Adaptive Model-Independent Tracking of Rigid Body Position and Attitude Motion with Mass and Inertia Matrix Identification using Dual Quaternions

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In this paper, we propose a nonlinear adaptive position and attitude tracking controller for a rigid body that requires no information about the mass and inertia matrix of the body. Moreover, we provide sufficient conditions on the reference trajectory that guarantee mass and inertia matrix identification. The controller is shown to be *almost* globally asymptotically stable and can handle large error angles and displacements. One of the novelties of this paper is the use of unit dual quaternions to represent the position and attitude of the rigid body. We show that dual quaternions can be used to extend existing attitude-only controllers based on quaternions into combined position and attitude controllers with similar properties.

I. Introduction

Unit quaternions, as a global nonsingular parameterization of the attitude with the least number of parameters, have been extensively used in the past to develop *almost* globally asymptotically stable attitude controllers for rigid bodies.[1,2] In particular, the literature about adaptive attitude-tracking controllers based on quaternions is extensive. In Ref. 1, an adaptive attitude-tracking controller is given that depends on a parameter that is required to be smaller than some function of the minimum and maximum eigenvalues of the inertia matrix, which is assumed to be unknown. Likewise, in Ref. 3, an adaptive attitude-tracking controller without angular velocity measurements is given that also requires knowledge of the minimum and maximum eigenvalues of the inertia matrix. In addition, it also requires the initial value of the scalar part of the quaternion to be zero. Hence, initialization may be a problem for some cases of practical interest. Finally, the control gains depend on an upper bound of the initial state. More recently, Ref. 4 proposes an adaptive attitude-tracking controller that eliminates the degradation of the closed-loop dynamics caused by the estimation of the inertia matrix and that stops the estimation process if the true inertia matrix is found. However, the controller is a 27th-order dynamic compensator, which may limit its applicability for cases of small satellites with limited on-board computational resources. None of the above controllers can guarantee that the inertia matrix estimate will converge to the true inertia matrix.

In Ref. 5, an adaptive attitude-tracking controller is given that does not require any information about the unknown inertia matrix, works for all initial conditions, and has only 6 states (as many as the unknown elements of the inertia matrix). Moreover, unlike many of the previous controllers, it can identify the true inertia matrix if proper reference trajectories are used. Although adaptive controllers may guarantee convergence even if the parameter estimates do not converge to their true values, there are applications where the true moment of inertia might be needed to achieve additional performance objectives. The proposed adaptive scheme achieves mass and moment of inertia parameter identification without any additional effort.

In this paper, we extend the results presented in Ref. 5 to include position-tracking and mass identification. We do that by representing the attitude and position of a rigid body with unit *dual* quaternions. Unit dual quaternions have successfully been applied in the past to several fields: inertial navigation,[6] rigid body control,[7–14] spacecraft formation flying,[15] inverse kinematic analysis,[16] and computer vision[17,18] and animation.[19] Several properties make unit dual quaternions appealing. First, they have been argued to be

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the most compact and efficient way to simultaneously express the translation and rotation of robotic kinematic chains.[20,21] Moreover, they allow attitude and position controllers to be written as a single control law. It has also been shown that they automatically take into account the natural coupling between the rotational and translational motions.[12,13] However, the most useful property of dual quaternions is that the combined translational and rotational kinematic and dynamic equations of motion written in terms of dual quaternions have the same form as the rotational-only kinematic and dynamic equations of motion written in terms of quaternions.[8] Hence, as it will be shown in this paper, there exists a natural analogy between rotational controllers based on unit quaternions and combined translational and rotational controllers based on unit dual quaternions. This can be extremely useful for designing control laws for combined translational and rotational motion, as it is the case with satellite proximity operations, such as rendezvous and docking in space.[7,8]

Unit dual quaternions have been used in the past to solve the attitude and position tracking problem. In Ref. 12, for instance, a position and attitude tracking law is suggested based on the feedback of the relative linear and angular velocities and of the logarithm of the dual error quaternion. However, the control law is not written in terms of required forces and torques, but in terms of a dual vector, from which the required forces and torques must be derived. Moreover, the control law is not adaptive in the sense that the mass and inertia matrix of the rigid body must be known. In Ref. 15, for instance, an adaptive terminal sliding mode tracking law for the relative position and attitude of a leader-follower spacecraft formation is given in terms of required forces and torques. However, the control gains depend on upper bounds on the mass and on the maximum eigenvalue of the inertia matrix. This adaptive controller is also not guaranteed to estimate the mass and inertia matrix of the rigid body. Furthermore, the set of initial conditions for which the controller achieves tracking is not specified. Finally, the tracking law in Ref. 15 is based on the special *dual inertia operator*, first introduced in Ref. 22.

In the current paper we propose an adaptive position and attitude tracking law, written in terms of the required forces and torques, which does not require any knowledge about the mass and inertia matrix of the rigid body. The proposed controller is guaranteed to identify the mass and inertia matrix, if suitable reference trajectories are used. We also show that our controller is *almost* globally asymptotically stable. Another appealing feature of our controller is that instead of using the special *dual inertia operator* used in Ref. 22, it uses the *dual inertia matrix*.[7] The dual inertia matrix is a standard 8-by-8 matrix and, thus, it is easier to work with than the dual inertia operator.

This paper is organized as follows. In Section II, unit quaternions and unit dual quaternions are introduced. The relative rigid body kinematic and dynamic equations in terms of dual quaternions are also given. In Section III, the adaptive attitude and position tracking controller is deduced and proved to be *almost* globally asymptotically stable. In Section IV, it is shown that the controller is able to identify the mass and inertia matrix of the rigid body, if proper periodic reference trajectories are used. Finally, in Section V, the proposed controller is analyzed and validated through a numerical example.

II. Mathematical Preliminaries

II.A. Quaternions

A quaternion is classically defined as $q = q_1 i + q_2 j + q_3 k + q_4$, where q_1, q_2, q_3, $q_4 \in \mathbb{R}$ and i, j, and k satisfy $i^2 = j^2 = k^2 = -1$, $i = jk = -kj$, $j = ki = -ik$, and $k = ij = -ji$.[12] They can be viewed as an extension of complex numbers to $\mathbb{R}^4$. A quaternion can also be represented as the ordered pair $q = (\bar{q}, q_4)$, where $\bar{q} = [q_1\ q_2\ q_3]^\mathsf{T} \in \mathbb{R}^3$ denotes the *vector part* of the quaternion and $q_4 \in \mathbb{R}$ denotes the *scalar part* of the quaternion. Quaternions with scalar part equal to zero will be referred to as *vector quaternions*, whereas quaternions with vector part equal to zero will be referred to as *scalar quaternions*. The symbols $\mathbb{H} = \{q : \ q = q_1 i + q_2 j + q_3 k + q_4,\ q_1, q_2, q_3, q_4 \in \mathbb{R}\}$, $\mathbb{H}^v = \{q \in \mathbb{H} : \ q_4 = 0\}$, and $\mathbb{H}^s = \{q \in \mathbb{H} : \ q_1 = q_2 = q_3 = 0\}$ will be used to represent the set of quaternions, vector quaternions, and scalar quaternions, respectively.

The basic operations on quaternions are defined as follows:

$$\text{Addition: } a+b=(\bar{a}+\bar{b}, a_4+b_4),$$
$$\text{Multiplication by a scalar: } \lambda a=(\lambda\bar{a}, \lambda a_4),$$
$$\text{Multiplication: } ab=(a_4\bar{b}+b_4\bar{a}+\bar{a}\times\bar{b}, a_4b_4-\bar{a}\cdot\bar{b}),$$
$$\text{Conjugation: } a^*=(-\bar{a}, a_4),$$
$$\text{Dot product: } a\cdot b=\tfrac{1}{2}(a^*b+b^*a)=\tfrac{1}{2}(ab^*+ba^*)=(\bar{0}, a_4b_4+\bar{a}\cdot\bar{b}),$$
$$\text{Cross product: } a\times b=\tfrac{1}{2}(ab-b^*a^*)=(b_4\bar{a}+a_4\bar{b}+\bar{a}\times\bar{b}, 0),$$
$$\text{Norm: } \|a\|^2=aa^*=a^*a=a\cdot a=(\bar{0}, a_4^2+\bar{a}\cdot\bar{a}),$$
$$\text{Scalar part: } \mathrm{sc}(a)=(\bar{0}, a_4)\in\mathbb{H}^s,$$
$$\text{Vector part: } \mathrm{vec}(a)=(\bar{a}, 0)\in\mathbb{H}^v,$$

where $a, b\in\mathbb{H}$, $\lambda\in\mathbb{R}$, and $\bar{0}=[0\ 0\ 0]^\intercal$. Note that in general $ab\neq ba$.

Note that as vector spaces, $\mathbb{H}$, $\mathbb{H}^v$, and $\mathbb{H}^s$ are isomorphic to $\mathbb{R}^4$, $\mathbb{R}^3$, and $\mathbb{R}$, respectively. Under the natural isomorphism between $\mathbb{H}$ and $\mathbb{R}^4$, the square of the quaternion norm and the dot product on $\mathbb{H}$ correspond to the square of the Euclidean norm and to the dot (inner) product on $\mathbb{R}^4$, respectively. Moreover, under the natural isomorphism between $\mathbb{H}^s$ and $\mathbb{R}$, with a slight abuse of notation, $(\bar{0}, q_4)$ will often be denoted as q_4 for simplicity.

The multiplication of a matrix $M\in\mathbb{R}^{4\times4}$ with a quaternion $q\in\mathbb{H}$ will be defined analogously to the multiplication of a 4-by-4 matrix with a 4-dimensional vector as $M*q=(M_{11}\bar{q}+M_{12}q_4, M_{21}\bar{q}+M_{22}q_4)\in\mathbb{H}$, where

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix},$$

$M_{11}\in\mathbb{R}^{3\times3}$, $M_{12}\in\mathbb{R}^{3\times1}$, $M_{21}\in\mathbb{R}^{1\times3}$, and $M_{22}\in\mathbb{R}$.

It can be shown that the following properties follow from the previous definitions:

$$a\cdot(bc)=b\cdot(ac^*)=c\cdot(b^*a)\in\mathbb{H}^s,\quad a,b,c\in\mathbb{H},$$
$$\|ab\|=\|a\|\|b\|,\quad a,b\in\mathbb{H},$$
$$(M*a)\cdot b=a\cdot(M^\intercal*b),\quad a,b\in\mathbb{H},\quad M\in\mathbb{R}^{4\times4},$$
$$a\cdot(b\times c)=b\cdot(c\times a)=c\cdot(a\times b),\quad a,b,c\in\mathbb{H}^v,$$
$$a\times a=0,\quad a\in\mathbb{H}^v,$$
$$a\times b=-b\times a,\quad a,b\in\mathbb{H}^v,$$
$$\|a^*\|=\|a\|,\quad a\in\mathbb{H},$$
$$|a\cdot b|\leq\|a\|\|b\|,\quad a,b\in\mathbb{H}.$$

II.A.1. Attitude Representation with Unit Quaternions

The relative orientation between a frame fixed to a rigid body and the inertial frame can be represented by the *unit* quaternion $q_{\mathrm{B/I}}=\left(\sin(\frac{\phi}{2})\bar{n}, \cos(\frac{\phi}{2})\right)$, where the body frame is said to be rotated with respect to the inertial frame about the unit vector $\bar{n}$ by an angle ϕ. Note that $q_{\mathrm{B/I}}$ is a unit quaternion because it belongs to the set $\mathbb{H}^u=\{q\in\mathbb{H}:\ q\cdot q=1\}$. The body coordinates of a vector, $\bar{v}^{\mathrm{B}}$, can be calculated from the inertial coordinates of that same vector, $\bar{v}^{\mathrm{I}}$, and vice-versa, through

$$v^{\mathrm{B}}=q^*_{\mathrm{B/I}}v^{\mathrm{I}}q_{\mathrm{B/I}}\qquad\text{and}\qquad v^{\mathrm{I}}=q_{\mathrm{B/I}}v^{\mathrm{B}}q^*_{\mathrm{B/I}},\tag{1}$$

where $v^{\mathrm{B}}=(\bar{v}^{\mathrm{B}}, 0)$ and $v^{\mathrm{I}}=(\bar{v}^{\mathrm{I}}, 0)$.

II.A.2. *Quaternion Representation of the Rotational Kinematic and Dynamic Equations*

The rotational kinematic equations of the body frame and of a frame with some desired attitude, both with respect to the inertial frame and represented by the unit quaternions $q_{\mathrm{B/I}}$ and $q_{\mathrm{D/I}}$, respectively, are given by

$$\dot{q}_{\mathrm{B/I}}=\tfrac{1}{2}q_{\mathrm{B/I}}\omega^{\mathrm{B}}_{\mathrm{B/I}}=\tfrac{1}{2}\omega^{\mathrm{I}}_{\mathrm{B/I}}q_{\mathrm{B/I}} \quad \text{and} \quad \dot{q}_{\mathrm{D/I}}=\tfrac{1}{2}q_{\mathrm{D/I}}\omega^{\mathrm{D}}_{\mathrm{D/I}}=\tfrac{1}{2}\omega^{\mathrm{I}}_{\mathrm{D/I}}q_{\mathrm{D/I}}, \tag{2}$$

where $\omega^{\mathrm{X}}_{\mathrm{Y/Z}} = (\bar{\omega}^{\mathrm{X}}_{\mathrm{Y/Z}}, 0)$, and $\bar{\omega}^{\mathrm{X}}_{\mathrm{Y/Z}}$ is the angular velocity of the Y-frame with respect to the Z-frame expressed in the X-frame. The error quaternion

$$q_{\mathrm{B/D}} = q^{*}_{\mathrm{D/I}}q_{\mathrm{B/I}} \tag{3}$$

is the unit quaternion that rotates the desired frame onto the body frame. By differentiating Eq. (3) and using Eq. (2), the kinematic equations of the error quaternion turn out to be

$$\dot{q}_{\mathrm{B/D}} = \tfrac{1}{2}q_{\mathrm{B/D}}\omega^{\mathrm{B}}_{\mathrm{B/D}} = \tfrac{1}{2}\omega^{\mathrm{D}}_{\mathrm{B/D}}q_{\mathrm{B/D}}, \tag{4}$$

where $\omega^{\mathrm{B}}_{\mathrm{B/D}} = \omega^{\mathrm{B}}_{\mathrm{B/I}} - \omega^{\mathrm{B}}_{\mathrm{D/I}}$ and $\omega^{\mathrm{D}}_{\mathrm{B/D}} = \omega^{\mathrm{D}}_{\mathrm{B/I}} - \omega^{\mathrm{D}}_{\mathrm{D/I}}$.

On the other hand, the quaternion representation of the rotational dynamic equations is given by[15]

$$\dot{\omega}^{\mathrm{B}}_{\mathrm{B/D}}=(I^{\mathrm{B}})^{-1} * \Big(\tau^{\mathrm{B}}-(\omega^{\mathrm{B}}_{\mathrm{B/D}}+\omega^{\mathrm{B}}_{\mathrm{D/I}})\times\big(I^{\mathrm{B}} * (\omega^{\mathrm{B}}_{\mathrm{B/D}}+\omega^{\mathrm{B}}_{\mathrm{D/I}})\big)-I^{\mathrm{B}} * (q^{*}_{\mathrm{B/D}}\dot{\omega}^{\mathrm{D}}_{\mathrm{D/I}}q_{\mathrm{B/D}})-I^{\mathrm{B}} * (\omega^{\mathrm{B}}_{\mathrm{D/I}}\times\omega^{\mathrm{B}}_{\mathrm{B/D}})\Big), \tag{5}$$

where $\tau^{\mathrm{B}} = (\bar{\tau}^{\mathrm{B}}, 0)$, $\bar{\tau}^{\mathrm{B}}$ is the total external moment vector applied to the body about its center of mass,

$$I^{\mathrm{B}}=\begin{bmatrix}\bar{I}^{\mathrm{B}} & 0_{3\times 1}\\ 0_{1\times 3} & 1\end{bmatrix}, \qquad \bar{I}^{\mathrm{B}}=\begin{bmatrix}I_{11} & I_{12} & I_{13}\\ I_{12} & I_{22} & I_{23}\\ I_{13} & I_{23} & I_{33}\end{bmatrix}, \tag{6}$$

and $\bar{I}^{\mathrm{B}} \in \mathbb{R}^{3\times 3}$ is the mass moment of inertia of the body about its center of mass written in the body frame.

II.B. Dual Quaternions

A dual quaternion is defined as $\hat{q} = q_r + \epsilon q_d$, where $q_r = q_1 i + q_2 j + q_3 k + q_4 \in \mathbb{H}$, $q_d = q_5 i + q_6 j + q_7 k + q_8 \in \mathbb{H}$, and ϵ is the *dual unit* defined as $\epsilon^2 = 0$ and $\epsilon \neq 0$. The quaternions q_r and q_d are the *real part* and *dual part* of the dual quaternion, respectively.

Dual quaternions formed from vector quaternions (i.e., $q_r, q_d \in \mathbb{H}^v$) will be referred to as *dual vector quaternions*, whereas dual quaternions formed from scalar quaternions (i.e., $q_r, q_d \in \mathbb{H}^s$) will be referred to as *dual scalar quaternions*. The symbols $\mathbb{H}_d = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d,\ q_r, q_d \in \mathbb{H}\}$, $\mathbb{H}^s_d = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d,\ q_r, q_d \in \mathbb{H}^s\}$, and $\mathbb{H}^v_d = \{\hat{q} : \ \hat{q} = q_r + \epsilon q_d,\ q_r, q_d \in \mathbb{H}^v\}$ will be used to represent the set of dual quaternions, dual scalar quaternions, and dual vector quaternions, respectively. Moreover, the set of dual scalar quaternions with zero dual part will be denoted by $\mathbb{H}^r_d = \{\hat{q} : \ \hat{q} = q_r + \epsilon(\bar{0}, 0),\ q_r \in \mathbb{H}^s\}$.

The basic operations on dual quaternions are defined as follows:[13,15]

$$\text{Addition: } \hat{a} + \hat{b} = (a_r + b_r) + \epsilon(a_d + b_d),$$
$$\text{Multiplication by a scalar: } \lambda\hat{a} = (\lambda a_r) + \epsilon(\lambda a_d),$$
$$\text{Multiplication: } \hat{a}\hat{b} = (a_r b_r) + \epsilon(a_r b_d + a_d b_r),$$
$$\text{Conjugation: } \hat{a}^{*} = a_r^{*} + \epsilon a_d^{*},$$
$$\text{Swap: } \hat{a}^{\mathsf{s}} = a_d + \epsilon a_r,$$
$$\text{Dot product: } \hat{a}\cdot\hat{b}=\tfrac{1}{2}(\hat{a}^{*}\hat{b} + \hat{b}^{*}\hat{a})=\tfrac{1}{2}(\hat{a}\hat{b}^{*} + \hat{b}\hat{a}^{*})=a_r \cdot b_r + \epsilon(a_d \cdot b_r + a_r \cdot b_d) \in \mathbb{H}^s_d,$$
$$\text{Cross product: } \hat{a}\times\hat{b} = \tfrac{1}{2}(\hat{a}\hat{b} - \hat{b}^{*}\hat{a}^{*}) = a_r \times b_r + \epsilon(a_d \times b_r + a_r \times b_d) \in \mathbb{H}^v_d,$$
$$\text{Dual norm: } \|\hat{a}\|^2_d = \hat{a}\hat{a}^{*} = \hat{a}^{*}\hat{a} = \hat{a}\cdot\hat{a} = (a_r \cdot a_r) + \epsilon(2a_r \cdot a_d) \in \mathbb{H}^s_d,$$
$$\text{Scalar part: } \mathrm{sc}(\hat{a}) = \mathrm{sc}(a_r) + \epsilon\,\mathrm{sc}(a_d) \in \mathbb{H}^s_d,$$
$$\text{Vector part: } \mathrm{vec}(\hat{a}) = \mathrm{vec}(a_r) + \epsilon\,\mathrm{vec}(a_d) \in \mathbb{H}^v_d,$$

where $\hat{a}, \hat{b} \in \mathbb{H}_d$ and $\lambda \in \mathbb{R}$. Note that in general $\hat{a}\hat{b} \neq \hat{b}\hat{a}$.

The dot product and dual norm of dual quaternions yield a dual number and not a real number, in general. Hence, the *dual quaternion norm* will be defined here as[15,22]

$$\|\hat{a}\|^2 = \hat{a} \circ \hat{a}, \tag{7}$$

where $\circ$ denotes the *dual quaternion circle product* given by

$$\hat{a} \circ \hat{b} = a_r \cdot b_r + a_d \cdot b_d, \tag{8}$$

where $\hat{a}, \hat{b} \in \mathbb{H}_d$.

Note that as vector spaces, $\mathbb{H}_d$, $\mathbb{H}_d^v$, $\mathbb{H}_d^s$, and $\mathbb{H}_d^r$ are isomorphic to $\mathbb{R}^8$, $\mathbb{R}^6$, $\mathbb{R}^2$, and $\mathbb{R}$, respectively. Under the natural isomorphism between $\mathbb{H}_d$ and $\mathbb{R}^8$, the square of the dual quaternion norm and the circle product on $\mathbb{H}_d$ correspond to the square of the Euclidean norm and to the dot (inner) product on $\mathbb{R}^8$, respectively. Moreover, under the natural isomorphism between $\mathbb{H}_d^r$ and $\mathbb{R}$, $(\bar{0}, q_4) + \epsilon(\bar{0}, 0)$ will often be denoted as q_4 for simplicity.

The multiplication of a matrix $M \in \mathbb{R}^{8\times 8}$ with a dual quaternion $\hat{q} \in \mathbb{H}_d$ will be defined analogously to the multiplication of a 8-by-8 matrix with a 8-dimensional vector as

$$M \star \hat{q} = (M_{11} * q_r + M_{12} * q_d) + \epsilon(M_{21} * q_r + M_{22} * q_d),$$

where

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad M_{11}, M_{12}, M_{21}, M_{22} \in \mathbb{R}^{4\times 4}.$$

It can be shown that the following properties[8] follow from the previous definitions:

$$\hat{a}\circ(\hat{b}\hat{c}) = \hat{b}^{\mathsf{s}}\circ(\hat{a}^{\mathsf{s}}\hat{c}^*) = \hat{c}^{\mathsf{s}}\circ(\hat{b}^*\hat{a}^{\mathsf{s}}) \in \mathbb{R}, \quad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d, \tag{9}$$

$$\hat{a}\circ(\hat{b}\times\hat{c}) = \hat{b}^{\mathsf{s}}\circ(\hat{c}\times\hat{a}^{\mathsf{s}}) = \hat{c}^{\mathsf{s}}\circ(\hat{a}^{\mathsf{s}}\times\hat{b}), \quad \hat{a}, \hat{b}, \hat{c} \in \mathbb{H}_d^v, \tag{10}$$

$$\hat{a} \times \hat{a} = 0, \quad \hat{a} \in \mathbb{H}_d^v, \tag{11}$$

$$\hat{a} \times \hat{b} = -\hat{b} \times \hat{a}, \quad \hat{a}, \hat{b} \in \mathbb{H}_d^v, \tag{12}$$

$$\hat{a}^{\mathsf{s}} \circ \hat{b}^{\mathsf{s}} = \hat{a} \circ \hat{b}, \quad \hat{a}, \hat{b} \in \mathbb{H}_d, \tag{13}$$

$$\|\hat{a}^{\mathsf{s}}\| = \|\hat{a}\|, \quad \hat{a} \in \mathbb{H}_d, \tag{14}$$

$$\|\hat{a}^*\| = \|\hat{a}\|, \quad \hat{a} \in \mathbb{H}_d, \tag{15}$$

$$(M \star \hat{a}) \circ \hat{b} = \hat{a} \circ (M^{\mathsf{T}} \star \hat{b}), \ \hat{a}, \hat{b} \in \mathbb{H}_d, \ M \in \mathbb{R}^{8\times 8}, \tag{16}$$

$$|\hat{a} \circ \hat{b}| \leq \|\hat{a}\|\|\hat{b}\|, \quad \hat{a}, \hat{b} \in \mathbb{H}_d, \tag{17}$$

$$\|\hat{a}\hat{b}\| \leq \sqrt{3/2}\,\|\hat{a}\|\|\hat{b}\|, \quad \hat{a}, \hat{b} \in \mathbb{H}_d. \tag{18}$$

Finally, the $\mathcal{L}_\infty$-norm of a function $\hat{u} : [0, \infty) \to \mathbb{H}_d$ is defined as $\|\hat{u}\|_\infty = \sup_{t\geq 0} \|\hat{u}(t)\|$. Moreover, the dual quaternion $\hat{u} \in \mathcal{L}_\infty$, if and only if $\|\hat{u}\|_\infty < \infty$.

II.B.1. Attitude and Position Representation with Unit Dual Quaternions

The position and orientation (i.e., pose) of the body frame with respect to the inertial frame can be represented by a unit quaternion $q_{\mathrm{B/I}} \in \mathbb{H}^u$ and by a translation vector $\bar{r}_{\mathrm{B/I}} \in \mathbb{R}^3$. Alternatively, the pose of the body frame with respect to the inertial frame can be represented more compactly by the *unit* dual quaternion[6]

$$\hat{q}_{\mathrm{B/I}} = q_{\mathrm{B/I}} + \epsilon\frac{1}{2}r^{\mathrm{I}}_{\mathrm{B/I}}q_{\mathrm{B/I}} = q_{\mathrm{B/I}} + \epsilon\frac{1}{2}q_{\mathrm{B/I}}r^{\mathrm{B}}_{\mathrm{B/I}}, \tag{19}$$

where $r^{\mathrm{X}}_{\mathrm{Y/Z}} = (\bar{r}^{\mathrm{X}}_{\mathrm{Y/Z}}, 0)$ and $\bar{r}^{\mathrm{X}}_{\mathrm{Y/Z}} = [x^{\mathrm{X}}_{\mathrm{Y/Z}}\ y^{\mathrm{X}}_{\mathrm{Y/Z}}\ z^{\mathrm{X}}_{\mathrm{Y/Z}}]^{\mathsf{T}}$ is the translation vector from the origin of the Z-frame to the origin of the Y-frame expressed in the X-frame. Note that $\hat{q}_{\mathrm{B/I}}$ is a unit dual quaternion because it belongs to the set[7] $\mathbb{H}_d^u = \{\hat{q} \in \mathbb{H}_d : \ \hat{q} \cdot \hat{q} = \hat{q}\hat{q}^* = \hat{q}^*\hat{q} = \|\hat{q}\|_d = 1\}$.

II.B.2. Dual Quaternion Representation of the Rigid Body Kinematic and Dynamic Equations

The rigid body kinematic equations of the body frame and of a frame with some desired position and attitude, both with respect to the inertial frame and represented by the unit dual quaternions $\hat{q}_{\rm B/I}$ and $\hat{q}_{\rm D/I} = q_{\rm D/I} + \epsilon\frac{1}{2}r^{\rm I}_{\rm D/I}q_{\rm D/I} = q_{\rm D/I} + \epsilon\frac{1}{2}q_{\rm D/I}r^{\rm D}_{\rm D/I}$, respectively, are given by[6]

$$\dot{\hat{q}}_{\rm B/I}=\tfrac{1}{2}\hat{\omega}^{\rm I}_{\rm B/I}\hat{q}_{\rm B/I}=\tfrac{1}{2}\hat{q}_{\rm B/I}\hat{\omega}^{\rm B}_{\rm B/I} \quad \text{and} \quad \dot{\hat{q}}_{\rm D/I}=\tfrac{1}{2}\hat{\omega}^{\rm I}_{\rm D/I}\hat{q}_{\rm D/I}=\tfrac{1}{2}\hat{q}_{\rm D/I}\hat{\omega}^{\rm D}_{\rm D/I}, \tag{20}$$

where $\hat{\omega}^{\rm X}_{\rm Y/Z}$ is the *dual velocity* of the Y-frame with respect to the Z-frame expressed in the X-frame, so that $\hat{\omega}^{\rm X}_{\rm Y/Z} = \omega^{\rm X}_{\rm Y/Z} + \epsilon(v^{\rm X}_{\rm Y/Z} + \omega^{\rm X}_{\rm Y/Z} \times r^{\rm X}_{\rm X/Y})$, $v^{\rm X}_{\rm Y/Z} = (\bar{v}^{\rm X}_{\rm Y/Z}, 0)$, and $\bar{v}^{\rm X}_{\rm Y/Z}$ is the linear velocity of the Y-frame with respect to the Z-frame expressed in the X-frame.

By direct analogy to Eq. (3), the *dual error quaternion*[12,15] is defined as

$$\hat{q}_{\rm B/D} \triangleq \hat{q}^*_{\rm D/I}\hat{q}_{\rm B/I} = q_{\rm B/D} + \epsilon\tfrac{1}{2}q_{\rm B/D}r^{\rm B}_{\rm B/D} \tag{21}$$

where $r^{\rm B}_{\rm B/D} = r^{\rm B}_{\rm B/I} - r^{\rm B}_{\rm D/I}$. As illustrated in Figure 1, the dual error quaternion represents the rotation ($q_{\rm B/D}$) and the translation ($r^{\rm B}_{\rm B/D}$) necessary to align the desired frame with the body frame. It can be shown[7] that $\hat{q}_{\rm B/D}$ is a unit dual quaternion. By differentiating Eq. (21) and using Eq. (20), the kinematic equations of

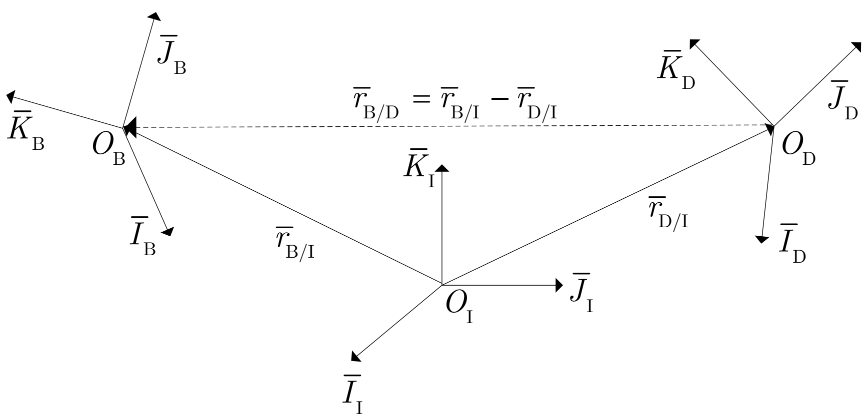

Figure 1. Relation between frames.

the dual error quaternion turn out to be[15]

$$\dot{\hat{q}}_{\rm B/D} = \tfrac{1}{2}\hat{q}_{\rm B/D}\hat{\omega}^{\rm B}_{\rm B/D} = \tfrac{1}{2}\hat{\omega}^{\rm D}_{\rm B/D}\hat{q}_{\rm B/D}, \tag{22}$$

where $\hat{\omega}^{\rm B}_{\rm B/D} = \hat{\omega}^{\rm B}_{\rm B/I} - \hat{\omega}^{\rm B}_{\rm D/I}$ is the *dual relative velocity* between the body frame and the desired frame expressed in the body frame. Note that $\hat{\omega}^{\rm B}_{\rm D/I} = \hat{q}^*_{\rm B/D}\hat{\omega}^{\rm D}_{\rm D/I}\hat{q}_{\rm B/D}$ and $\hat{\omega}^{\rm D}_{\rm B/I} = \hat{q}_{\rm B/D}\hat{\omega}^{\rm B}_{\rm B/I}\hat{q}^*_{\rm B/D}$. Note also that the kinematic equations of the dual error quaternion, Eq. (22), and of the error quaternion, Eq. (4), have the same form.

Finally, the dual quaternion representation of the rigid body dynamic equations is given by[8,15]

$$(\dot{\hat{\omega}}^{\rm B}_{\rm B/D})^{\rm s}=(M^{\rm B})^{-1}\star\Big(\hat{f}^{\rm B}-(\hat{\omega}^{\rm B}_{\rm B/D}+\hat{\omega}^{\rm B}_{\rm D/I})\times\big(M^{\rm B}\star((\hat{\omega}^{\rm B}_{\rm B/D})^{\rm s}+(\hat{\omega}^{\rm B}_{\rm D/I})^{\rm s})\big)-M^{\rm B}\star(\hat{q}^*_{\rm B/D}\dot{\hat{\omega}}^{\rm D}_{\rm D/I}\hat{q}_{\rm B/D})^{\rm s}-M^{\rm B}\star(\hat{\omega}^{\rm B}_{\rm D/I}\times\hat{\omega}^{\rm B}_{\rm B/D})^{\rm s}\Big), \tag{23}$$

where $\hat{f}^{\rm B} = f^{\rm B} + \epsilon\tau^{\rm B}$ is the total external *dual force* applied to the body about its center of mass expressed in body coordinates, $f^{\rm B} = (\bar{f}^{\rm B}, 0)$, and $\bar{f}^{\rm B}$ is the total external force vector applied to the body. Finally, $M^{\rm B} \in \mathbb{R}^{8\times 8}$ is the *dual inertia matrix*[7] defined as

$$M^{\rm B}=\begin{bmatrix} mI_3 & 0_{3\times 1} & 0_{3\times 3} & 0_{3\times 1} \\ 0_{1\times 3} & 1 & 0_{1\times 3} & 0 \\ 0_{3\times 3} & 0_{3\times 1} & \bar{I}^{\rm B} & 0_{3\times 1} \\ 0_{1\times 3} & 0 & 0_{1\times 3} & 1 \end{bmatrix} \tag{24}$$

and m is the mass of the body.

Note the similarity between the dual quaternion representation of the combined rotational and translational dynamic equations given by Eq. (23) and the quaternion representation of the rotational(-only) dynamic equations given by Eq. (5).

III. Adaptive Controller

The main result of this paper is an adaptive pose-tracking controller that does not need mass and moment of inertia information. This controller is described in the next theorem.

Theorem 1. *Consider the rigid body relative kinematic and dynamic equations given by Eq.* (22) *and Eq.* (23)*. Let the input dual force be defined by the feedback control law*

$$\begin{aligned}\hat{f}^{\mathrm{B}} =& -\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)\big) - K_d \star \hat{s}^{\mathsf{s}} + \hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times (\widehat{M}^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}) \\ &+ \widehat{M}^{\mathrm{B}} \star (\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} + \widehat{M}^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} - \widehat{M}^{\mathrm{B}} \star \Big(K_p \star \frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))\Big)^{\mathsf{s}}\end{aligned} \tag{25}$$

where

$$\hat{s} = \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} + (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))^{\mathsf{s}}, \tag{26}$$

$$K_p = \begin{bmatrix} K_r & 0_{4\times4} \\ 0_{4\times4} & K_q \end{bmatrix}, \quad K_d = \begin{bmatrix} K_v & 0_{4\times4} \\ 0_{4\times4} & K_\omega \end{bmatrix}, \tag{27}$$

$$K_r = \begin{bmatrix} \bar{K}_r & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \quad K_q = \begin{bmatrix} \bar{K}_q & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \quad K_v = \begin{bmatrix} \bar{K}_v & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \quad K_\omega = \begin{bmatrix} \bar{K}_\omega & 0_{3\times1} \\ 0_{1\times3} & 0 \end{bmatrix}, \tag{28}$$

$\bar{K}_r, \bar{K}_q, \bar{K}_v, \bar{K}_\omega \in \mathbb{R}^{3\times3}$ *are positive definite matrices,* $\widehat{M}^{\mathrm{B}}$ *is an estimate of the dual inertia matrix updated according to*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\widehat{M}^{\mathrm{B}}) = K_i \Big[\mathrm{h}(\hat{s}^{\mathsf{s}}, -(\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} - (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} + (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))^{\mathsf{s}}) - \mathrm{h}((\hat{s} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}, (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})\Big], \tag{29}$$

$K_i \in \mathbb{R}^{7\times7}$ *is a positive definite matrix,*

$$\mathrm{v}(M^{\mathrm{B}}) = [I_{11}\ I_{12}\ I_{13}\ I_{22}\ I_{23}\ I_{33}\ m]^{\mathsf{T}} \tag{30}$$

is a vectorized version of the dual inertia matrix M^{B}*, the function* $\mathrm{h} : \mathbb{H}^v_d \times \mathbb{H}^v_d \to \mathbb{R}^7$ *is defined as*

$$\hat{a} \circ (M^B \star \hat{b}) = \mathrm{h}(\hat{a}, \hat{b})^{\mathsf{T}}\mathrm{v}(M^B) = \mathrm{v}(M^B)^{\mathsf{T}}\mathrm{h}(\hat{a}, \hat{b}) \tag{31}$$

or, equivalently,

$$\mathrm{h}(\hat{a}, \hat{b}) = [a_5b_5\ \ a_6b_5{+}a_5b_6\ \ a_7b_5{+}a_5b_7\ \ a_6b_6\ \ a_7b_6{+}a_6b_7\ \ a_7b_7\ \ a_1b_1{+}a_2b_2{+}a_3b_3]^{\mathsf{T}}, \tag{32}$$

and assume that $\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$*. Then, for all initial conditions,* $\lim_{t\to\infty} \hat{q}_{\mathrm{B/D}} = \pm1$ *(i.e.,* $\lim_{t\to\infty} q_{\mathrm{B/D}} = \pm1$ *and* $\lim_{t\to\infty} r^{\mathrm{B}}_{\mathrm{B/D}} = 0$*),* $\lim_{t\to\infty} \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$ *(i.e.,* $\lim_{t\to\infty} \omega^{\mathrm{B}}_{\mathrm{B/D}} = 0$ *and* $\lim_{t\to\infty} v^{\mathrm{B}}_{\mathrm{B/D}} = 0$*), and* $\mathrm{v}(\widehat{M}^{\mathrm{B}}) \in \mathcal{L}_\infty$*.*

Proof. First, define the dual inertia matrix estimation error as

$$\Delta M^{\mathrm{B}} = \widehat{M}^{\mathrm{B}} - M^{\mathrm{B}}, \tag{33}$$

and note that $\hat{q}_{\mathrm{B/D}} = \pm1$, $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$, and $\mathrm{v}(\Delta M^{\mathrm{B}}) = 0$ are the equilibrium conditions of the closed-loop system formed by Eqs. (23), (22), and (29). Consider now the following candidate Lyapunov function for the equilibrium point $\hat{q}_{\mathrm{B/D}} = +1$, $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$, and $\mathrm{v}(\Delta M^{\mathrm{B}}) = 0$:

$$V(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = (\hat{q}_{\mathrm{B/D}} - 1) \circ (\hat{q}_{\mathrm{B/D}} - 1) + \tfrac{1}{2}\hat{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star \hat{s}^{\mathsf{s}}) + \tfrac{1}{2}\mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}}K_i^{-1}\mathrm{v}(\Delta M^{\mathrm{B}}).$$

Note that V is a valid candidate Lyapunov function since $V(\hat{q}_{\mathrm{B/D}} = 1, \hat{s} = 0, \mathrm{v}(\Delta M^{\mathrm{B}}) = 0) = 0$ and $V(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) > 0$ for all $(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) \in \mathbb{H}^u_d \times \mathbb{H}^v_d \times \mathbb{R}^7 \backslash \{1, 0, 0\}$. The time derivative of V is equal to

$$\dot{V}(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = 2(\hat{q}_{\mathrm{B/D}} - 1) \circ \dot{\hat{q}}_{\mathrm{B/D}} + \hat{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star \dot{\hat{s}}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}}K_i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}).$$

Then, since from Eq. (22), $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 2\hat{q}^*_{\mathrm{B/D}}\dot{\hat{q}}_{\mathrm{B/D}}$, Eq. (26) can be rewritten as

$$\dot{\hat{q}}_{\mathrm{B/D}} = \tfrac{1}{2}\hat{q}_{\mathrm{B/D}}\hat{s} - \tfrac{1}{2}\hat{q}_{\mathrm{B/D}}(K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))^{\mathsf{s}}, \tag{34}$$

which can then by plugged into $\dot{V}$, together with the time derivative of Eq. (26), to yield

$$\begin{aligned}\dot{V}(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = &(\hat{q}_{\mathrm{B/D}} - 1) \circ (\hat{q}_{\mathrm{B/D}}\hat{s} - \hat{q}_{\mathrm{B/D}}(K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))^{\mathsf{s}}) + \hat{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (\dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})\\ &+\hat{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))}{\mathrm{d}t})^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}).\end{aligned}$$

Applying Eq. (9) and inserting Eq. (23) yields

$$\begin{aligned}\dot{V}(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = &\hat{s}^{\mathsf{s}} \circ (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)) - (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))) \circ (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))\\ &+\hat{s}^{\mathsf{s}} \circ (\hat{f}^{\mathrm{B}} - \hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times \big(M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\big) - M^{\mathrm{B}} \star (\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} - M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}})\\ &+\hat{s}^{\mathsf{s}} \circ (M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))}{\mathrm{d}t})^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}).\end{aligned}$$

By introducing the feedback control law, Eq. (25), and using Eq. (10), $\dot{V}$ becomes

$$\begin{aligned}\dot{V}(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = &-(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)) \circ (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))\\ &+\hat{s}^{\mathsf{s}} \circ (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times \big(\Delta M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}\big) + \Delta M^{\mathrm{B}} \star (\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} + \Delta M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\\ &-\Delta M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))}{\mathrm{d}t})^{\mathsf{s}}) - \hat{s}^{\mathsf{s}} \circ (K_d \star \hat{s}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}})\\ = &-(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)) \circ (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))\\ &+(\hat{s} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}} \circ (\Delta M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}}) + \hat{s}^{\mathsf{s}} \circ (\Delta M^{\mathrm{B}} \star (\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} + \Delta M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}}\\ &-\Delta M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))}{\mathrm{d}t})^{\mathsf{s}}) - \hat{s}^{\mathsf{s}} \circ (K_d \star \hat{s}^{\mathsf{s}}) + \mathrm{v}(\Delta M^{\mathrm{B}})^{\mathsf{T}} K_i^{-1} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}}).\end{aligned}$$

Therefore, if $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\Delta M^{\mathrm{B}})$ is defined as in Eq. (29), it follows that

$$\dot{V}(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) = -(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)) \circ (K_p \star (\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon))) - \hat{s}^{\mathsf{s}} \circ (K_d \star \hat{s}^{\mathsf{s}}) \leq 0, \tag{35}$$

for all $(\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}})) \in \mathbb{H}^u_d \times \mathbb{H}^v_d \times \mathbb{R}^7 \backslash \{1, 0, 0\}$. Hence, $\hat{q}_{\mathrm{B/D}}$, $\hat{s}$, and $\mathrm{v}(\Delta M^{\mathrm{B}})$ are uniformly bounded, i.e., $\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}}) \in \mathcal{L}_\infty$. Moreover, from Eqs. (26) and (33), this means that also $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \mathrm{v}(\widehat{M}^{\mathrm{B}}) \in \mathcal{L}_\infty$. Since $V \geq 0$ and $\dot{V} \leq 0$, $\lim_{t\to\infty} V(t)$ exists and is finite. Hence,

$$\lim_{t\to\infty} \int_0^t \dot{V}(\tau)\,\mathrm{d}\tau = \lim_{t\to\infty} V(t) - V(0) \tag{36}$$

also exists and is finite. Since $\hat{q}_{\mathrm{B/D}}, \hat{s}, \mathrm{v}(\Delta M^{\mathrm{B}}), \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}, \mathrm{v}(\widehat{M}^{\mathrm{B}}), \dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \hat{\omega}^{\mathrm{B}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$, then from Eqs. (22), (25), and (23), $\dot{\hat{q}}_{\mathrm{B/D}}, \hat{f}^{\mathrm{B}}, \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}, \dot{\hat{s}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)\big) \to 0$ and $\hat{s} \to 0$ as $t \to \infty$. In Ref. 7, it is shown that $\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)\big) \to 0$ is equivalent to $\hat{q}_{\mathrm{B/D}} \to \pm 1$. Finally, calculating the limit as $t \to \infty$ of both sides of Eq. (26) yields $\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$. □

Remark 1. Theorem 1 states that $\hat{q}_{\mathrm{B/D}}$ converges to either $+1$ or -1. Note that $\hat{q}_{\mathrm{B/D}} = +1$ and $\hat{q}_{\mathrm{B/D}} = -1$ represent the same physical relative position and attitude between frames, so either equilibrium is acceptable. However, this can lead to the so-called *unwinding phenomenon* where a large rotation (greater than 180 degrees) is performed, despite the fact that a smaller rotation (less than 180 degrees) to the equilibrium exits. This problem of quaternions is well documented and possible solutions exist in literature.[13,15,23,24]

Remark 2. It can be easily shown that if the control law given by Eq. (25) is replaced by its nonadaptive model-dependent version, where the estimate of the dual inertia matrix is replaced by the true dual inertia matrix of the rigid body, i.e.,

$$\begin{aligned}\hat{f}^{\mathrm{B}} = &-\mathrm{vec}\big(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)\big) - K_d \star \hat{s}^{\mathsf{s}} + \hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}} \times (M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{B}}_{\mathrm{B/I}})^{\mathsf{s}})\\ &+M^{\mathrm{B}} \star (\hat{q}^*_{\mathrm{B/D}}\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}\hat{q}_{\mathrm{B/D}})^{\mathsf{s}} + M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}} \times \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}})^{\mathsf{s}} - M^{\mathrm{B}} \star (K_p \star \frac{\mathrm{d}}{\mathrm{d}t}(\hat{q}^*_{\mathrm{B/D}}(\hat{q}^{\mathsf{s}}_{\mathrm{B/D}} - \epsilon)))^{\mathsf{s}},\end{aligned} \tag{37}$$

in Theorem 1, then it is still true that, for all initial conditions, $\lim_{t\to\infty} \hat{q}_{\mathrm{B/D}} = \pm 1$ and $\lim_{t\to\infty} \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$.

Remark 3. The dual part of the control law given by Eq. (25) is

$$\begin{aligned}\tau^{\mathrm{B}} =& -\mathrm{vec}\big(q_{\mathrm{B/D}}\big) - K_\omega * \omega^{\mathrm{B}}_{\mathrm{B/D}} - (K_\omega K_q) * q_{\mathrm{B/D}} \\ & + \omega^{\mathrm{B}}_{\mathrm{B/I}} \times (\widehat{I}^{\mathrm{B}} * \omega^{\mathrm{B}}_{\mathrm{B/I}}) + \widehat{I}^{\mathrm{B}} * (q^*_{\mathrm{B/D}} \dot{\omega}^{\mathrm{D}}_{\mathrm{D/I}} q_{\mathrm{B/D}}) + \widehat{I}^{\mathrm{B}} * (\omega^{\mathrm{B}}_{\mathrm{D/I}} \times \omega^{\mathrm{B}}_{\mathrm{B/D}}) - (\widehat{I}^{\mathrm{B}} K_q) * \frac{\mathrm{d}}{\mathrm{d}t}(q_{\mathrm{B/D}}),\end{aligned} \tag{38}$$

where $\widehat{I}^{\mathrm{B}}$ is an estimate of the inertia matrix of the rigid body. Eq. (38) is identical to the adaptive attitude(-only) tracking law proposed in Ref. 5.

Remark 4. Note that although the control law given by Eq. (25) does not require knowledge about the mass and inertia matrix of the rigid body, it still requires knowledge about the position of the center of mass of the rigid body. This is because Eq. (23) is only valid if the origin of the body frame coincides with the center of mass of the rigid body. If the origin of the body frame was placed at any other point fixed to the body, Eq. (23) would no longer be valid.

IV. Mass and Inertia Matrix Identification

In this section, we show that by properly choosing the reference pose (i.e., the reference position and attitude), the estimate of the dual inertia matrix will converge to the true dual inertia matrix. In other words, the adaptive pose tracking controller proposed in Theorem 1 can also be used to identify the mass and inertia matrix of a rigid body.

Proposition 1. *Let* $\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}$ *be periodic and assume that* $\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}, \dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \ddot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}} \in \mathcal{L}_\infty$. *Furthermore, let* $\mathrm{W} : [0, \infty) \to \mathbb{R}^{8\times 7}$ *be defined as*

$$\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}(t) \times (\mathit{\Delta} M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}(t))^{\mathsf{s}}) + \mathit{\Delta} M^{\mathrm{B}} \star (\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}(t))^{\mathsf{s}} = \mathrm{W}(t)\mathrm{v}(\mathit{\Delta} M^{\mathrm{B}}), \tag{39}$$

or, equivalently,

$$\mathrm{W}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dot{u} + qw - rv \\ 0 & 0 & 0 & 0 & 0 & 0 & \dot{v} - pw + ru \\ 0 & 0 & 0 & 0 & 0 & 0 & \dot{w} + pv - qu \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \dot{p} & \dot{q} - pr & \dot{r} + pq & -qr & q^2 - r^2 & qr & 0 \\ pr & \dot{p} + qr & -p^2 + r^2 & \dot{q} & \dot{r} - pq & -pr & 0 \\ -pq & p^2 - q^2 & \dot{p} - qr & pq & \dot{q} + pr & \dot{r} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

where $\bar{v}^{\mathrm{D}}_{\mathrm{D/I}} = [u\ v\ w]^{\mathsf{T}}$ *and* $\bar{\omega}^{\mathrm{D}}_{\mathrm{D/I}} = [p\ q\ r]^{\mathsf{T}}$. *Let also* $0 \le t_1 \le t_2 \le ... \le t_n$ *be such that*

$$\mathrm{rank}\begin{bmatrix} \mathrm{W}(t_1) \\ \vdots \\ \mathrm{W}(t_n) \end{bmatrix} = 7. \tag{41}$$

Then, under the control law given by Eq. (25), $\lim_{t\to\infty} \widehat{M}^{\mathrm{B}} = M^{\mathrm{B}}$.

Proof. We start by showing that $\lim_{t\to\infty} \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} = 0$. (Note that $\lim_{t\to\infty} \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = 0$ does not imply that $\lim_{t\to\infty} \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} = 0$.) First, note that $\lim_{t\to\infty} \int_0^t \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}(\tau)\, \mathrm{d}\tau = \lim_{t\to\infty} \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(t) - \hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(0) = -\hat{\omega}^{\mathrm{B}}_{\mathrm{B/D}}(0)$ exits and is finite. Furthermore, since $\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}, \dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \ddot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}, \dot{\hat{q}}_{\mathrm{B/D}}, \ddot{\hat{q}}_{\mathrm{B/D}}, \dot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}}, \frac{\mathrm{dv}(\widehat{M}^{\mathrm{B}})}{\mathrm{d}t} \in \mathcal{L}_\infty$, we have that $\ddot{\hat{\omega}}^{\mathrm{B}}_{\mathrm{B/D}} \in \mathcal{L}_\infty$. Hence, by Barbalat's lemma, $\lim_{t\to\infty} \dot{\hat{\omega}}^{B}_{\mathrm{B/D}} = 0$. Now, we calculate the limit as $t \to \infty$ of both sides of Eq. (23) to get

$$\lim_{t\to\infty} (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}(t) \times (\mathit{\Delta} M^{\mathrm{B}} \star (\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}(t))^{\mathsf{s}}) + \mathit{\Delta} M^{\mathrm{B}} \star (\dot{\hat{\omega}}^{\mathrm{D}}_{\mathrm{D/I}}(t))^{\mathsf{s}}) = 0 \tag{42}$$

Finally, noting that $\lim_{t\to\infty} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{v}(\widehat{M}^{\mathrm{B}}) = 0$ from Eq. (29) and Theorem 1, under the conditions of Proposition 1, Eq. (42) implies that $\lim_{t\to\infty} \mathrm{v}(\mathit{\Delta} M^{\mathrm{B}}) = 0$ or, equivalently, $\lim_{t\to\infty} \widehat{M}^{\mathrm{B}} = M^{\mathrm{B}}$. □

Remark 5. It is a well known fact that the rotational motion is independent from the translational motion, but the translational motion depends on the rotational motion. It is interesting to note from Eq. (40) that, likewise, while the inertia matrix identification is independent from the translational motion, the mass identification depends on the rotational motion.

Remark 6. An alternative, and more general, sufficient condition than Eq. (41) for dual inertia matrix identification, which does not require $\hat{\omega}^{\mathrm{D}}_{\mathrm{D/I}}$ to be periodic, is that the 7×7 matrix

$$\int_t^{t+T_2} \mathrm{W}^{\mathsf{T}}(t)\mathrm{W}(t)\,\mathrm{d}t, \tag{43}$$

is positive definite for all $t \geq T_1$ for some $T_1 \geq 0$ and $T_2 > 0$.[25,26]

V. Simulation Results

To illustrate that the control law given by Eq. (25) can achieve position and attitude tracking and identify the mass and inertia matrix of a rigid body, a simple example is considered here. The performance of the control law given by Eq. (25) is also compared to its nonadaptive model-dependent version given by Eq. (37). A rigid body with true inertia matrix

$$\bar{I}^{\mathrm{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.63 & 0 \\ 0 & 0 & 0.85 \end{bmatrix} \text{ kg.m}^2$$

and mass $m = 1$ kg is assumed. The origin of the body frame (coincident with the center of mass of the body) is positioned relatively to the origin of the desired frame at $\bar{r}^{\mathrm{B}}_{\mathrm{B/D}}=[10,\ 10,\ 10]^{\mathsf{T}}$ m. The initial error quaternion, relative linear velocity, and relative angular velocity of the body frame with respect to the desired frame are set to $q_{\mathrm{B/D}} = [q_{\mathrm{B/D}1}\ q_{\mathrm{B/D}2}\ q_{\mathrm{B/D}3}\ q_{\mathrm{B/D}4}]^{\mathsf{T}} = [0.4618,\ 0.1917,\ 0.7999,\ 0.3320]^{\mathsf{T}}$, $\bar{v}^{\mathrm{B}}_{\mathrm{B/D}} = [u^{\mathrm{B}}_{\mathrm{B/D}}\ v^{\mathrm{B}}_{\mathrm{B/D}}\ w^{\mathrm{B}}_{\mathrm{B/D}}]^{\mathsf{T}} = [0.1,\ 0.1,\ 0.1]^{\mathsf{T}}$ m/s, and $\bar{\omega}^{\mathrm{B}}_{\mathrm{B/D}} = [p^{\mathrm{B}}_{\mathrm{B/D}}\ q^{\mathrm{B}}_{\mathrm{B/D}}\ r^{\mathrm{B}}_{\mathrm{B/D}}]^{\mathsf{T}} = [0.1,\ 0.1,\ 0.1]^{\mathsf{T}}$ rad/s, respectively.

The linear and angular velocity of the desired frame with respect to the inertial frame, expressed in the desired frame, are defined as $\bar{v}^{\mathrm{D}}_{\mathrm{D/I}} = [u^{\mathrm{D}}_{\mathrm{D/I}}\ v^{\mathrm{D}}_{\mathrm{D/I}}\ w^{\mathrm{D}}_{\mathrm{D/I}}]^{\mathsf{T}} = [0.1,\ 0.2,\ 0.3]^{\mathsf{T}}\cos(2\pi[10^{-1},\ 20^{-1},\ 30^{-1}]^{\mathsf{T}}t + \frac{\pi}{180}[0,\ 20,\ 40]^{\mathsf{T}})$ m/s and $\bar{\omega}^{\mathrm{D}}_{\mathrm{D/I}} = [p^{\mathrm{D}}_{\mathrm{D/I}}\ q^{\mathrm{D}}_{\mathrm{D/I}}\ r^{\mathrm{D}}_{\mathrm{D/I}}]^{\mathsf{T}} = [0.1,\ 0.2,\ 0.3]^{\mathsf{T}}\cos(2\pi[10^{-1},\ 20^{-1},\ 30^{-1}]^{\mathsf{T}}t + \frac{\pi}{180}[0,\ 20,\ 40]^{\mathsf{T}})$ rad/s, respectively. They are illustrated in Figure 2. Note that these signals satisfy the conditions of Proposition 1 since

$$\operatorname{rank}\begin{bmatrix} \mathrm{W}(0) \\ \mathrm{W}(\pi/2) \end{bmatrix} = 7. \tag{44}$$

The initial estimate of the mass and inertia matrix of the rigid body are chosen to be zero, whereas the control gains are chosen to be $\bar{K}_r = 0.1I_3$, $\bar{K}_q = 0.5I_3$, $\bar{K}_v = 8I_3$, $\bar{K}_w = 8I_3$, and $K_i = 100I_7$.

The relative position and attitude of the body frame with respect to the desired frame using the controller given by Eq. (25) (adaptive) and the controller given by Eq. (37) (nonadaptive) are compared in Figure 3. In both cases, $q_{\mathrm{B/D}} \to 1$ and $\bar{r}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ as $t \to \infty$, as expected.

Figure 4 shows the relative linear and angular velocity of the body frame with respect to the desired frame for the same two cases studied in Figure 3. As predicted, $\bar{\omega}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ and $\bar{v}^{\mathrm{B}}_{\mathrm{B/D}} \to 0$ as $t \to \infty$.

Finally, Figure 5 shows that, for this reference motion, the adaptive controller identifies the mass and inertia matrix of the rigid body, in accordance to Proposition 1. Note that the controller needs more time to identify the parameters than to track the reference motion. The same observation has been made in Ref. 5.

For completeness, Figure 6 shows the control force, $\bar{f}^{\mathrm{B}}=[f^{\mathrm{B}}_1\ f^{\mathrm{B}}_2\ f^{\mathrm{B}}_3]^{\mathsf{T}}$, and the control torque, $\bar{\tau}^{\mathrm{B}}=[\tau^{\mathrm{B}}_1\ \tau^{\mathrm{B}}_2\ \tau^{\mathrm{B}}_3]^{\mathsf{T}}$, applied to the body for the same two cases.

VI. Conclusion

An adaptive tracking controller for the relative position and attitude of a rigid body with respect to some desired frame is presented in this paper. The controller requires no information about the mass and inertia matrix of the rigid body. The controller is shown to be *almost* globally asymptotically stable and is

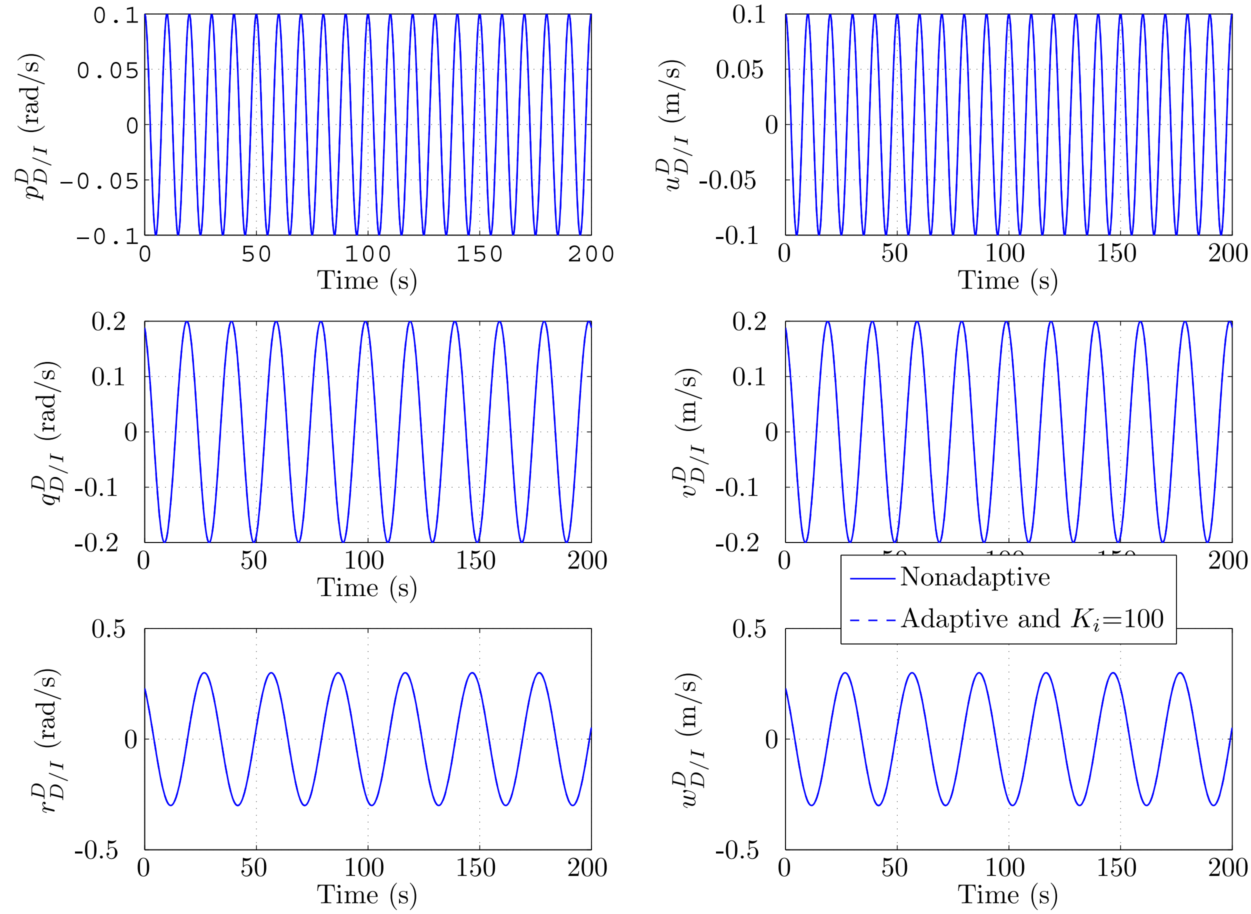

Figure 2. Desired linear and angular velocity expressed in the desired frame.

also capable of identifying the mass and inertia matrix. Conditions on the reference motion are given that guarantee identification. Hence, this controller can be used to asymptotically track time-varying attitude and position profiles when little or no information about the mass and/or inertia matrix of a rigid body is available. In addition, this controller can be used to simply identify the mass and inertia matrix of a rigid body. Future work includes applying the proposed controller to satellite proximity operations, which will require taking into account, for example, the gravitational force, the gravity-gradient torque, and the perturbing force due to Earth's oblateness, which are functions of the body's mass and inertia matrix.

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References

[1] Wen, J. T.-Y. and Kreutz-Delgado, K., "The Attitude Control Problem," *IEEE Transactions on Automatic Control*, Vol. 36, No. 10, October 1991, pp. 1148–1162.

[2] Lizarralde, F. and Wen, J. T., "Attitude Control Without Angular Velocity Measurement: A Passivity Approach," *IEEE Transactions on Automatic Control*, Vol. 41, No. 3, March 1996, pp. 468–472.

[3] Costic, B., Dawson, D., de Queiroz, M. S., and Kapila, V., "Quaternion-Based Adaptive Attitude Tracking Controller Without Velocity Measurements," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 6, November-December 2001, pp. 1214–2429.

[4] Seo, D. and Akella, "High-Performance Spacecraft Adaptive Attitude-Tracking Control Through Attracting-Manifold

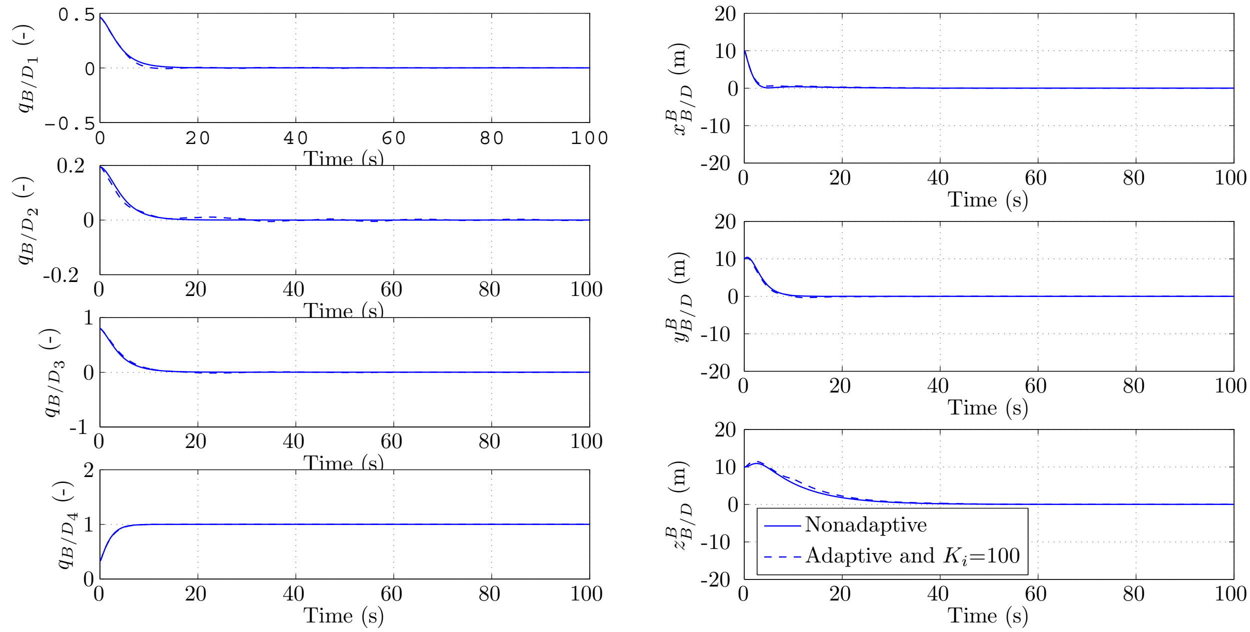

Figure 3. Relative attitude and position.

Design," *Journal of Guidance, Control, and Dynamics*, Vol. 31, No. 4, July-August 2008, pp. 884–891.

[5] Ahmed, J., Coppola, V. T., and Bernstein, D. S., "Adaptive Asymptotic Tracking of Spacecraft Attitude Motion with Inertia Matrix Identification," *Journal of Guidance, Control, and Dynamics*, Vol. 21, No. 5, September-October 1998, pp. 684–691.

[6] Wu, Y., Hu, X., Hu, D., Li, T., and Lian, J., "Strapdown Inertial Navigation System Algorithms Based on Dual Quaternions," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 41, No. 1, January 2005, pp. 110–132.

[7] Filipe, N. and Tsiotras, P., "Simultaneous Position and Attitude Control Without Linear and Angular Velocity Feedback Using Dual Quaternions," *American Control Conference*, Washington, DC, June 17-19 2013, pp. 4815–4820.

[8] Filipe, N. and Tsiotras, P., "Rigid Body Motion Tracking Without Linear and Angular Velocity Feedback Using Dual Quaternions," *European Control Conference*, Zurich, Switzerland, July 17-19 2013.

[9] Pham, H.-L., Perdereau, V., Adorno, B. V., and Fraisse, P., "Position and Orientation Control of Robot Manipulators Using Dual Quaternion Feedback," *IEEE/RSJ International Conference on Intelligent Robost and Systems*, Taipei, Taiwan, October 18–22 2010, pp. 658–663.

[10] Perez, A. and McCarthy, J., "Dual Quaternion Synthesis of Constrained Robotic Systems," *Journal of Mechanical Design*, Vol. 126, No. 3, September 2004, pp. 425–435.

[11] Dooley, J. and McCarthy, J., "Spatial Rigid Body Dynamics using Dual Quaternion Components," *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, Sacramento, California, April 9-11 1991, pp. 90–95.

[12] Han, D., Wei, Q., Li, Z., and Sun, W., "Control of Oriented Mechanical systems: A Method Based on Dual Quaternions," *Proceeding of the 17th World Congress, The International Federation of Automatic Control*, Seoul, Korea, July 6–11 2008, pp. 3836–3841.

[13] Han, D.-P., Wei, Q., and Li, Z.-X., "Kinematic Control of Free Rigid Bodies Using Dual Quaternions," *International Journal of Automation and Computing*, Vol. 5, No. 3, July 2008, pp. 319–324.

[14] Wang, X. and Yu, C., "Feedback Linearization Regulator with Coupled Attitude and Translation Dynamics Based on Unit Dual Quaternion," *2010 IEEE International Symposium on Intelligent Control, Part of 2010 IEEE Multi-Conference on Systems and Control*, Yohohama, Japan, September 8–10 2010, pp. 2380–2384.

[15] Wang, J. and Sun, Z., "6–DOF robust adaptive terminal sliding mode control for spacecraft formation flying," *Acta Astronautica*, Vol. 73, April-May 2012, pp. 676–87.

[16] Gan, D., Liao, Q., Wei, S., Dai, J., and Qiao, S., "Dual quaternion-based inverse kinematics of the general spatial 7R mechanics," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, Vol. 222, No. 8, 2008, pp. 1593–1598.

[17] Daniilidis, K., "Hand-eye calibration using dual quaternions," *The International Journal of Robotics Research*, Vol. 18, No. 3, March 1999, pp. 286–298.

[18] Goddard, J. S., *Pose and Motion Estimation from Vision Using Dual Quaternion-based Extended Kalman Filtering*, Ph.D. thesis, The University of Tennessee, Knoxville, 1997.

[19] Ge, Q. and Ravani, B., "Computer Aided Geometric Design of Motion Interpolants," *Transactions of the ASME*, Vol. 116, No. 3, September 01 1994, pp. 756–762.

[20] Aspragathos, N. and Dimitros, J., "A comparative study of three methods for robotic kinematics," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, Vol. 28, No. 2, April 1998, pp. 135–145.

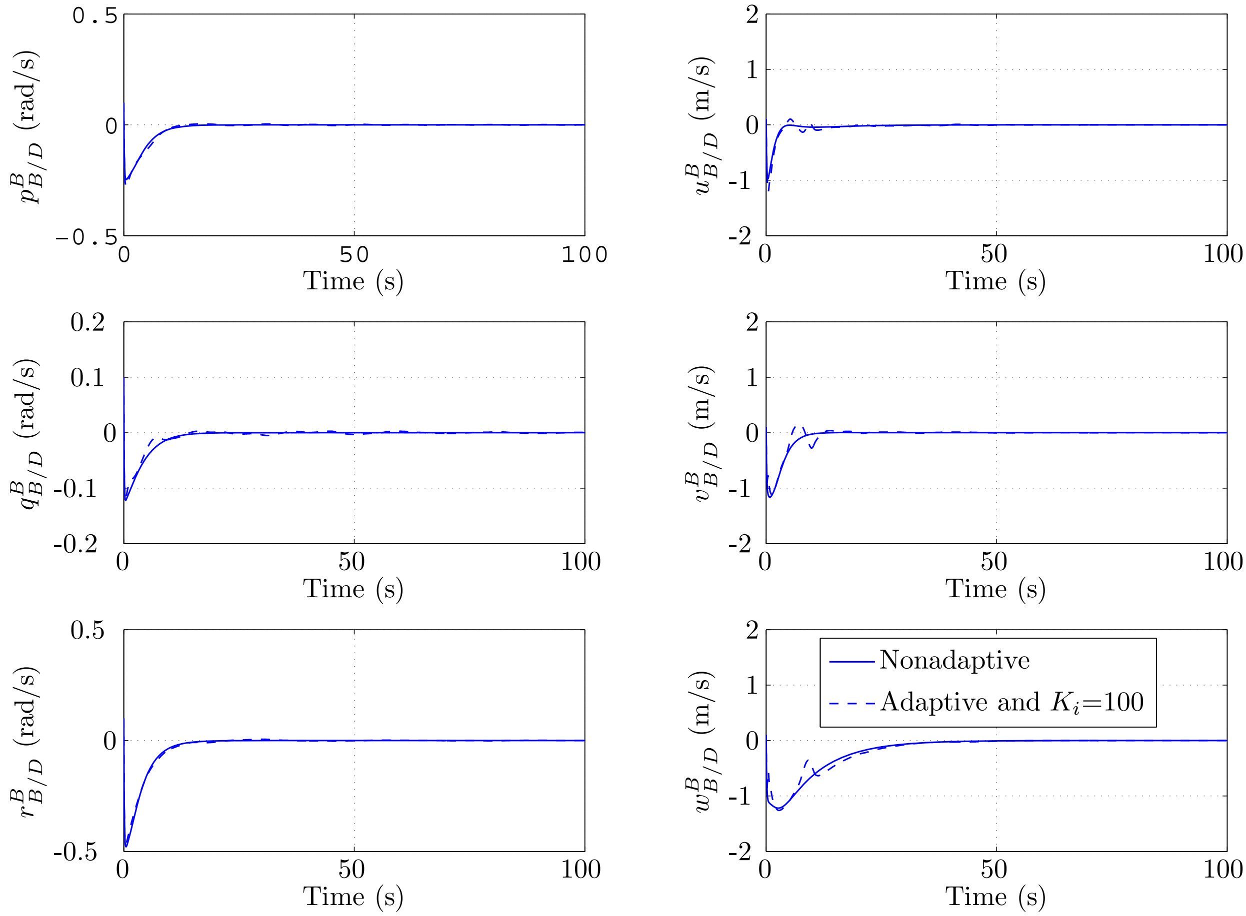

Figure 4. Relative linear and angular velocity expressed in the body frame.

[21]Funda, J., Taylor, R., and Paul, R., "On homogeneous transforms, quaternions, and computational efficiency," *IEEE Transactions on Robotics and Automation*, Vol. 6, No. 3, June 1990, pp. 382–388.

[22]Brodsky, V. and Shoham, M., "Dual numbers representation of rigid body dynamics," *Mechanism and Machine Theory*, Vol. 34, No. 5, July 1999, pp. 693–718.

[23]Bhat, S. P. and Bernstein, D. S., "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, Vol. 39, No. 1, January 28 2000, pp. 63 – 70.

[24]Mayhew, C. G., Sanfelice, R. G., and Teel, A. R., "Robust Global Asymptotic Attitude Stabilization of a Rigid Body by Quaternion-based Hybrid Feedback," *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, Shanghai, P.R. China, December 16-18 2009, pp. 2522–2527.

[25]Thienel, J. K., Luquette, R. J., and Sanner, R. M., "Estimation of Spacecraft Inertia Parameters," *AIAA Guidance, Navigation and Control Conference and Exhibit*, AIAA 2008-6454, Honolulu, Hawaii, August 18-21 2008.

[26]Slotine, J.-J. E. and Li, W., *Applied Nonlinear Control*, Prentice Hall, 1991.

[27]Cho, D.-M., Jung, D., and Tsiotras, P., "A 5-DOF Experimental Platform for Spacecraft Rendezvous and Docking," *AIAA Infotech @ Aerospace Conference*, Seattle, WA, April 6–9 2009, AIAA Paper 2009-1869.

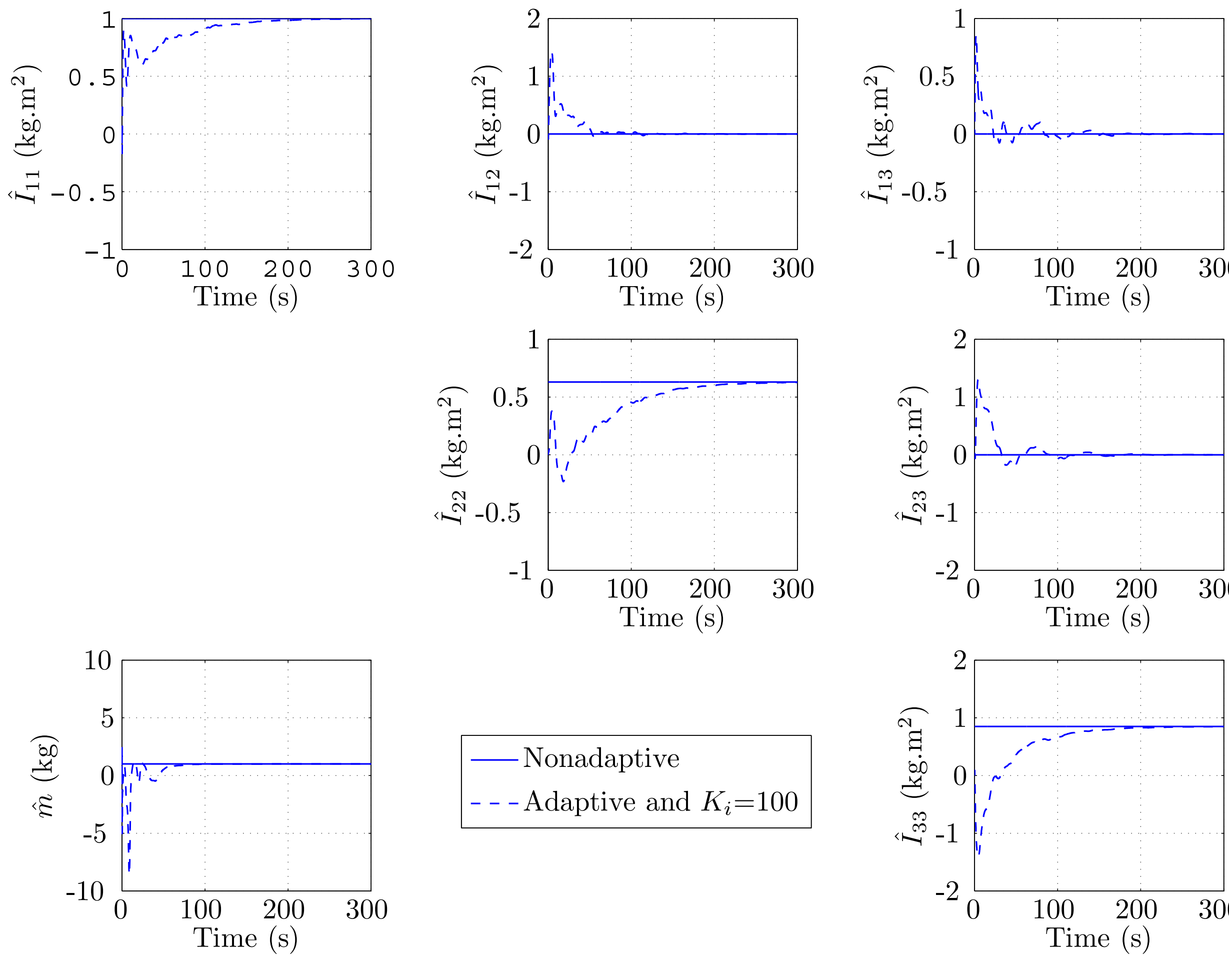

Figure 5. Mass and inertia matrix identification.

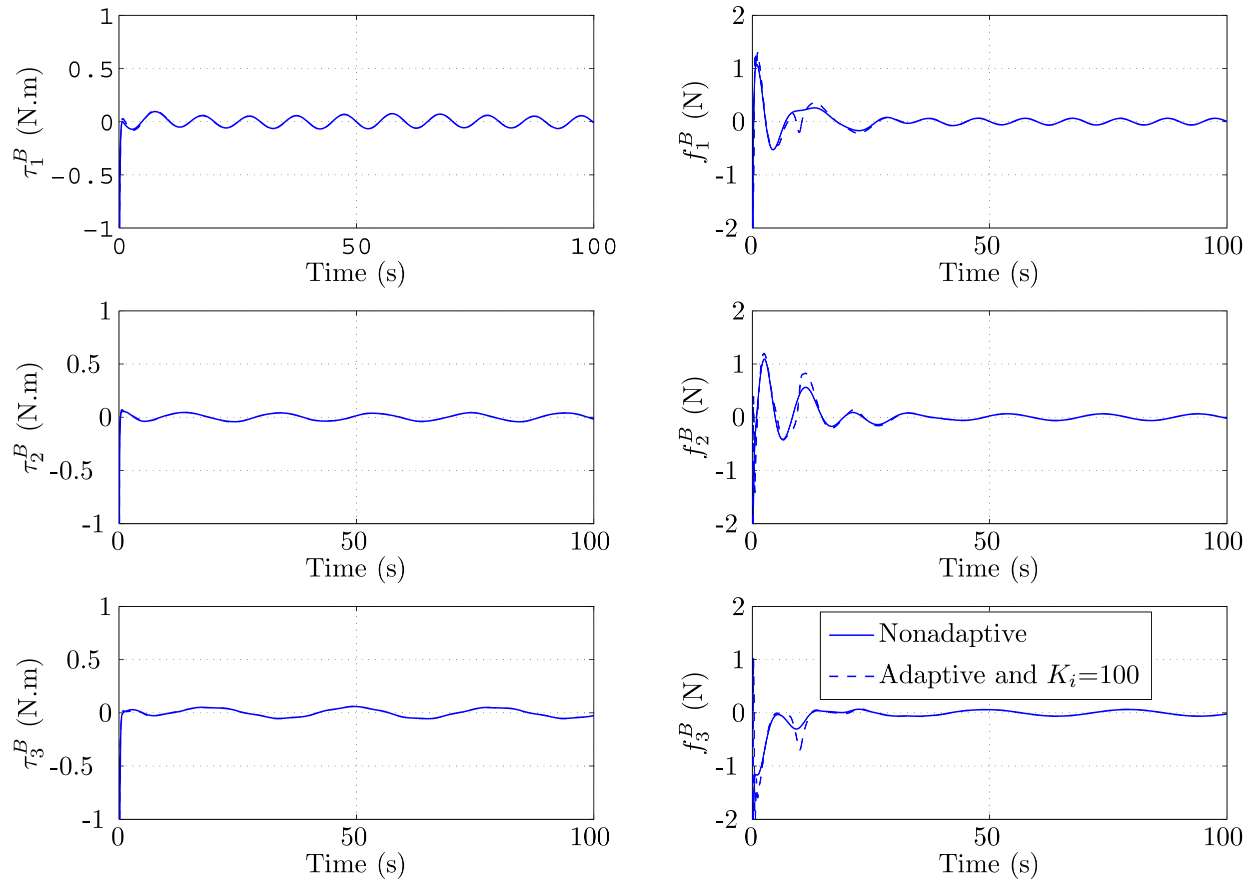

Figure 6. Control force and torque.