Pursuit-Evasion Problems Involving Two Pursuers and One Evader

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Time-optimal evading strategies for pursuit-evasion problems involving two pursuers and one evader are analyzed. It is assumed that both pursuers are identical and possess higher speed capabilities when compared to the evader. The problems are categorized into two scenarios, which are based on the evader's information provided to the pursuers. In the first scenario, the pursuers have information of the evader's position and velocity at each instant of time and follow a constant bearing strategy. In the second scenario, the pursuers have only positional information of the evader and follow a pure pursuit strategy.

Nomenclature

t_c	Time-to-capture
p	Player's position
x	Player's x-coordinate
y	Player's y-coordinate
r	Relative distance between a pursuer and evader
θ	Heading angle
φ	Line-of-sight angle
u	Pursuer's speed
v	Evader's speed
H	Hamiltonian
λ	Co-state

Subscript

1	Closer Pursuer
2	Farther Pursuer
E	Evader

I. Introduction

The amount of research that went into understanding pursuit-evasion games involving one pursuer and one evader, starting from its inception in 1950s through Isaacs [1], is immense. Domains of application include collision avoidance, missile guidance, and several other application that involve mainly defense and strategic elements. As the research on swarm robotics and formation control is becoming more prevalent currently, the idea of performing a task efficiently with multiple agents is gaining prominence. Along these lines, a study on two-pursuer/one evader problem is a first step in understanding cooperation to capture a target using multiple pursuers.

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H. J. Kelley initially studied the two-pursuer one-evader problem [2], and discussed some aspects of two-on-one team tactics. The problem was later posed as a differential game with linear dynamics and quadratic cost, and it was investigated for Nash equilibrium solutions by Foley and Schmitendorf [3]. They identified partitions in the state-space and their categorization is similar to the degenerate and the non-degenerate regions, subsequently discussed in this paper. A linear differential game formulation of the problem with the motion of the players restricted to a straight line was studied and presented in great detail [4–6]. In this type of formulation, the two pursuers coordinate to reduce the miss distance. Furthermore, to construct level sets of the value function of a finite horizon problem, an algorithm was proposed [7]. Some variations of the two-pursuer one-evader problem include: non-convex payoff functions [8], two identical inertial pursuers (second order dynamics) against a non-inertial evader (first order dynamics) [9], a faster evader that must pass between two pursuers [10]. Yavin analyzed the stochastic version of the two-pursuer one-evader differential game [11]. Finally, Sun and Tsiotras discussed a version of the relay pursuit problem presented in this paper, and proposed a suboptimal strategy [12].

The majority of the existing literature on the two-pursuer one-evader problem assumes linear dynamics, with optimization of the miss distance by formulating a quadratic cost function with fixed final time. In this paper, we are motivated to find time-optimal strategies for an evader that is being pursued by two pursuers and examine the outcomes under two different information structures. Time-optimal solutions were briefly investigated in the book by Isaacs [1] but, apart from that discussion, there are no comprehensive results existing in the current literature. The two scenarios, dealt in this paper, differ in the information that the pursuers contain about the evader. The first scenario deals with the pursuers that follow a constant bearing strategy by accessing the position and velocity of the evader at each instant of time. However, in the second scenario, the pursuers know only the instantaneous position of the evader and use this information to follow a pure pursuit strategy. It is assumed that both pursuers are identical and the speed of a pursuer is higher compared to the evader's speed, which means capture is guaranteed. In this regard, the evader tries to maximize the capture time using its control input which is the heading angle.

At first, the value associated in employing two pursuers to capture an evader is studied in both scenarios, which is done by characterizing the degenerate and the non-degenerate regions for the problem. Afterwards, using optimal control theory, the expressions of the corresponding optimal evading strategies are derived. Some nice geometric interpretations of the optimal evading strategies can be made with the help of the closed form solution obtained for the first scenario, and from the numerical simulations to the second scenario. This study can then be used to propose an analytical suboptimal strategy for the evader that is easy to implement in practice.

The paper is organized into six sections. Section II formally presents the problem statement discussed in this paper along with the equations of motion for the players. Section III discusses the notion of degenerate/non-degenerate problems, and the construction of the regions of non-degeneracy for both scenarios. Section IV analyzes the optimal strategies of the evader obtained by formulating the problem, for both scenarios, in their corresponding reduced state spaces. The discussion on the suboptimal evading strategy for second scenario, in which the pursuers follow a pure pursuit strategy, is also included in this section. Section V presents numerical results, simulated for both scenarios, along with a comparative study performed on the suboptimal strategy against the optimal one for the second scenario. The concluding remarks are made in Section VI.

II. Problem Formulation

Consider a pursuit-evasion game with two pursuers and one evader in the plane. The game terminates when one of the pursuers enters the evader's capture zone, assumed here to be a disk of radius $\epsilon > 0$ centered at the current position of the evader. The subscripts 1 and 2 will be used for the two pursuers (P_1 and P_2) while the subscript E will be used for the evader. The equations of motion for all the players involved in the game are given below

$$\dot{x}_1 = u\cos\theta_1, \quad \dot{y}_1 = u\sin\theta_1, \tag{1}$$

$$\dot{x}_2 = u\cos\theta_2, \quad \dot{y}_2 = u\sin\theta_2, \tag{2}$$

$$\dot{x}_E = v\cos\theta_E, \quad \dot{y}_E = v\sin\theta_E, \tag{3}$$

where $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, and $p_E = (x_E, y_E)$ denote the positions of pursuer P_1, pursuer P_2, and the evader E, respectively. Similarly, $\theta_1,\ \theta_2,\ \theta_E \in (-\pi, \pi]$ denote the control inputs of the players, u and v

are the speeds (constant) of the pursuers and the evader, respectively, with $u > v$. The game evolves in the six-dimensional state-space, $[x_1, y_1, x_2, y_2, x_E, y_E]^\intercal \in \mathbb{R}^6$.

Now, consider Scenario 1 when the pursuers know the instantaneous position and velocity of the evader. Under these circumstances, a constant bearing pursuit is an efficient strategy for the pursuers to guarantee capture [13]. As the pursuers follow a constant bearing strategy, their control inputs θ_1 and θ_2 depend on the evader's heading and the instantaneous positions of all the players. Scenario 2 deals with the case where the pursuers have access only to the evader's instantaneous position but not its velocity. Pure pursuit is a reasonable strategy to capture the evader in this case [13]. The control inputs θ_1 and θ_2 are then solely dependent on the instantaneous positions of the players, where

$$\cos\theta_1 = \frac{x_E - x_1}{\|p_E - p_1\|}, \quad \sin\theta_1 = \frac{y_E - y_1}{\|p_E - p_1\|}, \tag{4}$$

$$\cos\theta_2 = \frac{x_E - x_2}{\|p_E - p_2\|}, \quad \sin\theta_2 = \frac{y_E - y_2}{\|p_E - p_2\|}, \tag{5}$$

$\|p_E - p_1\| = \sqrt{(x_E - x_1)^2 + (y_E - y_1)^2}$, and $\|p_E - p_2\| = \sqrt{(x_E - x_2)^2 + (y_E - y_2)^2}$.

The problem we wish to address in this paper can be stated as follows:

Problem: Find the optimal control input for the evader, $\theta_E \in (-\pi, \pi]$ that maximizes the time of capture t_c in the two aforementioned scenarios.

Since it is assumed that the pursuers are identical, without loss of generality, we consider P_1 to be the closer pursuer to the evader at the initial time ($t = 0$). The problem is symmetric when the evader is on the perpendicular bisector of the line segment joining the two pursuers, both pursuers are equidistant to the evader. In such a case, P_1 is assigned at random.

III. The Regions of Non-Degeneracy

In the case of a one-pursuer/one-evader problem, pure pursuit is the optimal strategy for the pursuer, while for the evader the optimal strategy is pure evasion [1]. In the case of the two-pursuer/one-evader problem, assuming that each of the pursuers follows either a constant bearing strategy or a pure pursuit strategy may result in degenerate cases as follows.

The perpendicular bisector between the two pursuers defines the regions of dominance (or Voronoi cells [14]) for each pursuer. The pursuer whose Voronoi cell contains the evader is the closer pursuer (P_1). The optimal strategy of the evader with respect to P_1 (alone) is to move away from P_1 along the corresponding line-of-sight (LoS). However, there is a second pursuer (P_2 - farther pursuer) also trying to capture the evader. If the evader follows a pure evasion strategy with respect to P_1 and never hits the perpendicular bisector before getting captured by P_1, it follows that P_2's presence does not affect the game. Given the initial locations of P_1 and the evader, it is of interest then to find the set of initial positions for P_2 that makes the problem non-degenerate, that is, when the evader needs to take into account both pursuers, as long as they are not equidistant from the evader. These are all the locations where the evader will have to consider the presence of the second pursuer (P_2 in this case), and for these initial locations of P_2, the evader reaches the perpendicular bisector before it gets captured by P_1 under pure evasion. In the non-degenerate region, the farther pursuer has a role to play in the problem, and the optimal strategy is not pure evasion anymore. We can compute the non-degenerate regions for both scenarios under consideration as follows.

Assume momentarily that there is only a single pursuer (i.e., the closer pursuer P_1). The capture time given the problem formulation in Section II, and assuming that the evader follows a pure evasion strategy, is given by

$$t_f = \frac{\|p_E(0) - p_1(0)\|}{u - v}. \tag{6}$$

Here t_f represents the capture time assuming that a second pursuer is not present. Also, the capture point is $C = (x_E(t_f), y_E(t_f))$ where $x_E(t_f)$ and $y_E(t_f)$ are given by

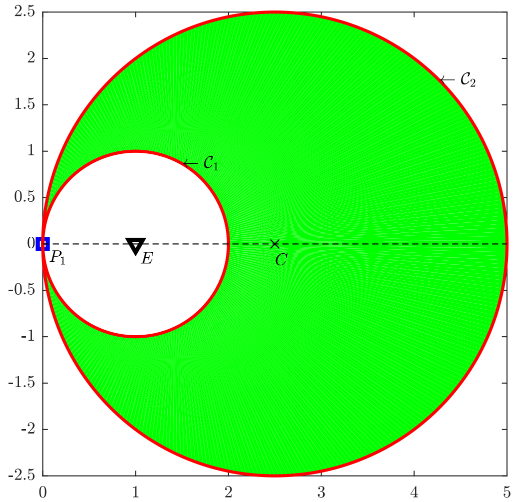
$$x_E(t_f) = x_E(0) + vt_f \frac{x_E(0) - x_1(0)}{\|p_E(0) - p_1(0)\|}, \tag{7a}$$

$$y_E(t_f) = y_E(0) + vt_f \frac{y_E(0) - y_1(0)}{\|p_E(0) - p_1(0)\|}. \tag{7b}$$

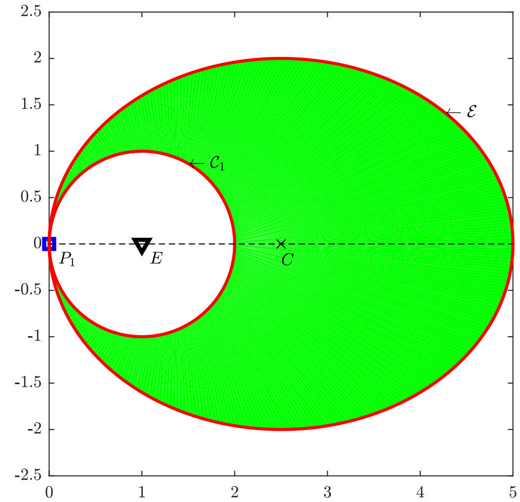
Next, define the circle $\mathcal{C}_1$

$$\mathcal{C}_1 = \{\mathsf{x} \in \mathbb{R}^2 : \|\mathsf{x} - p_E(0)\| = \|p_1(0) - p_E(0)\|\}. \tag{8}$$

The exterior of the circle $\mathcal{C}_1$ denotes the allowable region for the initial position of P_2 (since P_1 is the closer pursuer by definition), given the initial positions of P_1 and the evader.



(a) Scenario 1



(b) Scenario 2

Figure 1. Regions of non-degeneracy for $\epsilon = 0$ (point capture).

III.A. Scenario 1

The non-degenerate region for this scenario can be constructed by defining another circle $\mathcal{C}_2$ as follows

$$\mathcal{C}_2 = \{\mathsf{x} \in \mathbb{R}^2 : \|\mathsf{x} - C\| = \|p_1(0) - C\|\}. \tag{9}$$

The circle $\mathcal{C}_2$ is an isochrone that contains the set of initial positions for a pursuer that guarantees capture at time t_f (at C) under a constant bearing strategy for the given initial position of the evader and its heading, assuming that the evader is non-maneuvering. The center of this circle is the capture point C and its radius is ut_f. If the initial position of a pursuer (following a constant bearing strategy) lies inside $\mathcal{C}_2$, then it can capture the evader in a time less than t_f for the given initial position of the evader and its heading.

Proposition III.1. *For the pursuit-evasion problem formulated in Section II, the set of initial locations of the farther pursuer (P_2), following a constant bearing strategy, for which the evader reaches the perpendicular bisector at a time $0 \leq \tau < t_f$ (i.e., the non-degenerate region) is given by the area bounded by the circles $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1$ inclusive.*

Proof. The condition that needs to be satisfied for the problem to be non-degenerate is that there exists a time $0 \leq \tau < t_f$, and $\varepsilon > 0$ such that

$$\varepsilon < \|p_E(\tau) - p_1(\tau)\| = \|p_E(\tau) - p_2(\tau)\|. \tag{10}$$

The limiting cases occur for $\tau = 0$ and $\tau = t_f$. The curve corresponding to the instant $\tau = 0$ is the circle $\mathcal{C}_1$. Hence, the farther pursuer (P_2) has to lie on, or outside, this circle. The limiting curve (set of points) for the case $\tau = t_f$ should allow P_2 to intercept the evader that is moving at a constant heading exactly at time t_f under a constant bearing strategy. This corresponds to the circle $\mathcal{C}_2$. That is, given that P_1 and

P_2 initially lie on $\mathcal{C}_2$, they both capture the evader simultaneously at t_f (at location C), assuming that the evader follows pure evasion from P_1. Furthermore, if P_2 initially lies inside $\mathcal{C}_2$, and assuming that the evader still follows pure evasion from P_1, then there will exist a time $0 \leq \tau < t_f$ when the evader is equidistant to both the pursuers. Since the capture time for P_2 will be less than t_f before the evader gets captured at some time $\tau < t_f$ by P_2, it will be equidistant to both the pursuers. It follows that since the evader is equidistant to both pursuers (i.e., it meets the perpendicular bisector) at $0 \leq \tau < t_f$, the region bounded by $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1$ inclusive, is the non-degenerate region. □

To demonstrate the corresponding geometry pictorially, consider a problem with an evader initially stationed at $(1, 0)$ with the closer pursuer (P_1) located at $(0, 0)$ having speeds $v = 0.6$ and $u = 1$, respectively. The shaded region in Figure 1(a) represents the set of initial points for pursuer P_2 that make the problem non-degenerate. The inner circle, centered at the evader's position, sets the lower bound for P_2 in terms of its initial distance from the evader. The outer circle, with its center at the capture point $(2.5, 0)$, denotes the limiting case for pursuer P_2, and beyond which P_2 plays no role in the game.

III.B. Scenario 2

To compute the non-degenerate region in this scenario, first define the ellipse $\mathcal{E}$ as

$$\mathcal{E} = \{\mathsf{x} \in \mathbb{R}^2 : \|\mathsf{x} - p_E(0)\| + \|\mathsf{x} - p'_E\| = 2ut_f\}, \tag{11}$$

where $p'_E = (x'_E, y'_E)$ from

$$x'_E = x_E(0) + 2vt_f \frac{x_E(0) - x_1(0)}{\|p_E(0) - p_1(0)\|}, \tag{12a}$$

$$y'_E = y_E(0) + 2vt_f \frac{y_E(0) - y_1(0)}{\|p_E(0) - p_1(0)\|}. \tag{12b}$$

The ellipse $\mathcal{E}$ is an isochrone that contains the set of initial positions for a pursuer that guarantee capture at time t_f (at C) under a pure pursuit strategy for the given initial position of the evader and its heading, assuming that the evader is non-maneuvering. In the literature, $\mathcal{C}_2$ and $\mathcal{E}$ are called t_f-isochrones [13]. It can be seen from (11) that $\mathcal{E}$ is an ellipse centered at C having the initial position of the evader at one of its foci. For any initial position of a pursuer (following a pure pursuit strategy) inside $\mathcal{E}$, the capture time is less than t_f, for the given initial position of the evader and its heading.

Proposition III.2. *For the pursuit-evasion problem formulated in Section II, the set of initial points of the farther pursuer (P_2), following a pure pursuit strategy, for which the evader reaches the perpendicular bisector at a time $0 \leq \tau < t_f$ (i.e., the non-degenerate region), is given by the area bounded by the circle $\mathcal{C}_1$ and the ellipse $\mathcal{E}$, $\mathcal{C}_1$ inclusive.*

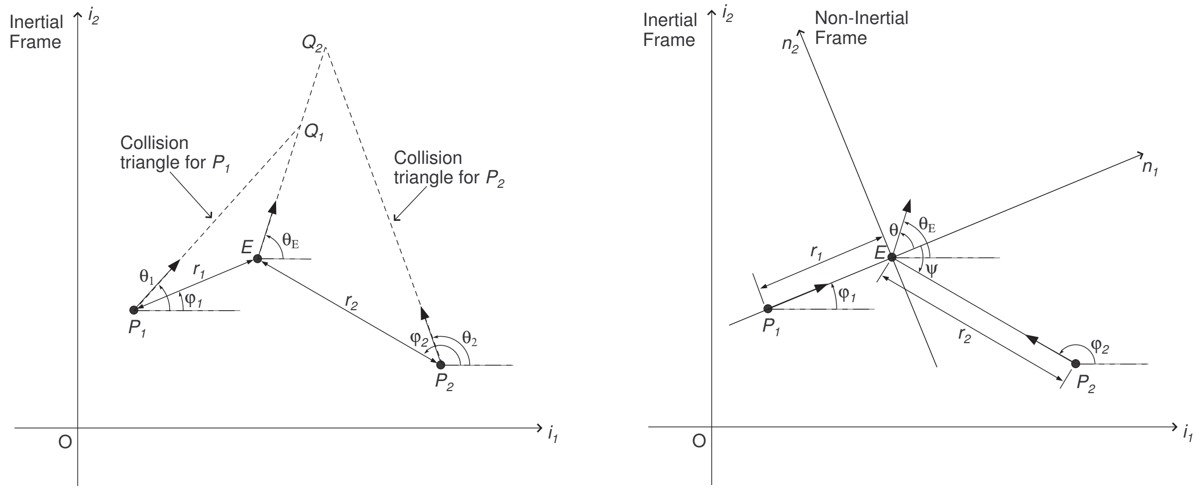
Proof. The proof is identical to the one of Proposition III.1, by simply replacing constant bearing strategy and the circle $\mathcal{C}_2$, in the discussion, with pure pursuit strategy and the ellipse $\mathcal{E}$, respectively. □

The geometry of the non-degenerate region in this scenario is depicted with the same example used for Scenario 1. The shaded region in Figure 1(b) represents the set of initial points for the second pursuer (P_2) that make the problem non-degenerate when P_2 follows a pure pursuit strategy. The inner curve which is a circle, centered at the evader's position, sets the lower bound for P_2 in terms of its initial distance from the evader. The outer curve which is an ellipse, with the evader at one of its foci, denotes the limiting case for P_2 beyond which P_2 plays no role in the game. Note that the non-degenerate region in Scenario 2 will always be smaller compared to its counterpart in Scenario 1, as can be observed from Figure 1.

IV. Optimal Strategies

IV.A. Scenario 1

As per the initial formulation in Section II, it can be seen that the game evolves in the six-dimensional state-space. However, the problem formulation can be reduced to the two-dimensional state-space in the



(a) Scenario 1

(b) Scenario 2

Figure 2. Schematics of the proposed pursuit-evasion problems.

following manner. Consider the relative distances between the evader and each of the pursuers ($r_1 - P_1$ and E, $r_2 - P_2$ and E), and the corresponding LoS angles (φ_1, φ_2), shown in Figure 2(a). Using equations (1)-(3), the dynamics can be expressed as

$$\dot{r}_1 = v\cos(\theta_E - \varphi_1) - u\cos(\theta_1 - \varphi_1), \tag{13}$$

$$\dot{\varphi}_1 = \frac{1}{r_1}\left[v\sin(\theta_E - \varphi_1) - u\sin(\theta_1 - \varphi_1)\right], \tag{14}$$

$$\dot{r}_2 = v\cos(\theta_E - \varphi_2) - u\cos(\theta_2 - \varphi_2), \tag{15}$$

$$\dot{\varphi}_2 = \frac{1}{r_2}\left[v\sin(\theta_E - \varphi_2) - u\sin(\theta_2 - \varphi_2)\right]. \tag{16}$$

Furthermore, it is assumed that the pursuers follow a constant bearing strategy and hence the LoS for a given pursuer does not rotate, i.e., $\dot{\varphi}_1 = 0$ and $\dot{\varphi}_2 = 0$. That is, $\varphi_1(t) = \varphi_{10}$, $\varphi_2(t) = \varphi_{20}$, for all $t \geq 0$, where φ_{10} and φ_{20} are the LoS angles at the initial time $t = 0$. Therefore, r_1 and r_2 are the only states that have to be taken into consideration to solve for the optimal evading strategy in this scenario.

The problem can now be solved using tools from optimal control theory. Since we are dealing with a time maximization problem, one way to formulate the problem is to introduce a payoff function

$$\min_{\theta_E} \mathcal{J}(\theta_E) = -\int_0^{t_c} \mathrm{d}t, \tag{17}$$

subject to the dynamics

$$\dot{r}_1 = v\cos(\theta_E - \varphi_{10}) - u\cos(\theta_1 - \varphi_{10}), \tag{18}$$

$$\dot{r}_2 = v\cos(\theta_E - \varphi_{20}) - u\cos(\theta_2 - \varphi_{20}). \tag{19}$$

Note that θ_1, θ_2 are functions of θ_E. They can be determined at each instant of time, given θ_E, using the equations (14) and (16), and the fact that $\dot{\varphi}_1 = \dot{\varphi}_2 = 0$. Therefore,

$$u\sin(\theta_1 - \varphi_{10}) = v\sin(\theta_E - \varphi_{10}), \tag{20}$$

$$u\sin(\theta_2 - \varphi_{20}) = v\sin(\theta_E - \varphi_{20}). \tag{21}$$

Each of the above relations have two possible solutions for θ_1 and θ_2, given θ_E, and each pursuer chooses that solution for which $\dot{r}_1 < 0$ and $\dot{r}_2 < 0$, respectively. The initial conditions of the problem are

$$r_1(0) = r_{10} = \|p_E(0) - p_1(0)\|, \tag{22}$$

$$r_2(0) = r_{20} = \|p_E(0) - p_2(0)\|. \tag{23}$$

The terminal condition for capture is

$$\Psi(r_1(t_c), r_2(t_c)) = \min\{r_1(t_c), r_2(t_c)\} - \epsilon = 0. \tag{24}$$

The Hamiltonian for this problem can be expressed as

$$\begin{aligned} H &= -1 + \lambda_1 \dot{r}_1 + \lambda_2 \dot{r}_2 \\ &= -1 + \lambda_1 \left[v\cos(\theta_{\scriptscriptstyle E} - \varphi_{10}) - u\cos(\theta_1 - \varphi_{10})\right] + \lambda_2 \left[v\cos(\theta_{\scriptscriptstyle E} - \varphi_{20}) - u\cos(\theta_2 - \varphi_{20})\right], \end{aligned} \tag{25}$$

where λ_1 and λ_2 are the co-states. The corresponding adjoint equations are given by

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial r_1} = 0, \tag{26a}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial r_2} = 0. \tag{26b}$$

Therefore, $\lambda_1(t) = c_1$, $\lambda_2(t) = c_2$, $t \in [0, t_c]$, where c_1 and c_2 are constants. The transversality conditions are given by

$$\lambda_1(t_c) = \nu \frac{\partial \Psi}{\partial r_1}\Big|_{t=t_c}, \qquad \lambda_2(t_c) = \nu \frac{\partial \Psi}{\partial r_2}\Big|_{t=t_c}, \qquad H(t_c) = 0, \tag{27}$$

where $\nu \in \mathbb{R}$. Since the Hamiltonian has no explicit dependency on time,

$$H(t) = 0, \quad t \in [0, t_c]. \tag{28}$$

Note that the terminal condition is not fully differentiable and it can be written as

$$\begin{bmatrix} \frac{\partial \Psi}{\partial r_1}\Big|_{t=t_c} \\ \frac{\partial \Psi}{\partial r_2}\Big|_{t=t_c} \end{bmatrix}^{\top} = \begin{cases} [1 \quad 0], & r_1(t_c) = \epsilon < r_2(t_c), \\ [0 \quad 1], & r_2(t_c) = \epsilon < r_1(t_c). \end{cases} \tag{29}$$

At $r_1(t_c) = r_2(t_c) = \epsilon$, the partial derivatives are undefined. Using Pontryagin's minimum principle,

$$\frac{\partial H}{\partial \theta_{\scriptscriptstyle E}} = 0, \tag{30}$$

the following result is obtained,

$$\lambda_1 \left[-v\sin(\theta_{\scriptscriptstyle E} - \varphi_{10}) + u\sin(\theta_1 - \varphi_{10})\frac{\partial \theta_1}{\partial \theta_{\scriptscriptstyle E}}\right] + \lambda_2 \left[-v\sin(\theta_{\scriptscriptstyle E} - \varphi_{20}) + u\sin(\theta_2 - \varphi_{20})\frac{\partial \theta_2}{\partial \theta_{\scriptscriptstyle E}}\right] = 0, \tag{31}$$

where from (20) and (21),

$$\frac{\partial \theta_1}{\partial \theta_{\scriptscriptstyle E}} = \frac{v\cos(\theta_{\scriptscriptstyle E} - \varphi_{10})}{u\cos(\theta_1 - \varphi_{10})}, \quad \cos(\theta_1 - \varphi_{10}) \neq 0, \tag{32}$$

$$\frac{\partial \theta_2}{\partial \theta_{\scriptscriptstyle E}} = \frac{v\cos(\theta_{\scriptscriptstyle E} - \varphi_{20})}{u\cos(\theta_2 - \varphi_{20})}, \quad \cos(\theta_2 - \varphi_{20}) \neq 0. \tag{33}$$

Since λ_1, λ_2 are constants, θ_1, θ_2 and their partials from (32), (33) are dependent only on $\theta_{\scriptscriptstyle E}$, we can conclude from (31) that the optimal heading of the evader $\theta_{\scriptscriptstyle E}$ is constant in time, and hence the headings of the pursuers are constant as well.

Proposition IV.1. *Consider the optimal control problem expressed using the equations (17)-(24). If the farther pursuer (P_2) initially lies inside the non-degenerate region, then the optimal control strategy of the evader involves simultaneous capture.*

Proof. First, consider the case when $r_1(t_c) = \epsilon < r_2(t_c)$, i.e, only P_1 captures the evader at the final time. In this case, $\lambda_2(t_c) = 0$, which implies $c_2 = 0$. From (25), (28), and (31), we have that

$$-1 + c_1 \left[v \cos(\theta_E - \varphi_{10}) - u \cos(\theta_1 - \varphi_{10})\right] = 0, \tag{34}$$

$$c_1 \left[-v \sin(\theta_E - \varphi_{10}) + u \sin(\theta_1 - \varphi_{10}) \frac{\partial \theta_1}{\partial \theta_E}\right] = 0, \tag{35}$$

and c_1 cannot be equal to zero as it will lead to a contradiction in (34). Therefore,

$$-v \sin(\theta_E - \varphi_{10}) + u \sin(\theta_1 - \varphi_{10}) \frac{\partial \theta_1}{\partial \theta_E} = 0. \tag{36}$$

From (32), $\sin(\theta_1 - \varphi_{10}) \cos(\theta_E - \varphi_{10}) - \sin(\theta_E - \varphi_{10}) \cos(\theta_1 - \varphi_{10}) = 0$, which implies

$$\sin(\theta_E - \theta_1) = 0. \tag{37}$$

Further analysis leads to the result

$$\theta_E^* = \varphi_{10}, \tag{38}$$

i.e, the optimal strategy is a pure evasion from P_1. This is the solution for a degenerate case of the problem. It has been proven that in the non-degenerate case with this strategy, P_2 will capture the evader prior to P_1, leading to a contradiction. Similarly, the strategy for the instance when $r_2(t_c) = \epsilon < r_1(t_c)$ turns out to be a pure evasion from P_2. But since P_1 is closer to the evader, it lies inside the circle of equal time-to-capture corresponding to P_2, and therefore P_1 reaches the evader before P_2, again leading to a contradiction. Hence, the optimal evading strategy in a non-degenerate case should involve $r_1(t_c) = r_2(t_c) = \epsilon$ namely, simultaneous capture. □

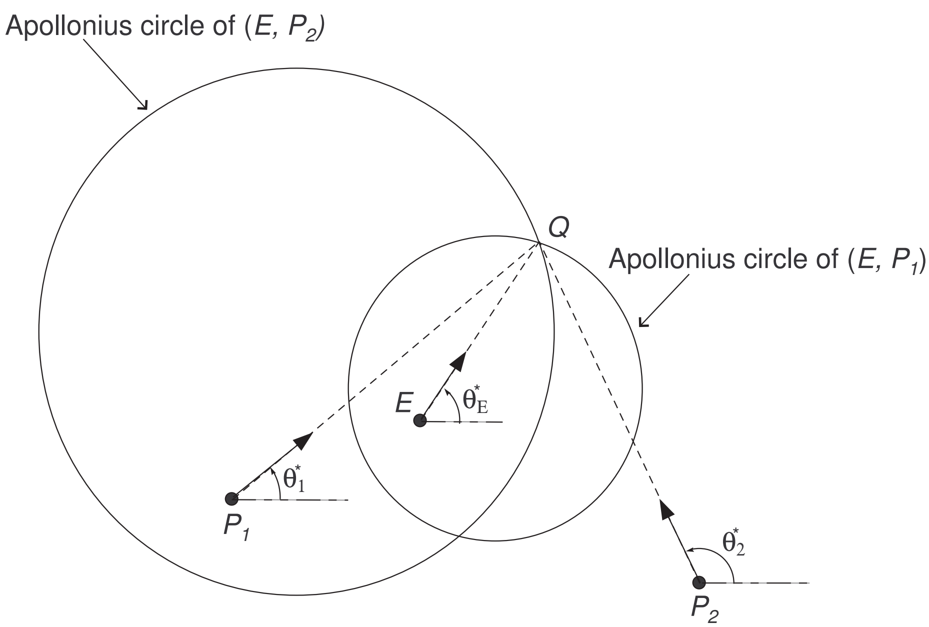


Figure 3. A schematic of finding the optimal heading of the evader using Apollonius circles.

Since the optimal heading θ_E^* is constant, and involves a simultaneous capture in the non-degenerate case, an easy way to obtain the heading is to use the well-known Apollonius circles [1], see Figure 3. The Apollonius circles of the pairs (E, P_1) and (E, P_2) at the initial time can be constructed from the players' initial positions. If the problem is non-degenerate, then there always exist two intersection points. During optimal play, the evader should head towards one of the intersection points, namely, the one that is farther away. If both the points are equidistant, then the evader can choose either point. This completes the analysis on the optimal evading strategy for Scenario 1.

IV.B. Scenario 2

In this scenario, the problem can be examined in the three-dimensional state-space that makes the analysis simpler. A schematic of the geometry of the proposed pursuit-evasion problem is shown in Figure 2(b). First, we translate the problem into a rotating/non-inertial frame with the origin fixed on the evader (E) and with the x-axis along the line joining P_1 and E. The velocity vector of P_1 is along the x-axis, as it follows a pure pursuit strategy. In this frame P_1 is restricted to move only along the x-axis. The positions of the players expressed in polar coordinates are given by $p_1 = (r_1, \pi)$, $p_2 = (r_2, \psi)$, and $p_E = (0, 0)$, $-\pi < \psi \leq \pi$. Since the pursuers follow a pure pursuit strategy, their headings are along their corresponding LoS directions, i.e., $\theta_1 = \varphi_1$, $\theta_2 = \varphi_2$, see Figure 2(b). The angle between the velocity vectors of P_1 and E is $\theta = \theta_E - \varphi_1$. The rotation rate of the non-inertial frame is given by

$$\dot{\varphi}_1 = \frac{v \sin\theta}{\|p_E - p_1\|} = \frac{v \sin\theta}{r_1}. \tag{39}$$

In the reduced state-space, the number of states is just three, and the corresponding equations of motion are given by

$$\dot{r}_1 = -u + v\cos\theta, \tag{40}$$
$$\dot{r}_2 = -u - v\cos(\psi - \theta), \tag{41}$$
$$\dot{\psi} = \frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta. \tag{42}$$

The initial conditions for the states can be computed as follows

$$r_1(0) = \|p_E(0) - p_1(0)\|, \tag{43}$$
$$r_2(0) = \|p_E(0) - p_2(0)\|, \tag{44}$$
$$\psi(0) = \pi - \varphi_{10} + \varphi_{20}, \tag{45}$$

where φ_{10} and φ_{20} are now the initial headings of P_1 and P_2, respectively, which can be obtained from the initial positions of the players. The terminal condition remains the same as

$$\Psi(r_1(t_c), r_2(t_c)) = \min\{r_1(t_c), r_2(t_c)\} - \epsilon = 0. \tag{46}$$

The problem statement is then to find the optimal control $\theta^*(t)$ that maximizes capture time t_c given the equations of motion (40)-(42), the initial conditions (43)-(45), and the terminal condition (46). It is assumed that the initial conditions are such that the problem is non-degenerate for the given speeds of the players. Otherwise, the pursuit strategy for the evader is pure evasion from pursuer P_1.

The problem can be solved using the framework of optimal control theory by considering the payoff function

$$\min_{\theta} \mathcal{J}(\theta) = -\int_0^{t_c} \mathrm{d}t. \tag{47}$$

The Hamiltonian can then be written as

$$\begin{aligned} H &= -1 + \lambda_1\dot{r}_1 + \lambda_2\dot{r}_2 + \lambda_3\dot{\psi} \\ &= -1 + \lambda_1(-u + v\cos\theta) + \lambda_2[-u - v\cos(\psi - \theta)] + \lambda_3[\frac{v}{r_2}\sin(\psi - \theta) - \frac{v}{r_1}\sin\theta], \end{aligned} \tag{48}$$

where λ_1, λ_2, and λ_3 are the co-states. The corresponding adjoint equations are given by

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial r_1} = -\frac{\lambda_3 v \sin\theta}{r_1^2}, \tag{49}$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial r_2} = \frac{\lambda_3 v \sin(\psi - \theta)}{r_2^2}, \tag{50}$$
$$\dot{\lambda}_3 = -\frac{\partial H}{\partial \psi} = -\lambda_2 v \sin(\psi - \theta) - \frac{\lambda_3 v \cos(\psi - \theta)}{r_2}. \tag{51}$$

Since $\psi(t_c)$ is not specified and is free, the transversality conditions are given by

$$\lambda_1(t_c) = \nu \frac{\partial \Psi}{\partial r_1}\bigg|_{t=t_c}, \tag{52}$$

$$\lambda_2(t_c) = \nu \frac{\partial \Psi}{\partial r_2}\bigg|_{t=t_c}, \tag{53}$$

$$\lambda_3(t_c) = 0, \tag{54}$$

$$H(t_c) = 0, \tag{55}$$

where $\nu \in \mathbb{R}$. Since the Hamiltonian has no explicit dependency on time,

$$H(t) = 0, \quad t \in [0, t_c]. \tag{56}$$

Note that since the terminal condition is the same as in (24), its derivatives are implicit from (29). From,

$$\frac{\partial H}{\partial \theta} = 0, \tag{57}$$

the following expression can be derived

$$-\lambda_1 v \sin\theta - \lambda_2 v \sin(\psi - \theta) - \lambda_3 \left[\frac{v}{r_2}\cos(\psi - \theta) + \frac{v}{r_1}\cos\theta\right] = 0. \tag{58}$$

Proposition IV.2. *Consider the optimal control problem expressed using the equations (40)-(47). If the farther pursuer (P_2) initially lies inside the non-degenerate region, then the optimal control strategy of the evader involves simultaneous capture.*

Proof. Consider the case when $r_1(t_c) = \epsilon < r_2(t_c)$. This implies $\lambda_1(t_c) = \nu$, and $\lambda_2(t_c) = 0$. Note that the adjoint equations are linear in the co-states λ_2, λ_3, and since $\lambda_2(t_c) = \lambda_2(t_c) = 0$, the co-states are constant in time, i.e., $\lambda_1(t) = \nu$, $\lambda_2(t) = 0$, $\lambda_3(t) = 0$. From (48), (56), (58), it follows that

$$-1 + \nu(-u + v\cos\theta) = 0, \tag{59}$$

$$\nu v \sin\theta = 0. \tag{60}$$

Since $\nu = 0$ leads to a contradiction in (59), $\sin\theta = 0$, and thus

$$\theta^*(t) = 0, \tag{61}$$

which means that the optimal strategy is pure evasion from P_1. However, this is true only when the problem is degenerate. In a non-degenerate case, it will lead to early capture by P_2. In the second instance, when $r_2(t_c) = \epsilon < r_1(t_c)$, $\lambda_1(t_c) = 0$, $\lambda_2(t_c) = \nu$. Furthermore, from (48), (56), (58), at $t = t_c$.

$$-1 + \nu(-u - v\cos(\psi(t_c) - \theta(t_c))) = 0, \tag{62}$$

$$\nu v \sin(\psi(t_c) - \theta(t_c)) = 0. \tag{63}$$

Since $\nu \neq 0$, it follows that $\sin(\psi(t_c) - \theta(t_c)) = 0$. With this terminal condition, it can be seen that the co-states are constant and similarly is the optimal heading, which in this case is given by $\theta^*(t) = \pi + \psi$. This means that the optimal evading strategy is a pure evasion from P_2. However, this strategy is infeasible in the non-degenerate case. Hence, the optimal evading strategy in a non-degenerate case should involve $r_1(t_c) = r_2(t_c) = \epsilon$ i.e., simultaneous capture. □

Proposition IV.3. *Consider the optimal control problem expressed using the equations (40)-(47). The optimal control strategy of the evader can be summarized as follows:*

$$\tan\theta^* = \begin{cases} 0, & r_2 \geq \dfrac{p}{1 - e\cos\psi}, \\ \Theta(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3), & r_1 < r_2 < \dfrac{p}{1 - e\cos\psi}, \\ \tan\left(\dfrac{\psi + \pi}{2}\right), & r_1 = r_2,\ \psi \leq 0, \\ \tan\left(\dfrac{\psi - \pi}{2}\right), & r_1 = r_2,\ \psi \geq 0, \end{cases} \tag{64}$$

where

$$\Theta(r_1, r_2, \psi, \lambda_1, \lambda_2, \lambda_3) = -\frac{\lambda_2 \sin\psi + (\lambda_3/r_1) + (\lambda_3 \cos\psi/r_2)}{\lambda_1 - \lambda_2 \cos\psi + (\lambda_3 \sin\psi/r_2)}, \tag{65}$$

$e = v/u$ is the eccentricity, and $p = r_1(1 + v/u)$ is the semi-latus rectum of the ellipse $\mathcal{E}$ in (11).

Proof. The regions of degeneracy and non-degeneracy, bounded by $\mathcal{E}$, given the distance r_1 (between P_1 and E), can be expressed using the pair (r_2, ψ). The size of the ellipse, having semi-major axis $a = r_1/(1 - v/u)$ grows with the distance between P_1 and the evader, and also with the speed ratio v/u. For a given r_1, P_2 (its position is expressed using the pair (r_2, ψ), $r_2 \geq r_1$), is in the non-degenerate region if it is inside $\mathcal{E}$ and vice-versa. In this regard, from the properties of the ellipse, it can be noticed that if

$$r_2 < \frac{p}{1 - e\cos\psi}, \tag{66}$$

then P_2 is inside $\mathcal{E}$, i.e, the non-degenerate region, and if

$$r_2 \geq \frac{p}{1 - e\cos\psi}, \tag{67}$$

then P_2 is in the degenerate region. As discussed earlier, the optimal strategy is a pure evasion from P_1, if the problem is degenerate.

There is a special case where both pursuers are equidistant to the evader when the problem is non-degenerate. In this case, the optimal strategy for the evader is to move along the perpendicular bisector and away form the line joining the two pursuers. When $\psi = 0$, the problem is symmetric and the evader has two choices, $\theta^* = \pi/2$ or $-\pi/2$. Thus, the optimal control for the case $r_1 = r_2$ can be summarized as

$$\theta^* = \begin{cases} \dfrac{\psi + \pi}{2}, & r_1 = r_2,\ \psi \leq 0, \\ \dfrac{\psi - \pi}{2}, & r_1 = r_2,\ \psi \geq 0. \end{cases} \tag{68}$$

When the problem is non-degenerate and the evader is not equidistant with respect to P_1 and P_2 ($r_1 \neq r_2$), it follows from (58) that

$$\tan\theta^* = -\frac{\lambda_2 \sin\psi + (\lambda_3/r_1) + (\lambda_3 \cos\psi/r_2)}{\lambda_1 - \lambda_2 \cos\psi + (\lambda_3 \sin\psi/r_2)}. \tag{69}$$

This completes the proof. □

The optimal control problem for Scenario 2 can be solved using the above analysis and a numerical solver. A closed-form solution to the equations of optimality subject to arbitrary initial conditions is at this point intractable. Section V presents some simulation results for both scenarios. In the next subsection, a suboptimal strategy for Scenario 2, that is easy to implement in practice, is discussed. This suboptimal strategy of the evader is based on geometric arguments and interpretations obtained from the above analysis.

IV.C. A Suboptimal Strategy for Scenario 2

The optimal strategy for Scenario 2 can be intuitively understood as one where the evader chooses its heading so that it does not favor any one of the two pursuers, and which finally results in simultaneous capture. With this motivation, a suboptimal strategy is constructed and its performance is compared with the optimal one. Given that each pursuer follows a pure pursuit strategy, the capture time for a non-maneuvering evader (constant heading) is given by

$$t_f = \frac{r_o(u + v\cos\theta)}{u^2 - v^2}, \quad v \neq 0, \tag{70}$$

where r_o is the initial distance between the pursuer and the evader, and θ is the evader's heading measured with respect to the line-of-sight from the pursuer to the evader [13]. For a non-degenerate problem, the evader's heading for which both P_1 and P_2 take equal time to reach the evader can be found from their initial positions using the expression

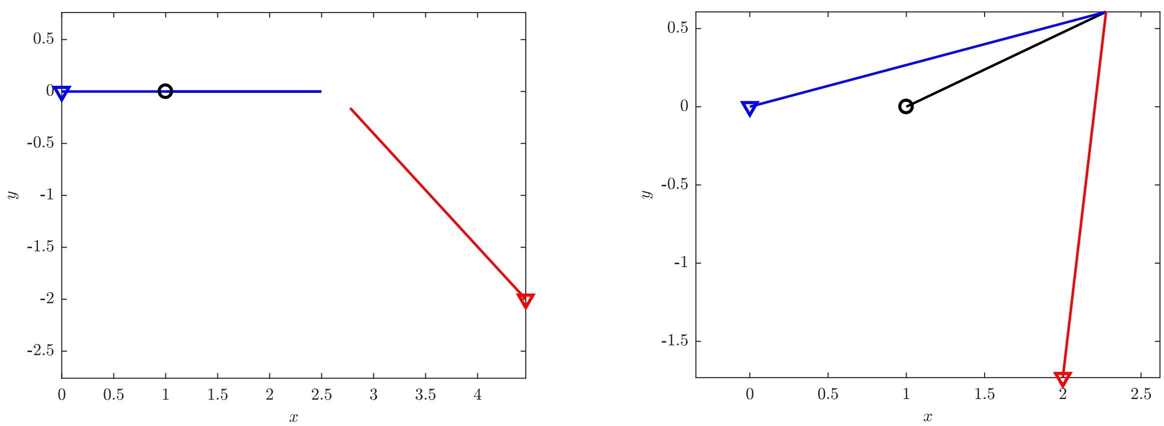
$$r_1(0)(u + v\cos\theta) = r_2(0)(u + v\cos(\psi - \theta)). \tag{71}$$

In general, equation (71) has two solutions resulting in simultaneous capture, assuming the evader follows a constant heading. For the given initial conditions, the solution to (71) that provides maximum capture time is chosen as the suboptimal strategy. If the problem is degenerate, i.e., P_2 is outside the ellipse, then (71) has no solution.

V. Numerical Simulations

In this section, the aforementioned strategies are demonstrated using simulations performed for both scenarios. The speeds of the players are chosen to be $u = 1$, $v = 0.6$, unless specified otherwise. The radius of capture is $\epsilon = 0.001$.

The optimal strategy for Scenario 1 is straightforward. The software package GPOPS-II [15] was used to simulate the test cases and validate the presented theory. Figure 4(a) presents the trajectories of the players for the initial conditions, $p_1 = (0, 0)$, $p_2 = (4.464, -2)$, $p_E = (1, 0)$, that makes the problem degenerate. Clearly, the optimal strategy is a pure evasion from P_1, and P_2 does not affect the evader's trajectory. An example for the non-degenerate case is presented in Figure 4(a) for the initial conditions, $p_1 = (0, 0)$, $p_2 = (2, -1.732)$, $p_E = (1, 0)$. It can be observed that the optimal evading strategy involves simultaneous capture with a constant heading.



(a) A degenerate case

(b) A non-degenerate case

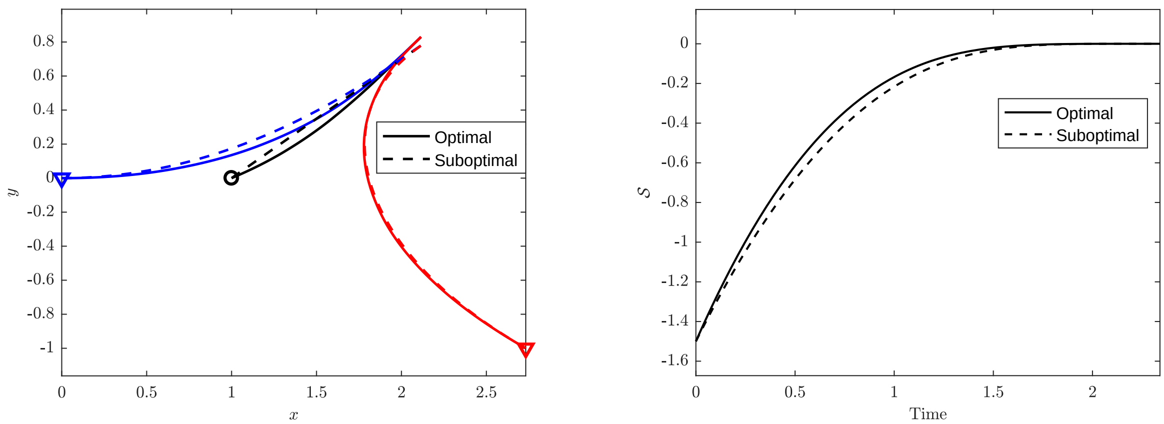
Figure 4. Trajectories of the players for optimal control inputs in Scenario 1: black - evader, blue - P_1, red - P_2.

The simulation results for the non-degenerate cases of Scenario 2, obtained using GPOPS-II, can be seen in Figure 5. Figure 5(a) presents the trajectories of the players for initial conditions $p_1 = (0, 0)$, $p_2 = (2.732, -1)$, $p_E = (1, 0)$. In the reduced state-space, these positions correspond to $r_1(0) = 1$, $r_2(0) = 2$, and $\psi(0) = -\pi/6$. The optimal capture time is $t_c = 2.340$.

As expected, simultaneous capture is observed in these figures. Also, it can be observed that the constraint is met only at the final time. This suggests that the evader is approaching the perpendicular bisector just before it gets captured. The same behavior has been observed in all the simulations that were carried out. The suboptimal strategy is also compared against the optimal strategy in Figure 5. The (constant) heading obtained from the suboptimal strategy is $\theta = 0.6099$ ($34.94°$) with a capture time of $t_c = 2.329$. Note that the capture time and the time variation of the constraint are comparable to the corresponding results obtained using the optimal strategy, see Figure 5(b).

Furthermore, a comparative study was carried out to gauge the performance of the suboptimal strategy. For this purpose, the following parameters were chosen: $r_1(0) = 1$, $u = 1$. The speed of the evader v was varied from 0.3 to 0.7. For each v, 140 different initial conditions ($r_2(0)$, $\psi(0)$) were considered spanning the non-degenerate area for the chosen $r_1(0)$ and u. Table 1 presents the results of this comparative study. Though the average percentage variation of the time-to-capture increases with the evader's speed v, the variation is less than 1% for all the evader speeds considered. The maximum percentage variation is only 2%. It can be observed that the suboptimal strategy is easily implementable and its performance is similar to the optimal one. Hence, the suboptimal strategy can be considered for all practical purposes.

The elegance of the optimal strategy lies in the empirical observation that the evader is able to choose



(a) Trajectories

(b) Time variation of the constraint ($\mathcal{S}$)

Figure 5. Performance of the optimal and suboptimal strategies for a non-degenerate case in Scenario 2: black - evader, blue - P_1, red - P_2.

Table 1. Comparison table for optimal and suboptimal strategies of Scenario 2

v	Average percentage variation in t_c	Maximum percentage variation in t_c
0.3	0.0337 %	0.4451 %
0.4	0.0727 %	0.7653 %
0.5	0.1277 %	1.2182 %
0.6	0.1704 %	1.6883 %
0.7	0.2343 %	2.2487 %

a control that lets it hit the perpendicular bisector only at the instant of capture (this is true everywhere except when P_1 and P_2 are initially equidistant from the evader). In this regard, some trends are observed and are plotted in Figure 6. The fixed parameters for this set of simulations are $r_1 = 1$, $u = 1$, $v = 0.5$. For a given ψ, r_2 varies as $1 \leq r_2 \leq 3/(2 - \cos\psi)$, from (66). The corresponding optimal headings of the evader are plotted for different values of ψ, $\psi \in (-\pi, \pi]$ in Figure 6. As the initial position of P_2 approaches the ellipse for a given ψ, θ^* approaches 0, and in the case of $r_2 = r_1 = 1$, the heading is along the perpendicular bisector, in agreement with (64).

VI. Conclusion

We have analyzed the two-pursuer/one-evader problem considering two scenarios that differ in their information structure. In both scenarios, it is assumed that the pursuers are superior to the evader in terms of their speed capabilities, and hence capture is guaranteed. Optimal evading strategies that maximize capture time are derived. In the process, the corresponding regions of non-degeneracy are identified for the case of identical pursuers. These regions can be used to inspect the potency in employing two pursuers to capture the evader. If the problem is degenerate, then optimal evading strategy is pure evasion from the closer pursuer, and the second pursuer is a mere spectator and cannot affect the outcome of the game. However, if the problem is non-degenerate, then both pursuers should participate in the game to reduce capture time. In this case it is established that the optimal evading strategy entails one that leads to simultaneous capture. Since obtaining a closed-form solution for the second scenario is intractable, an efficient sub-optimal strategy is proposed and is compared with its optimal counterpart.

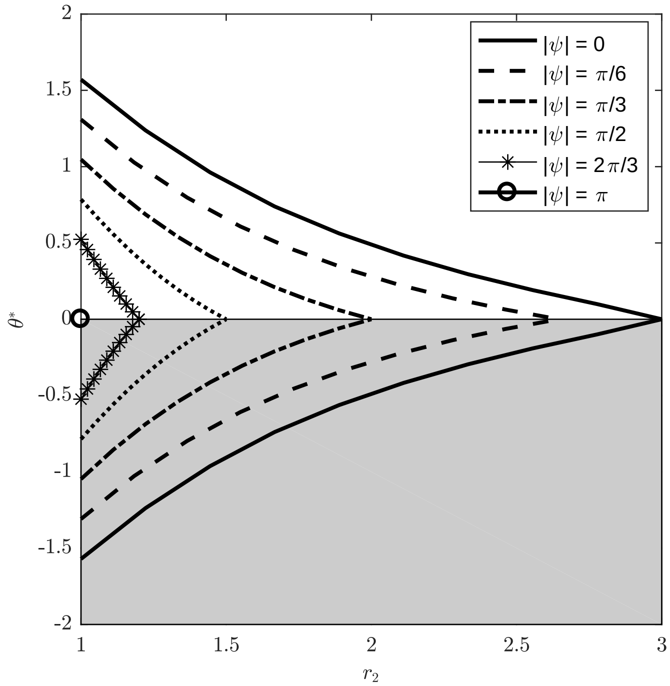


Figure 6. Optimal control trends in Scenario 2: $r_1 = 1$, $u = 1$, $v = 0.5$. $\psi \geq 0$ in the shaded region and $\psi \leq 0$ in the unshaded part.

Acknowledgment

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