Spacecraft Trajectory Tracking with Identification of Mass Properties Using Dual Quaternions

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The problem of estimating the mass properties of a spacecraft while tracking a 6-DOF reference is addressed using dual quaternions. Dual quaternions provide a position and attitude (pose) representation, which has proven to be advantageous over other, more conventional, parameterizations. An adaptive controller for 6-DOF tracking is proposed using concepts from the concurrent learning framework. The latter is a recently proposed methodology to incorporate current and recorded system data from measurements into the update of an adaptive controller's parameters. Asymptotic convergence of the parameters is ensured through an easily verifiable rank condition of the matrix formed from a finite set of collected data, contrary to the rather stringent, but more common requirement of persistency of excitation. Simulation results for the tracking of a non-persistently exciting, 6-DOF reference are provided and compared to the baseline adaptive controller.

Nomenclature

k	=	time index
t_k	=	time
$\mathbb{H}$	=	space of quaternions
$\mathbb{H}_d$	=	space of dual quaternions
$r^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	position quaternion from point Y to point X, in Z frame coordinates
$v^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	linear velocity quaternion of point X with respect to frame Y, in Z frame coordinates
$\omega^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	angular velocity quaternion of frame X relative to frame Y, in Z frame coordinates
$q_{\mathrm{X/Y}}$	=	quaternion describing attitude change from frame Y to frame X
$\mathsf{1}$	=	$(1,\bar{0})$
$\mathsf{0}$	=	$(0,\bar{0})$
ϵ	=	dual unit
$\boldsymbol{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	dual velocity of frame X relative to frame Y, in Z frame coordinates
$\boldsymbol{q}_{\mathrm{X/Y}}$	=	dual quaternion describing pose change from frame Y to frame X
$\mathbf{1}$	=	$\mathsf{1}+\epsilon\mathsf{0}$
$\mathbf{0}$	=	$\mathsf{0}+\epsilon\mathsf{0}$
m	=	satellite mass (kg)
$\bar{I}^{\mathrm{B}}$	=	satellite inertia about the center of mass in B frame coordinates $(kg.m^2)$
τ^{B}	=	external torque quaternion applied at body center of mass expressed in the body frame $(N.m)$
f^{B}	=	external force quaternion applied at body center of mass expressed in the body frame (N)
$\boldsymbol{f}^{\mathrm{B}}$	=	external dual force applied at body center of mass expressed in the body frame
M^{B}	=	dual inertia matrix about the center of mass in B frame coordinates

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$\widehat{M}^{\mathrm{B}}$	=	estimate of the dual inertia matrix
ΔM^{B}	=	estimate of the dual inertia matrix
ε_k	=	estimate error signal
$\mathcal{X}$	=	time-indexed storage of state information
$\mathcal{F}$	=	time-indexed storage of dual force information
$\mathrm{h}(\cdot,\cdot)$	=	function $\mathbb{H}_d^v \times \mathbb{H}_d^v \to \mathbb{R}^7$
$\mathrm{v}(\cdot)$	=	dual inertia matrix vectorization function, $\mathbb{R}^{8\times 8} \to \mathbb{R}^7$
R_k	=	regressor matrix function, $\mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^u \to \mathbb{R}^{8\times 7}$
K_p, K_d, K_i	=	gain matrices

I. Introduction

In the past, much of spacecraft control literature has focused on performing attitude reference tracking through the use of a wide range of techniques and attitude parameterizations.[1–5] With the advent of space missions (both commercial and military), spacecraft proximity operations have become increasingly common, and they remain among the most critical phases for space-related activities. Ranging from on-orbit servicing, asteroid sample return, or just rendezvous and docking, these maneuvers pose a challenging technological problem that requires addressing the natural coupling between the spacecraft's attitude and its position.

Originally, the modeling of rigid body motion to address proximity operations was decoupled into the corresponding attitude and position (pose) subproblems.[6,7] This tends to be the simpler approach, as it makes use of conventional techniques. The cost is usually efficiency and accuracy. New techniques treat attitude and position on the same footing and thus increase numerical efficiency and accuracy. The benefits have been especially dominant in the field of estimation where combined representations of pose have led to significant improvements in the estimation of position.[8–11]

Within the area of kinematic and dynamic modeling, *fixed-base* robotics literature has flourished, making extensive use of Lie-algebraic techniques. In particular, the space of homogeneous matrices $SE(3)$ has been adopted as a common resource in combination with standardizations such as the Denavit-Hartenberg parameters. However, modeling of a *freely-rotating* body and its dynamics under the same framework requires in-depth knowledge of Lie algebras and the associated geometric mechanics formalism for appropriate use and implementation (see, for example, Ref. 12). This added complexity makes quaternions and, in particular, dual quaternions an appealing alternative to work with for most practitioners.

Dual quaternions are an extension of quaternions in the context of dual algebra. While unit quaternions carry information about the relative attitude between two frames, dual quaternions contain relative pose information between two frames. Dual quaternions have been shown to have better computational efficiency and lower memory requirements than other conventional methods for kinematic modeling.[13–16] Since computational power is often limited in space-related tasks, this makes quaternions and dual quaternions more appropriate than, say, working directly with the more natural spaces $SO(3)$ (for attitude) or $SE(3)$ (for pose).

A large amount of literature exists that addresses the problem of the estimation of the inertia matrix of a spacecraft in orbit. It has been recurrently addressed and different solution approaches exist. The contribution by Ref. 17 is worth highlighting, in particular, since it provides assurances as the number of samples tends to infinity using least-correlation methods, without the assumption that the angular acceleration is known. Refs. 18–20 also provide least-squares solutions, posing the estimation as an optimization problem with convexity properties that aid convergence to a solution. Though theoretically sound, these approaches can be computationally costly, in many cases requiring matrix inversions or decompositions, make use of optimization software that may not be flight-rated, and are not necessarily incorporated into controllers that can track a 6-DOF time-varying references. In fact, an on-line update of the inertia matrix using these methods could introduce undesired discontinuities to the actuators, making them undesirable for actual on-board implementation.

Within the field of adaptive control, the requirement of persistency of excitation that frequently arises in the study of systems with structured uncertainties, as is the case with the estimation of the mass properties of a rigid body, is rather stringent.[21–24] The necessary rank conditions on the integral with respect to time of certain regressor matrices can be ensured by actuating the different axes of the spacecraft, as shown in

Ref. 22. However, this may lead to unnecessary maneuvering, thus wasting fuel and power.

The main objective of this paper is to address the estimation of the mass properties of a spacecraft through a data-based adaptive method, called *concurrent learning*, while following at the same time a desired pose reference. This technique, initially proposed by Chowdhary et al.,[25,26] bypasses the requirement of persistency of excitation that is so common in the field of adaptive control, and instead requires that the rank of a matrix built from a finite set of data be the same as the dimensionality of the uncertainty. Additionally, it avoids matrix inversion, which makes it more computationally favorable, while still being seamlessly integrated into an existing 6-DOF pose tracking controller.

This paper is structured as follows: Section II contains a review of the main equations for quaternions and dual quaternions. Section III lays out the theoretical framework for the use of concurrent learning. The main result of the paper is given in Section IV. Finally, Section V contains simulation results for the proposed control law in comparison with the baseline controller.

II. Mathematical Preliminaries

This section introduces the basic concepts of quaternions, dual quaternions, and their use in representing kinematics and dynamics of rigid bodies. For an exhaustive description the reader is referred to Refs. 27–30, from which the notation has been adopted.

II.A. Quaternions

Quaternions are a mathematical tool commonly used to represent rotations in three-dimensional space. Quaternions define an associative, non-commutative algebra, defined as $\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = ijk = -1,\ q_i \in \mathbb{R}\}$. In practice, quaternions are referred to by their scalar and vectors parts as $q = (q_0, \bar{q})$, where $q_0 \in \mathbb{R}$ and $\bar{q} = [q_1, q_2, q_3]^{\mathrm{T}} \in \mathbb{R}^3$. The properties of quaternion algebra are summarized in Table 1. Previous literature has defined quaternion multiplication as the multiplication between a 4×4 matrix and a vector in $\mathbb{R}^4$.

Table 1. Quaternion Operations

Operation	Definition
Addition	$a + b = (a_0 + b_0, \bar{a} + \bar{b})$
Multiplication by a scalar	$\lambda a = (\lambda a_0, \lambda a)$
Multiplication	$ab = (a_0 b_0 - \bar{a} \cdot \bar{b}, a_0 \bar{b} + b_0 \bar{a} + \bar{a} \times \bar{b})$
Conjugate	$a^* = (a_0, -\bar{a})$
Dot product	$a \cdot b = (a_0 b_0 + \bar{a} \cdot \bar{b}, 0_{3\times1})$
Cross product	$a \times b = (0, a_0 \bar{b} + b_0 \bar{a} + \bar{a} \times \bar{b})$
Norm	$\lVert a \rVert = \sqrt{a \cdot a}$

Since any rotation can be described by three parameters, the unit norm constraint is imposed on quaternions for attitude representation. *Unit* quaternions are closed under multiplication, but not under addition. A quaternion describing the orientation of frame X with respect to frame Y, $q_{\mathrm{X/Y}}$, will satisfy $q^*_{\mathrm{X/Y}} q_{\mathrm{X/Y}} = q_{\mathrm{X/Y}} q^*_{\mathrm{X/Y}} = \mathtt{1}$, where $\mathtt{1} = (1, 0_{3\times1})$. This quaternion can be constructed as $q_{\mathrm{X/Y}} = (\cos(\phi/2), \bar{n}\sin(\theta/2))$, where $\bar{n}$ and θ are the *unit* Euler axis, and Euler angle of the rotation respectively. It is worth emphasizing that $q^*_{\mathrm{Y/X}} = q_{\mathrm{X/Y}}$, and that $q_{\mathrm{X/Y}}$ and $-q_{\mathrm{X/Y}}$ represent the same rotation. Furthermore, given quaternions $q_{\mathrm{Y/X}}$ and $q_{\mathrm{Z/Y}}$, the quaternion describing the rotation from X to Z is given by $q_{\mathrm{Z/X}} = q_{\mathrm{Y/X}} q_{\mathrm{Z/Y}}$.

Three-dimensional vectors can be interpreted as quaternions. That is, given $\bar{s}^{\mathrm{X}} \in \mathbb{R}^3$, the coordinates of a vector expressed in frame X, its quaternion representation is given by $s^{\mathrm{X}} = (0, \bar{s}^{\mathrm{X}}) \in \mathbb{H}^v$, where $\mathbb{H}^v$ is the set of *vector* quaternions defined as $\mathbb{H}^v \triangleq \{(q_0, \bar{q}) \in \mathbb{H} :\ q_0 = 0\}$ (see Ref. 29 for further information). The change of reference frame on a vector quaternion is achieved by the adjoint operation, and is given by

$s^{\mathrm{Y}} = q^*_{\mathrm{Y/X}} s^{\mathrm{X}} q_{\mathrm{Y/X}}$. Additionally, given $s \in \mathbb{H}^v$, we can define the operation $[\cdot]^\times : \mathbb{H}^v \to \mathbb{R}^{4\times 4}$ as

$$[s]^\times = \begin{bmatrix} 0 & 0_{1\times 3} \\ 0_{3\times 1} & [\overline{s}]^\times \end{bmatrix}, \quad \text{where } [\overline{s}]^\times = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}. \tag{1}$$

In general, the attitude kinematics evolve as

$$\dot{q}_{\mathrm{X/Y}} = \tfrac{1}{2} q_{\mathrm{X/Y}} \omega^{\mathrm{X}}_{\mathrm{X/Y}} = \tfrac{1}{2} \omega^{\mathrm{Y}}_{\mathrm{X/Y}} q_{\mathrm{X/Y}}, \tag{2}$$

where $\omega^{\mathrm{Z}}_{\mathrm{X/Y}} \triangleq (0, \overline{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}}) \in \mathbb{H}^v$ and $\overline{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}} \in \mathbb{R}^3$ is the angular velocity of frame X with respect to frame Y expressed in Z-frame coordinates.

Let I be the inertial frame of reference, B a frame fixed on the rigid body, and D the desired reference frame to track. The kinematic equation of motion for the B and D frames, relative to the inertial frame is given, respectively, by

$$\dot{q}_{\mathrm{B/I}} = \tfrac{1}{2} q_{\mathrm{B/I}} \omega^{\mathrm{B}}_{\mathrm{B/I}}, \quad \text{and} \quad \dot{q}_{\mathrm{D/I}} = \tfrac{1}{2} q_{\mathrm{D/I}} \omega^{\mathrm{D}}_{\mathrm{D/I}}. \tag{3}$$

The kinematic equation for the rotational error between two non-inertial frames, whose relative orientation is described by $q_{\mathrm{B/D}}$, can be easily derived to be

$$\dot{q}_{\mathrm{B/D}} = \tfrac{1}{2} q_{\mathrm{B/D}} \omega^{\mathrm{B}}_{\mathrm{B/D}}, \tag{4}$$

where $\omega^{\mathrm{B}}_{\mathrm{B/D}} = \omega^{\mathrm{B}}_{\mathrm{B/I}} - \omega^{\mathrm{B}}_{\mathrm{D/I}} = \omega^{\mathrm{B}}_{\mathrm{B/I}} - q^*_{\mathrm{B/D}} \omega^{\mathrm{D}}_{\mathrm{D/I}} q_{\mathrm{B/D}}$.

II.B. Dual Quaternions

Dual quaternions are an extension of quaternions that arise in the study of dual numbers. A dual number can be described by $\boldsymbol{x} = x_r + \epsilon x_d$ for $x_r, x_d \in \mathbb{R}$, where ϵ is such that $\epsilon \neq 0, \epsilon^2 = 0$. A dual quaternion is a quaternion whose entries are dual numbers. This is analogous to defining the space of dual quaternions as $\mathbb{H}_d = \{\boldsymbol{q} = q_r + \epsilon q_d : q_r, q_d \in \mathbb{H}\}$. The nilpotent term ϵ commutes with the quaternion basis elements i, j, k, allowing us to define the basic properties listed in Table 2. Reference 31 also conveniently defines a multiplication between matrices and dual quaternions that resembles the well-known matrix-vector multiplication by simply representing the dual quaternion coefficients as a vector in $\mathbb{R}^8$. A property that arises from the

Table 2. Dual Quaternion Operations

Operation	Definition
Addition	$\boldsymbol{a} + \boldsymbol{b} = (a_r + b_r) + \epsilon(a_d + b_d)$
Multiplication by a scalar	$\lambda \boldsymbol{a} = (\lambda a_r) + \epsilon(\lambda a_d)$
Multiplication	$\boldsymbol{ab} = (a_r b_r) + \epsilon(a_d b_r + a_r b_d)$
Conjugate	$\boldsymbol{a}^* = (a_r^*) + \epsilon(a_d^*)$
Dot product	$\boldsymbol{a} \cdot \boldsymbol{b} = (a_r \cdot b_r) + \epsilon(a_d \cdot b_r + a_r \cdot b_d)$
Cross product	$\boldsymbol{a} \times \boldsymbol{b} = (a_r \times b_r) + \epsilon(a_d \times b_r + a_r \times b_d)$
Circle product	$\boldsymbol{a} \circ \boldsymbol{b} = (a_r \cdot b_r + a_d \cdot b_d) + \epsilon 0$
Swap	$\boldsymbol{a}^{\mathsf{s}} = a_d + \epsilon a_r$
Norm	$\|a\| = \sqrt{a \circ a}$
Vector part	$\mathrm{vec}(\boldsymbol{a}) = (0, \overline{a_r}) + \epsilon(0, \overline{a_d})$

definition of the circle product for dual quaternions is given by

$$\boldsymbol{a}^{\mathsf{s}} \circ \boldsymbol{b}^{\mathsf{s}} = \boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{b} \circ \boldsymbol{a}. \tag{5}$$

Analogous to the set of vector quaternions $\mathbb{H}^v$, we can define the set of vector dual quaternions as $\mathbb{H}_d^v \triangleq \{\boldsymbol{q} = q_r + \epsilon q_d : \; q_r, q_d \in \mathbb{H}^v\}$. Vector dual quaternions have special properties of interest in the study of kinematics, dynamics and control of rigid bodies. The two main properties are listed below, where $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{H}_d^v$:

$$\boldsymbol{a} \circ (\boldsymbol{b}\boldsymbol{c}) = \boldsymbol{b}^{\mathsf{s}} \circ (\boldsymbol{a}^{\mathsf{s}} \boldsymbol{c}^*) = \boldsymbol{c}^{\mathsf{s}} \circ (\boldsymbol{b}^* \boldsymbol{a}^{\mathsf{s}}), \tag{6}$$

$$\boldsymbol{a} \circ (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}^{\mathsf{s}} \circ (\boldsymbol{c} \times \boldsymbol{a}^{\mathsf{s}}) = \boldsymbol{c}^{\mathsf{s}} \circ (\boldsymbol{a}^{\mathsf{s}} \times \boldsymbol{b}). \tag{7}$$

Finally, for vector dual quaternions we will define the skew-symmetric operator $[\cdot]^\times : \mathbb{H}_d^v \to \mathbb{R}^{8\times 8}$,

$$[\boldsymbol{s}]^\times = \begin{bmatrix} [s_r]^\times & 0_{4\times 4} \\ [s_d]^\times & [s_r]^\times \end{bmatrix}. \tag{8}$$

Since rigid body motion has six degrees of freedom, a dual quaternion needs two constraints to parameterize it. The dual quaternion describing the relative pose of frame B relative to I is given by $\boldsymbol{q}_{\mathrm{B/I}} = q_{\mathrm{B/I},r} + \epsilon q_{\mathrm{B/I},d} = q_{\mathrm{B/I}} + \epsilon \frac{1}{2} q_{\mathrm{B/I}} r^{\mathrm{B}}_{\mathrm{B/I}}$, where $r^{\mathrm{B}}_{\mathrm{B/I}}$ is the position quaternion describing the location of the origin of frame B relative to that of frame I, expressed in B-frame coordinates. It can be easily observed that $q_{\mathrm{B/I},r} \cdot q_{\mathrm{B/I},r} = \mathsf{1}$ and $q_{\mathrm{B/I},r} \cdot q_{\mathrm{B/I},d} = \mathsf{0}$, where $\mathsf{0} = (0, \bar{0})$, providing the two necessary constraints. Thus, we say that a dual quaternion representing a pose transformation is a *unit* dual quaternion, since it satisfies $\boldsymbol{q} \cdot \boldsymbol{q} = \boldsymbol{q}^* \boldsymbol{q} = \mathbf{1}$, where $\mathbf{1} = \mathsf{1} + \epsilon \mathsf{0}$. For completeness purposes, let us also define $\mathbf{0} = \mathsf{0} + \epsilon \mathsf{0}$.

Furthermore, similar to quaternion relationships, the frame transformations laid out in Table 3 can be easily verified.

Table 3. Unit Dual Quaternion Operations

Composition of rotations	$\boldsymbol{q}_{\mathrm{Z/X}} = \boldsymbol{q}_{\mathrm{Y/X}} \boldsymbol{q}_{\mathrm{Z/Y}}$
Inverse, Conjugate	$\boldsymbol{q}^*_{\mathrm{Y/X}} = \boldsymbol{q}_{\mathrm{X/Y}}$

A useful equation is the generalization of velocity in dual form, which will contain a linear and an angular velocity term. The dual velocity is defined as

$$\boldsymbol{\omega}^{\mathrm{X}}_{\mathrm{Y/Z}} = \boldsymbol{q}^*_{\mathrm{X/Y}} \boldsymbol{\omega}^{\mathrm{Y}}_{\mathrm{Y/Z}} \boldsymbol{q}_{\mathrm{X/Y}} = \omega^{\mathrm{X}}_{\mathrm{Y/Z}} + \epsilon (v^{\mathrm{X}}_{\mathrm{Y/Z}} + \omega^{\mathrm{X}}_{\mathrm{Y/Z}} \times r^{\mathrm{X}}_{\mathrm{X/Y}}). \tag{9}$$

Thus, the dual velocity of a rigid body, assigned to frame B, with respect to the inertial frame is defined as $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}} = \omega^{\mathrm{B}}_{\mathrm{B/I}} + \epsilon v^{\mathrm{B}}_{\mathrm{B/I}}$.

In general, the dual quaternion kinematics can be expressed as

$$\dot{\boldsymbol{q}}_{\mathrm{X/Y}} = \tfrac{1}{2} \boldsymbol{q}_{\mathrm{X/Y}} \boldsymbol{\omega}^{\mathrm{X}}_{\mathrm{X/Y}} = \tfrac{1}{2} \boldsymbol{\omega}^{\mathrm{Y}}_{\mathrm{X/Y}} \boldsymbol{q}_{\mathrm{X/Y}}. \tag{10}$$

One of the key advantages of dual quaternions is the resemblance, in form, of the *pose* error kinematic equations of motion to the attitude-only error kinematics. The pose error kinematic equations of motion are given by

$$\dot{\boldsymbol{q}}_{\mathrm{B/D}} = \tfrac{1}{2} \boldsymbol{q}_{\mathrm{B/D}} \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \tag{11}$$

where $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}} - \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} = \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}} - \boldsymbol{q}^*_{\mathrm{B/D}} \boldsymbol{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \boldsymbol{q}_{\mathrm{B/D}}$.

In Ref. 31, the authors also represented the *pose* error dynamics in a manner that closely resembles the attitude(-only) error dynamic equations of motion through the introduction of the swap operator and the redefinition of the matrix-dual quaternion multiplication. These are given by

$$\begin{aligned}(\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} = (M^{\mathrm{B}})^{-1} \star \Big(\boldsymbol{f}^{\mathrm{B}} - (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}}) \times \big(M^{\mathrm{B}} \star ((\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} + (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}) \big) \\ - M^{\mathrm{B}} \star (\boldsymbol{q}^*_{\mathrm{B/D}} \dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}} \boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}} - M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} \Big),\end{aligned} \tag{12}$$

where $\boldsymbol{f}^{\mathrm{B}} = f^{\mathrm{B}} + \epsilon \tau^{\mathrm{B}}$ is the total external *dual force* expressed in the body frame, and $M^{\mathrm{B}} \in \mathbb{R}^{8\times 8}$ is the *dual inertia matrix* defined as

$$M^{\mathrm{B}} = \begin{bmatrix} 1 & 0_{1\times 3} & 0 & 0_{1\times 3} \\ 0_{3\times 1} & m I_{3\times 3} & 0_{3\times 1} & 0_{3\times 3} \\ 0 & 0_{1\times 3} & 1 & 0_{1\times 3} \\ 0_{3\times 1} & 0_{3\times 3} & 0_{3\times 1} & \bar{I}^{\mathrm{B}} \end{bmatrix}, \tag{13}$$

where $\bar{I}^{\mathrm{B}} \in \mathbb{R}^{3\times 3}$ is the mass moment of inertia of the body about the center of mass, and m is the mass of the body.

III. Concurrent Learning

Concurrent learning is a recently proposed approach that makes use of the current measured state of the system along with recorded data to modify the adaptation of the parameters. Section 3 of Ref. 26 lays out the fundamental results for the theory. An overview of how the concept feeds into Lyapunov stability theory is provided here for the reader's convenience, in the context of the estimation of the mass properties for a spacecraft.

The first step is to recast the dynamics from Eq.(12) in a way amenable to the concurrent learning framework. Specifically, we want the parameters to appear linearly with respect to the regressors, as is the case in most adaptive control approaches. In our case, we will define $\mathrm{v}(M^{\mathrm{B}}) = [I_{11}\ I_{12}\ I_{13}\ I_{22}\ I_{23}\ I_{33}\ m]^{\mathsf{T}}$, a vectorized version of the dual inertia matrix M^{B}, and the error in the estimation of the dual inertia matrix as

$$\Delta M^{\mathrm{B}} = \widehat{M^{\mathrm{B}}} - M^{\mathrm{B}}, \tag{14}$$

as they were originally defined in Ref. 22. This allows us to define the auxiliary function $r : \mathbb{H}_d^v \to \mathbb{R}^{8\times 7}$ that satisfies

$$M^{\mathrm{B}} \star \boldsymbol{a} = r(\boldsymbol{a})\mathrm{v}(M^{\mathrm{B}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_6 & a_7 & a_8 & 0 & 0 & 0 & 0 \\ 0 & a_6 & 0 & a_7 & a_8 & 0 & 0 \\ 0 & 0 & a_6 & 0 & a_7 & a_8 & 0 \end{bmatrix} \mathrm{v}(M^{\mathrm{B}}).$$

Using this expression to manipulate Eq.(12) yields the following affine representation with respect to $\mathrm{v}(M^{\mathrm{B}})$

$$\begin{aligned} \boldsymbol{f}^{\mathrm{B}} =& r((\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})\mathrm{v}(M^{\mathrm{B}}) + [(\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}})]^{\times} r((\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} + (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}})\mathrm{v}(M^{\mathrm{B}}) \\ &+ r((\boldsymbol{q}^*_{\mathrm{B/D}}\dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}})\mathrm{v}(M^{\mathrm{B}}) + r((\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})\mathrm{v}(M^{\mathrm{B}}) \\ =& \Big[r((\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{q}^*_{\mathrm{B/D}}\dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\boldsymbol{q}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + [(\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}})]^{\times} r((\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}})^{\mathsf{s}}) \Big] \mathrm{v}(M^{\mathrm{B}}) \\ \triangleq& \underbrace{R_k(\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}, \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \boldsymbol{\omega}^{\mathrm{D}}_{\mathrm{D/I}}, \boldsymbol{q}_{\mathrm{B/D}})}_{\text{regressor matrix}} \mathrm{v}(M^{\mathrm{B}}) = R_k \mathrm{v}(M^{\mathrm{B}}), \end{aligned} \tag{15}$$

where $R_k : \mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^v \times \mathbb{H}_d^u \to \mathbb{R}^{8\times 7}$, and the sub-index denotes that its arguments are taken at time t_k. Defining ε_k as

$$\varepsilon_k \triangleq R_k \mathrm{v}(\widehat{M^{\mathrm{B}}}) - \boldsymbol{f}^{\mathrm{B}}(t_k), \tag{16}$$

and using Eq. (15), the above equation can be re-expressed as

$$\begin{aligned} \varepsilon_k &\triangleq R_k \mathrm{v}(\widehat{M^{\mathrm{B}}}) - \boldsymbol{f}^{\mathrm{B}}(t_k) \\ &= R_k \mathrm{v}(\widehat{M^{\mathrm{B}}}) - R_k \mathrm{v}(M^{\mathrm{B}}) \\ &= R_k \mathrm{v}(\Delta M^{\mathrm{B}}), \end{aligned} \tag{17}$$

effectively making ε_k a signal that quantifies the error in the dual inertia matrix, which is in fact a key step in concurrent learning. It is worth emphasizing that in generating the variable ε_k there is no need for the true inertia matrix parameters; only the regressor matrix R_k, the estimated dual inertia, and the applied dual force are needed as in Eq. (16).

Let now $\mathcal{X} = \{(\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}, \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \boldsymbol{\omega}^{\mathrm{D}}_{\mathrm{D/I}}, \boldsymbol{q}_{\mathrm{B/D}})_j\}_{j=1}^{N_s}$ and $\mathcal{F} = \{(\boldsymbol{f}^{\mathrm{B}})_j\}_{j=1}^{N_s}$ be sets of recorded pairs of data as per Eq. (15) at times $\{t_j\}_{j=1}^{N_s}$. For our application, the cardinality of the sets $\mathcal{X}$ and $\mathcal{F}$ is $7 \leq N_s < \infty$, and is set by the user. It is worth emphasizing that these sets will be initially empty, and that data will be incorporated as the become available. Details on how this is done will be discussed in Section IV.

IV. Identification of Mass and Inertia Properties

The main result of this paper is an adaptive pose-tracking controller that uses concurrent learning to provide strong assurances on the convergence of the mass and the inertia matrix of the spacecraft. The result is an extension of the controller first described in Ref. 22, with the corrections incorporated in Ref. 29. The proof closely mimics the proof provided therein, incorporating the new concurrent learning term which leads to improved performance in the calculation of the mass properties. The next theorem presents this controller and shows that it ensures *almost* global asymptotic stability of the linear and angular motion relative to the desired reference, which is the strongest kind of stability that can be proven for this problem for the given parametrization.

Theorem 1. *Consider the relative kinematic and dynamic equations given by Eq.* (11) *and Eq.* (12)*. Let the dual control force be defined by the feedback control law*

$$\boldsymbol{f}_c^{\mathrm{B}} = -\mathrm{vec}\big(\boldsymbol{q}_{\mathrm{B/D}}^*(\boldsymbol{q}_{\mathrm{B/D}}^{\mathsf{s}} - \mathbf{1}^{\mathsf{s}})\big) - K_d \star \boldsymbol{s}^{\mathsf{s}} + \boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}} \times (\widehat{M^{\mathrm{B}}} \star (\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}})^{\mathsf{s}}) + \widehat{M^{\mathrm{B}}} \star (\boldsymbol{q}_{\mathrm{B/D}}^* \dot{\boldsymbol{\omega}}_{\mathrm{D/I}}^{\mathrm{D}} \boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}} + \widehat{M^{\mathrm{B}}} \star (\boldsymbol{\omega}_{\mathrm{D/I}}^{\mathrm{B}} \times \boldsymbol{\omega}_{\mathrm{B/D}}^{\mathrm{B}})^{\mathsf{s}}$$
$$- \widehat{M^{\mathrm{B}}} \star (K_p \star \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}_{\mathrm{B/D}}^*(\boldsymbol{q}_{\mathrm{B/D}}^{\mathsf{s}} - \mathbf{1}^{\mathsf{s}})))^{\mathsf{s}}, \tag{18}$$

where

$$\boldsymbol{s} = \boldsymbol{\omega}_{\mathrm{B/D}}^{\mathrm{B}} + (K_p \star (\boldsymbol{q}_{\mathrm{B/D}}^*(\boldsymbol{q}_{\mathrm{B/D}}^{\mathsf{s}} - \mathbf{1}^{\mathsf{s}})))^{\mathsf{s}}, \tag{19}$$

$$K_p = \begin{bmatrix} K_r & 0_{4\times4} \\ 0_{4\times4} & K_q \end{bmatrix},\ K_r = \begin{bmatrix} 0 & 0_{1\times3} \\ 0_{3\times1} & \bar{K}_r \end{bmatrix},\ K_q = \begin{bmatrix} 0 & 0_{1\times3} \\ 0_{3\times1} & \bar{K}_q \end{bmatrix}, \tag{20}$$

$$K_d = \begin{bmatrix} K_v & 0_{4\times4} \\ 0_{4\times4} & K_\omega \end{bmatrix},\ K_v = \begin{bmatrix} 0 & 0_{1\times3} \\ 0_{3\times1} & \bar{K}_v \end{bmatrix},\ K_\omega = \begin{bmatrix} 0 & 0_{1\times3} \\ 0_{3\times1} & \bar{K}_\omega \end{bmatrix}, \tag{21}$$

and $\bar{K}_r, \bar{K}_q, \bar{K}_v, \bar{K}_\omega \in \mathbb{R}^{3\times3}$ *are positive definite matrices,* $\widehat{M^{\mathrm{B}}}$ *is an estimate of* M^{B} *updated according to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\widehat{M^{\mathrm{B}}}) = -\alpha K_i \sum_{k=1}^{N_s} R_k^{\mathsf{T}} \varepsilon_k + K_i \big[-\mathrm{h}((\boldsymbol{s} \times \boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}})^{\mathsf{s}}, (\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}})^{\mathsf{s}})$$
$$-\mathrm{h}(\boldsymbol{s}^{\mathsf{s}}, (\boldsymbol{q}_{\mathrm{B/D}}^* \dot{\boldsymbol{\omega}}_{\mathrm{D/I}}^{\mathrm{D}} \boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}} + (\boldsymbol{\omega}_{\mathrm{D/I}}^{\mathrm{B}} \times \boldsymbol{\omega}_{\mathrm{B/D}}^{\mathrm{B}})^{\mathsf{s}} - K_p \star \frac{\mathrm{d}(\boldsymbol{q}_{\mathrm{B/D}}^*(\boldsymbol{q}_{\mathrm{B/D}}^{\mathsf{s}} - \mathbf{1}^{\mathsf{s}}))}{\mathrm{d}t})\big], \tag{22}$$

$K_i \in \mathbb{R}^{7\times7}$ *is a positive definite matrix, the function* $\mathrm{h} : \mathbb{H}_d^v \times \mathbb{H}_d^v \to \mathbb{R}^7$ *is defined as* $\boldsymbol{a} \circ (M^B \star \boldsymbol{b}) = \mathrm{h}(\boldsymbol{a}, \boldsymbol{b})^{\mathsf{T}} \mathrm{v}(M^B) = \mathrm{v}(M^B)^{\mathsf{T}} \mathrm{h}(\boldsymbol{a}, \boldsymbol{b})$ *or, equivalently,* $\mathrm{h}(\boldsymbol{a}, \boldsymbol{b}) = [a_6 b_6,\ a_7 b_6 + a_6 b_7,\ a_8 b_6 + a_6 b_8,\ a_7 b_7,\ a_8 b_7 + a_7 b_8,\ a_8 b_8,\ a_2 b_2 + a_3 b_3 + a_4 b_4]^{\mathsf{T}}$*, and* ε_k *is given by* (16)*, constructed from the data in the sets* $\mathcal{X}$ *and* $\mathcal{F}$*. Assume that* $\boldsymbol{q}_{\mathrm{D/I}}, \boldsymbol{\omega}_{\mathrm{D/I}}^{\mathrm{D}}, \dot{\boldsymbol{\omega}}_{\mathrm{D/I}}^{\mathrm{D}} \in \mathcal{L}_\infty$ *and*

$$\mathrm{rank} \sum_{k=1}^{N_s} R_k^{\mathsf{T}} R_k = 7. \tag{23}$$

Then, for all initial conditions, $\lim_{t\to\infty} \boldsymbol{q}_{\mathrm{B/D}} = \pm\mathbf{1}$ *(i.e.,* $\lim_{t\to\infty} q_{\mathrm{B/D}} = \pm 1$ *and* $\lim_{t\to\infty} r_{\mathrm{B/D}}^{\mathrm{B}} = 0$*),* $\lim_{t\to\infty} \boldsymbol{\omega}_{\mathrm{B/D}}^{\mathrm{B}} = \mathbf{0}$ *(i.e.,* $\lim_{t\to\infty} \omega_{\mathrm{B/D}}^{\mathrm{B}} = 0$ *and* $\lim_{t\to\infty} v_{\mathrm{B/D}}^{\mathrm{B}} = 0$*), and* $\mathrm{v}(\widehat{M^{\mathrm{B}}}) \to \mathrm{v}(M^{\mathrm{B}})$.

Proof. Note that $\boldsymbol{q}_{\mathrm{B/D}} = \pm\mathbf{1}$, $\boldsymbol{s} = \mathbf{0}$, and $\mathrm{v}(\Delta M^{\mathrm{B}}) = 0_{7\times1}$ are the equilibrium conditions of the closed-loop system formed by Eqs. (12), (11), (18), (22), and (16). Consider now the following candidate Lyapunov function for the equilibrium point $(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = (+\mathbf{1}, \mathbf{0}, 0_{7\times1})$:

$$V(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = (\boldsymbol{q}_{\mathrm{B/D}} - \mathbf{1}) \circ (\boldsymbol{q}_{\mathrm{B/D}} - \mathbf{1}) + \tfrac{1}{2} \boldsymbol{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star \boldsymbol{s}^{\mathsf{s}}) + \tfrac{1}{2} \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \mathrm{v}(\Delta M^{\mathrm{B}}). \tag{24}$$

Note that V is a valid candidate Lyapunov function since $V(\boldsymbol{q}_{\mathrm{B/D}} = \mathbf{1}, \boldsymbol{s} = \mathbf{0}, \mathrm{v}(\Delta M^{\mathrm{B}}) = 0_{7\times1}) = 0$ and $V(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) > 0$ for all $(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) \in \mathbb{H}_d^u \times \mathbb{H}_d^v \times \mathbb{R}^7 \backslash \{\mathbf{1}, \mathbf{0}, 0_{7\times1}\}$. The time derivative of V is equal to

$$\dot{V} = 2(\boldsymbol{q}_{\mathrm{B/D}} - \mathbf{1}) \circ \dot{\boldsymbol{q}}_{\mathrm{B/D}} + \boldsymbol{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star \dot{\boldsymbol{s}}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}\mathrm{v}(\Delta M^{\mathrm{B}})}{\mathrm{d}t}.$$

From Eq. (11) and Eq. (19) we can write $\dot{\boldsymbol{q}}_{\mathrm{B/D}} = \frac{1}{2}\boldsymbol{q}_{\mathrm{B/D}}\boldsymbol{s} - \frac{1}{2}\boldsymbol{q}_{\mathrm{B/D}}(K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})))^{\mathsf{s}}$, which can then be plugged into $\dot{V}$, together with the time derivative of Eq. (19), to yield

$$\dot{V} = (\boldsymbol{q}_{\mathrm{B/D}} - \mathbf{1}) \circ (\boldsymbol{q}_{\mathrm{B/D}}\boldsymbol{s} - \boldsymbol{q}_{\mathrm{B/D}}(K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})))^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}})$$
$$+ \boldsymbol{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + \boldsymbol{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))}{\mathrm{d}t})).$$

Applying Eq. (6) to the first term, evaluating the dynamics from Eq. (12), and using the identity $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} = \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}$ yields

$$\dot{V} = -(K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))) \circ (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})) + \boldsymbol{s}^{\mathsf{s}} \circ (\boldsymbol{f}^{\mathrm{B}} - \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times \big(M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\big) - M^{\mathrm{B}} \star (\boldsymbol{q}^*_{\mathrm{B/D}}\dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}}$$
$$-M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}) + \boldsymbol{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))}{\mathrm{d}t})) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}) + \boldsymbol{s}^{\mathsf{s}} \circ (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})).$$

Introducing the feedback control law given by Eq. (18) and using Eqs. (7) and (5) yields

$$\dot{V} = -(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})) \circ (K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))) + \boldsymbol{s}^{\mathsf{s}} \circ (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times \big(\Delta M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\big) + \Delta M^{\mathrm{B}} \star (\boldsymbol{q}^*_{\mathrm{B/D}}\dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}}$$
$$+\Delta M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} - \Delta M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))}{\mathrm{d}t})) - \boldsymbol{s}^{\mathsf{s}} \circ (K_d \star \boldsymbol{s}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}})$$

or

$$\dot{V} = -(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})) \circ (K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))) + (\boldsymbol{s} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}} \circ (\Delta M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}) + \boldsymbol{s}^{\mathsf{s}} \circ (\Delta M^{\mathrm{B}} \star (\boldsymbol{q}^*_{\mathrm{B/D}}\dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\boldsymbol{q}_{\mathrm{B/D}})^{\mathsf{s}}$$
$$+\Delta M^{\mathrm{B}} \star (\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} - \Delta M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))}{\mathrm{d}t})) - \boldsymbol{s}^{\mathsf{s}} \circ (K_d \star \boldsymbol{s}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}).$$

Therefore, if $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}})$ is defined as in Eq. (22), it follows that

$$\dot{V} = -(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})) \circ (K_p \star (\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}}))) - \boldsymbol{s}^{\mathsf{s}} \circ (K_d \star \boldsymbol{s}^{\mathsf{s}}) - \alpha \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} \sum_{k=1}^{N_s} R_k^{\mathrm{T}} R_k \mathrm{v}(\Delta M^{\mathrm{B}}) \leq 0$$

for all $(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) \in \mathbb{H}^u_d \times \mathbb{H}^v_d \times \mathbb{R}^7 \backslash \{\mathbf{1}, \mathbf{0}, 0_{7\times1}\}$. Hence, the equilibrium point $(\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = (+\mathbf{1}, \mathbf{0}, 0_{7\times1})$ is uniformly stable and the solutions are uniformly bounded, i.e., $\boldsymbol{q}_{\mathrm{B/D}}$, $\boldsymbol{s}$, $\mathrm{v}(\Delta M^{\mathrm{B}})$, $\in \mathcal{L}_\infty$. Moreover, from Eqs. (14) and (19) this also means that $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \mathrm{v}(\widehat{M^{\mathrm{B}}}) \in \mathcal{L}_\infty$. Since $V \geq 0$ and $\dot{V} \leq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. Hence, $\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} V(t) - V(0)$ also exists and is finite. Since $\boldsymbol{q}_{\mathrm{B/D}}, \boldsymbol{s}, \mathrm{v}(\Delta M^{\mathrm{B}}), \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \mathrm{v}(\widehat{M^{\mathrm{B}}}), \dot{\boldsymbol{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{D/I}}, \boldsymbol{q}_{\mathrm{D/I}} \in \mathcal{L}_\infty$, then from Eqs. (11), (18), and (12) and from Lemma 53 in Ref. 29, $r^{\mathrm{B}}_{\mathrm{B/I}}, \dot{\boldsymbol{q}}_{\mathrm{B/D}}, \boldsymbol{f}^{\mathrm{B}}, \dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}, \dot{\boldsymbol{s}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\mathrm{vec}\big(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})\big) \to \mathbf{0}$, $\boldsymbol{s} \to \mathbf{0}$ and $\mathrm{v}(\Delta M^{\mathrm{B}}) \to 0_{7\times1}$ as $t \to \infty$. In Ref. 31, it is shown that $\mathrm{vec}\big(\boldsymbol{q}^*_{\mathrm{B/D}}(\boldsymbol{q}^{\mathsf{s}}_{\mathrm{B/D}} - \mathbf{1}^{\mathsf{s}})\big) \to \mathbf{0}$ is equivalent to $\boldsymbol{q}_{\mathrm{B/D}} \to \pm\mathbf{1}$. Furthermore, calculating the limit as $t \to \infty$ of both sides of Eq. (19) yields $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to \mathbf{0}$. Finally, it is clear to see that $\mathrm{v}(\Delta M^{\mathrm{B}}) \to 0_{7\times1}$ implies by definition $\mathrm{v}(\widehat{M^{\mathrm{B}}}) \to \mathrm{v}(M^{\mathrm{B}})$ as $t \to \infty$. ■

Remark 1. It is possible to prove that $\mathrm{v}(\Delta M^{\mathrm{B}}) \to 0_{7\times1}$ because the term $-\mathrm{v}(\Delta M^{\mathrm{B}}) \sum_{j=1}^{N_s} R_k^{\mathrm{T}} R_k \mathrm{v}(\Delta M^{\mathrm{B}})$ appears in the derivative of the Lyapunov function, which is the main contribution of the concurrent learning framework. Note that to do this, Eq. (17) was key, and that the matrix $\sum_{k=1}^{N_s} R_k^{\mathsf{T}} \varepsilon_k$ is constructed from collected data in the sets $\mathcal{X}$ and $\mathcal{F}$.

Remark 2. Chapter 6 in Ref. 26 addresses how the matrices $\mathcal{X}$ and $\mathcal{F}$ should be populated. Algorithm 6.2 therein, which aims to maximize the minimum singular value of $\sum_{j=1}^{N_s} R_k^{\mathrm{T}} R_k$, was selected for the implementation of the proposed controller. It is worth emphasizing that for the algorithm to work we only require that Eq. (23) is satisfied. The maximization of the minimum singular value just speeds up the convergence of the parameters.

Remark 3. As pointed out in Ref. 17, linear and angular velocity measurements, among others, are inevitably corrupted by noise in real systems. This limitation is not considered in this paper and will be the subject of future research. However, the concurrent learning framework has already been successfully tested experimentally in Ref. 32.

Remark 4. In practice the derivatives of certain states might not be readily accessible through the measurements. This is the case, for example, with $\dot{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}$. An optimal fixed-point smoother can be used to estimate these variables, if needed.[26,32]

V. Numerical Results

The controller proposed by Eq. (18) was simulated using MATLAB R2015b and Simulink and its performance was compared to that of the nominal controller proposed in Ref. 27. The initial state of the system is given by $q_{\mathrm{B/D}}(0) = (0.8721, -[0.1178, 0.4621, 0.1097]^{\mathrm{T}})$, $\bar{r}^{\mathrm{B}}_{\mathrm{B/D}}(0) = [1, 2, 0.5]^{\mathrm{T}}$ (m), $\bar{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(0) = [0.5, 1, 1]^{\mathrm{T}}$ (rad/s), $\bar{v}^{\mathrm{B}}_{\mathrm{B/D}}(0) = [0.5, -0.5, 1]^{\mathrm{T}}$ (m/s), $\mathrm{v}(M^{\mathrm{B}}) = [5, 2, 3, 5, 1, 4, 10]^{\mathrm{T}}$, $\mathrm{v}(\widehat{M^{\mathrm{B}}}) = 0_{7\times1}$, with units of kg.m^2 and kg for the inertia elements and the mass respectively. The matrix gains were set to $\bar{K}_r = 0.74/3I_3$, $\bar{K}_q = 0.2/3I_3$, $\bar{K}_v = 84.37I_3$, $\bar{K}_\omega = 15I_3$, and $K_i = 10I_7$. In addition, to avoid a persistently exciting reference, a constant reference is selected as $\bar{\omega}^{\mathrm{D}}_{\mathrm{D/I}}(t) = [1, 0, 0]^{\mathrm{T}}$ and $\bar{v}^{\mathrm{D}}_{\mathrm{D/I}}(t) = [1, 0, 0]^{\mathrm{T}}$.

The parameters that concern the concurrent learning controller are set to $N_s = 50$, $\alpha = 0.0005$, and the minimum singular value of the sum of regressor matrices required to stop the search of new data points is set to a value of 20.

Figure 1 shows the evolution of the estimated mass properties as a function of time during the maneuver. It is clear that the mass converges quickly using the concurrent learning framework proposed in this paper, while the baseline controller does not converge. Similarly, all of the inertia parameters converge for the proposed algorithm, but four of the estimates do not converge for the baseline controller from Ref. 27. This behavior can be attributed to the lack of excitation induced by the desired linear and angular velocity references. This is reinforced by the fact that the regressor matrix $W(t)$ of Eq. 36 in Ref. 27 yields

$$W(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

for our simulated scenario. This implies that the baseline controller's convergence criterion,

$$\mathrm{rank}\begin{bmatrix} W(t_1)^{\mathrm{T}} & \ldots & W(t_n)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} = 7,$$

will never be satisfied, while the convergence criterion given in Eq. (23) is satisfied even for such a non-exciting reference. In fact, the criterion is achieved at $t = 0.0177s$. The difference in the behavior can be explained by the fact that the concurrent learning controller uses additional information (such as force, torque and acceleration) than the nominal controller to achieve this parameter convergence. When such information is readily available, the concurrent learning framework is a reliable alternative for adaptive feedback control with parameter convergence (see also Remark 1 related to this point).

Figure 2 shows the tracking error of the body frame relative to the desired frame. It is clear that both controllers are able to successfully track a 6-DOF reference. Additionally, Figure 3 shows the control effort (i.e., forces and torques) applied on the spacecraft. They are both similar and within acceptable limits.

This example highlights the significant advantage that the proposed framework can yield compared to others in terms of system identification and reliability in terms of tracking the desired reference trajectory. Finally, it is worth noting that the parameter N_s plays a significant role in the speed of convergence of the inertia parameters. For example, if instead of $N_s = 50$ we choose $N_s = 10$ instead, convergence is achieved in about 500 sec using the concurrent learning algorithm.

VI. Conclusions

On-line system identification tasks can be necessary beyond the mere requirement of performing reference tracking in the case of, for example, capture of another satellite or celestial body, under the assumption that the location of the center of mass is known or unchanged. The use of the concurrent learning framework avoids discontinuities in the estimate of the inertia matrix, while enabling precise pose control of the spacecraft. Additionally, it avoids computationally intensive operations, such as matrix inversion or matrix

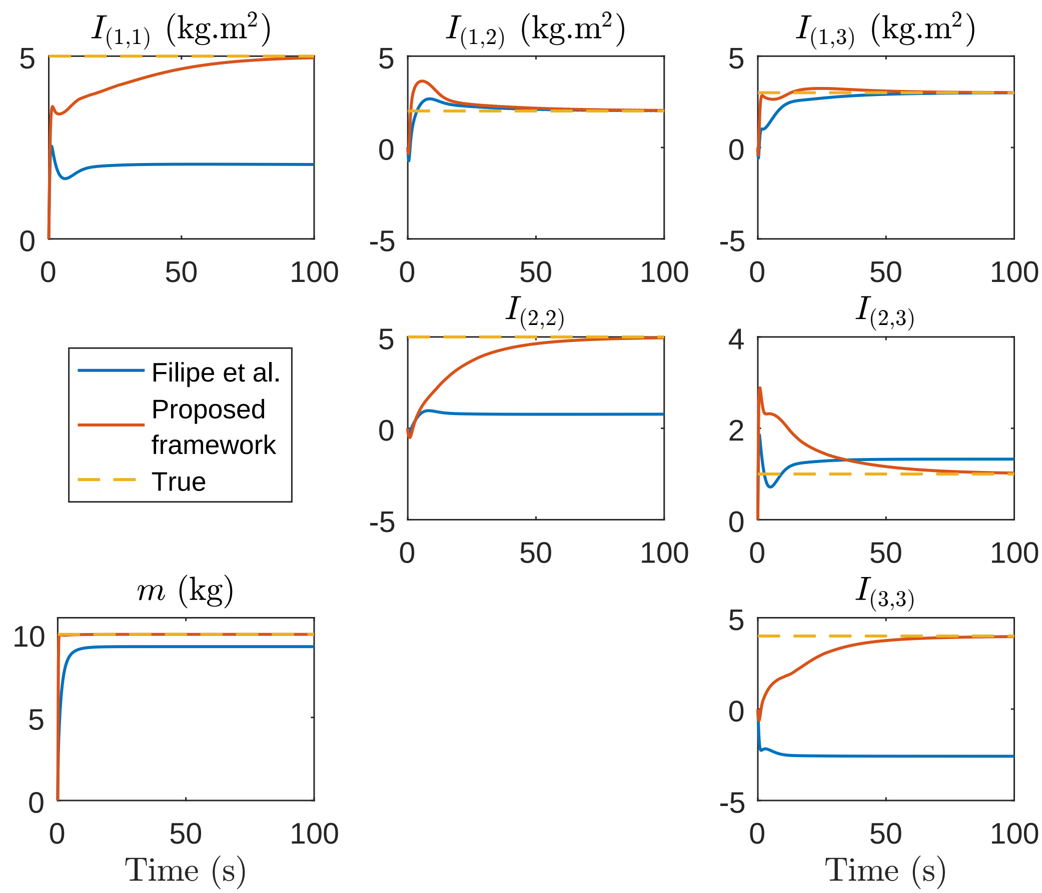

Figure 1. Evolution of estimated dual inertia matrix parameters.

decomposition, both tasks commonly found in optimization software or formulations deriving from least squares. In addition, the introduction of concurrent learning provides convergence assurances, given that an easily verifiable rank condition is satisfied. This is a simplification of the commonly stringent condition that the regressors are persistently exciting. Future work will include the study of the effects of noise in the measurements of the state and other signals necessary to implement concurrent learning, as well as system parameter identification for more complex systems.

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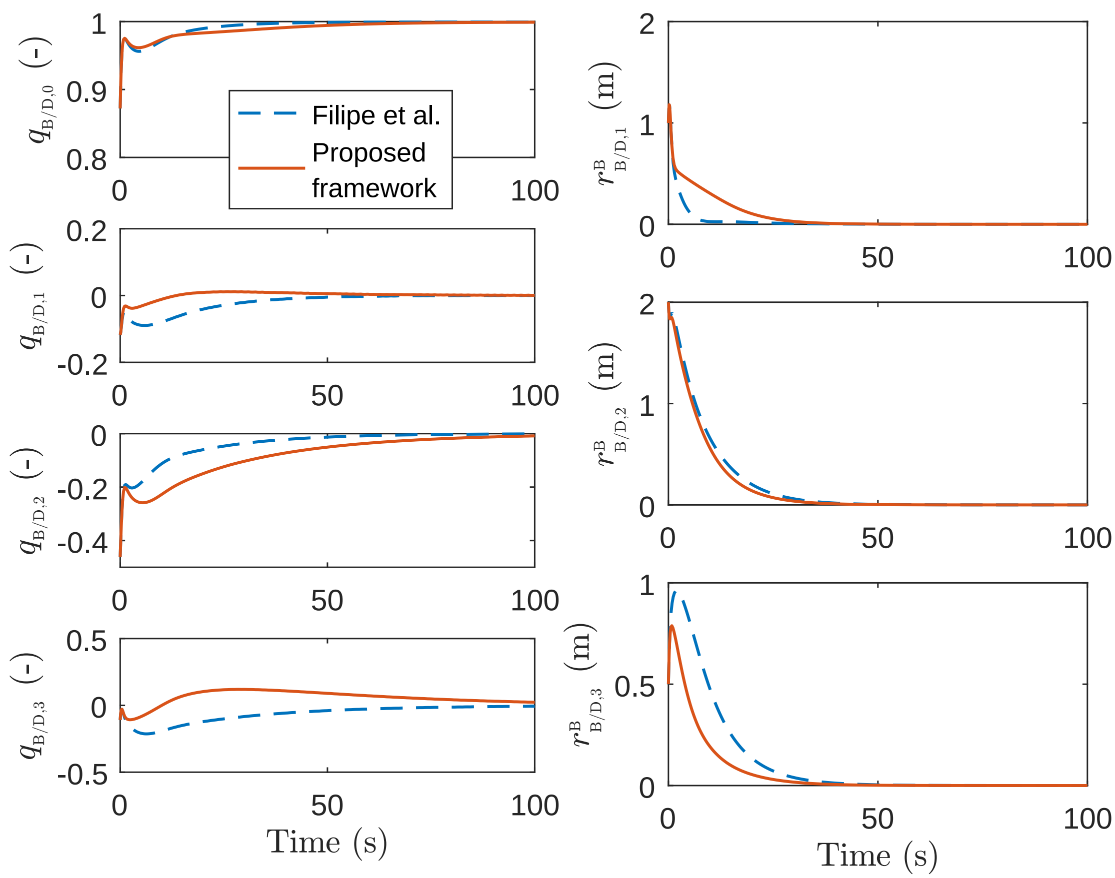

Figure 2. Attitude and position tracking error.

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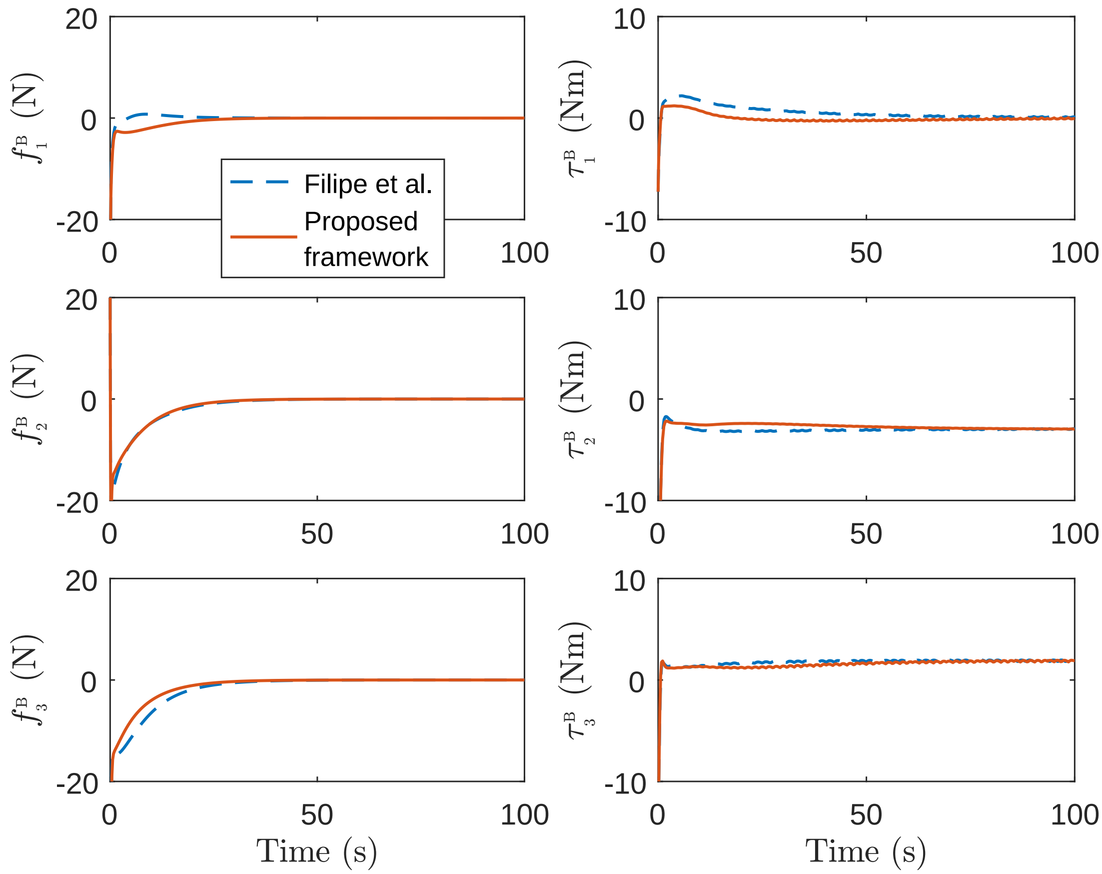

Figure 3. Control effort commanded by the controller.

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