = -\lambda z$$

minimizes the cost

$$\tilde{\mathcal{J}}_3(w, z, \tilde{v}) = \tfrac{1}{2} \int_0^\infty \{\|\tilde{v}\|^2 + 2k\|w\|^2 + \lambda^2\|z\|^2\}\, dt \quad (40)$$

subject to the dynamical constraints

$$\dot{z} = -\frac{w}{\lambda} + \tilde{v} \quad (41a)$$

$$\dot{w} = S(\omega_{30})w + F(w)(z - kw) \quad (41b)$$

That is, the first term in Eq. (40) includes a true penalty on the high gain portion of the controller. Moreover, notice that as $\lambda \to \infty$ then $v^* \to -\lambda z$ and $u^* \to u_{as}$ and we recover the results of the control law (27). Another interesting observation shows the different effect of increasing $\lambda$ for the two control laws (27) and (39). Recalling that the solution of the HJE is the cost-to-go and comparing Eqs. (31) and (38) one sees that increasing $\lambda$ has the effect of reducing the cost in Eq. (31), while in Eq. (38) the effect of large values of $\lambda$ is taken into consideration. Actually, the cost-to-go for the performance index (35) is proportional to $\lambda$, whereas for the performance index (18) is inversely proportional to $\lambda$. As it is evident from Eq. (40) the parameter $\lambda$ can be chosen to compromise between good performance (in the sense of small $z$) and acceptable control gain.

## 5. Numerical Example

We illustrate the theoretical results by means of numerical simulations. We consider an optimal regulation maneuver of an axi-symmetric rigid body from initial orientation

$$w_1(0) = w_2(0) = 10$$

These values correspond to a rigid body which is, initially, almost "up-side down." The body is assumed to be initially at rest. Therefore,

$$\omega_1(0) = \omega_2(0) = \omega_3(0) = 0$$

The inertia parameter is $a = 0.5$. The constants $r_1$ and $r_2$ in Eq. (22) were chosen to be equal to unity, which implies that also $k = 1$. The control law in Eq. (39) is implemented for different values of $\lambda$.

These results are shown in Figs. 6-8. Figures 6 and 7 show the response for the first component of the angular velocity and the orientation parameter $w$, respectively. The control effort for different values of $\lambda$ is shown in Fig. 8. The corresponding plots for the control law $u_{as}$ in Eq. (27) are also shown in Figures 9-11, for comparison.

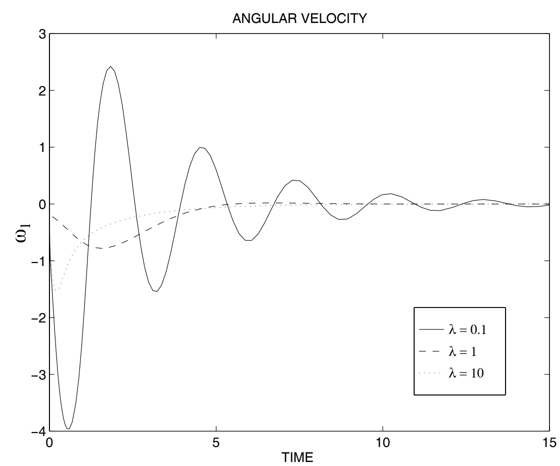


Figure 6: Angular velocity response using $u^*$.

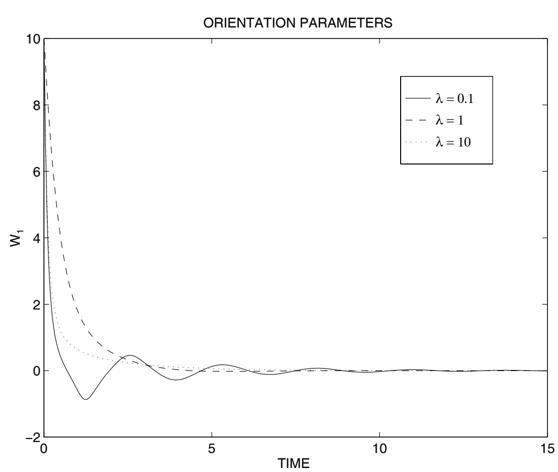


Figure 7: Orientation parameter response using $u^*$.

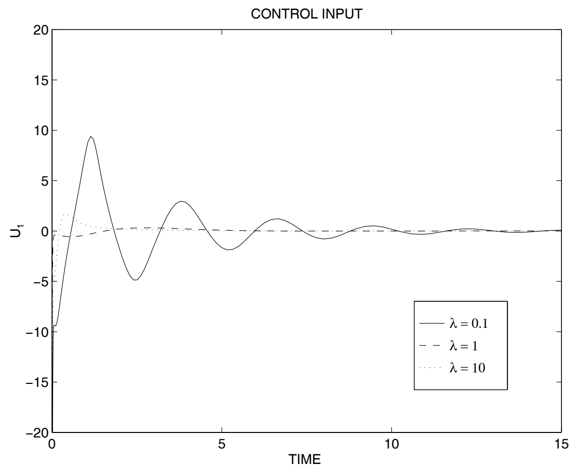


Figure 8: Control input response using $u^*$.

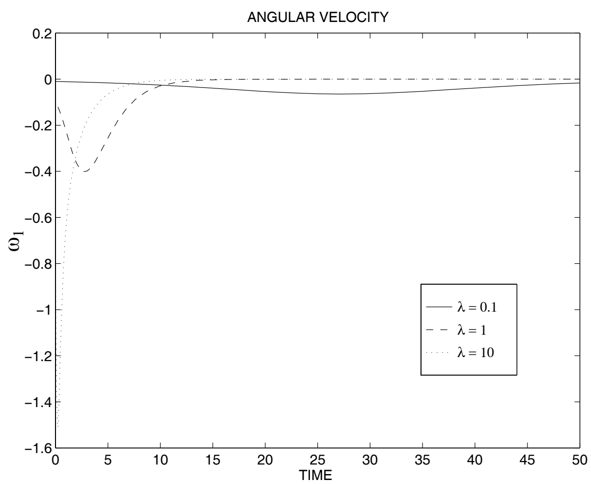


Figure 9: Angular velocity response using $u_{as}$.

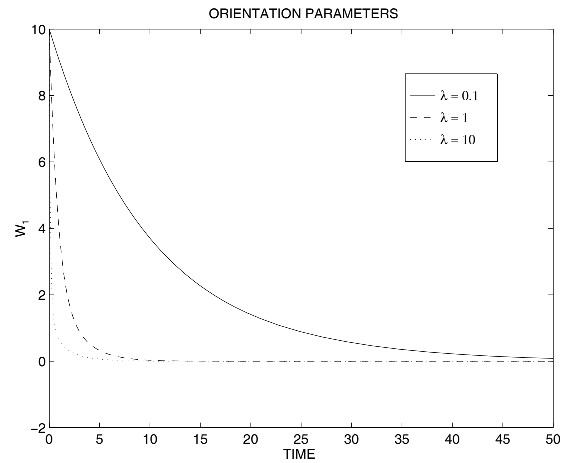


Figure 10: Orientation parameter response using $u_{as}$.

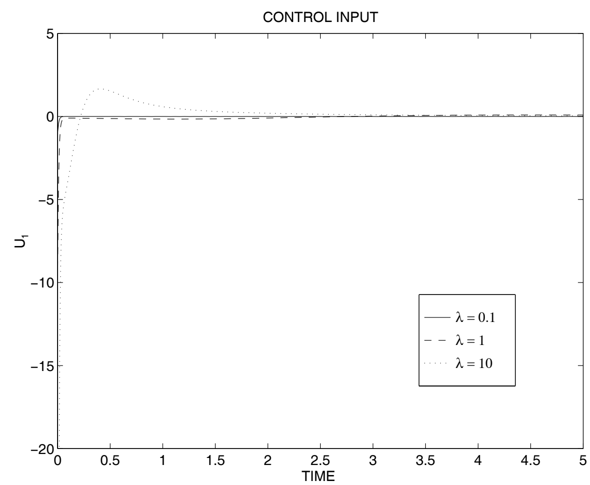


Figure 11: Control input response using $u_{as}$.

## 6. Conclusion

We have presented some new results for the optimal regulation of the symmetry axis of a spinning rigid body. Only two control torques are necessary if regulation of the relative rotation about the symmetry axis is not required. By using the natural decomposition of the system into its kinematics and dynamics subsystems and the inherent passivity properties of the two subsystems we derived an optimal controller in a two-step process. The optimal control for the kinematics is extremely simple (linear) and has the desirable characteristics. Direct implementation of this control through the dynamics may however require high gain. Finally, we modified this direct approach to obtain an optimal controller which tries to mimic the optimal controller for the kinematics by penalizing its high gain portion. The gain parameter can be used to compromise between speed of regulation and acceptable control effort.

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