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We consider the problem of attitude and angular velocity stabilization of a rigid spacecraft subject to a single actuator failure. Only two pairs of gas jet actuators are available to perform the control objectives. Various controllers have been proposed for this problem in the case that the spacecraft is axisymmetric about one of its axis. In this paper the spacecraft under consideration is a rigid body that is almost axisymmetric about its body 3-axis. A small parameter ε gives a measure of the non-symmetry of the spacecraft about this axis. A control law is introduced for a special subsystem of the complete dynamical system under consideration. This control law can be used either alone for detumbling maneuvers of nearly symmetric spacecraft, or as part of a more general control strategy to stabilize the complete attitude of non-symmetric spacecraft. Numerical examples demonstrate the success of the theoretical developments.

Introduction

The problem of attitude stabilization of a rigid spacecraft has been addressed by numerous papers and articles; see for example, Refs. 1–4. Typically, full control authority is assumed in these results. The problem of attitude stabilization using less than three control torques has only recently received serious attention, starting with the work of Crouch[5] and Byrnes and Isidori[6,7] and later by Krishnan et al.,[8] Tsiotras,[9–11] Morin and Samson,[12] and Coron and Keraï.[13] Apart from the obvious practical importance in case of thruster failure, the spacecraft stabilization problem with less than three control torques is of interest from a theoretical point of view as well, because the linearized system is not stabilizable. Thus, completely nonlinear control methods have to be employed.

The work by Krishnan et al.[8] and later by Tsiotras et al.[9] dealt with the (global) stabilization of an axisymmetric spacecraft for the special case when the initial spin-rate is $\omega_3(0) = 0$ using *time-invariant* feedback control laws. The work of Morin and Samson,[12] and Morin et al.,[14] on the other hand, dealt with the (local) stabilization of a general (non-symmetric) rigid spacecraft (i.e. one with no axis of symmetry) using *time-varying* controllers. Time-varying control laws are used in order to circumvent the topological obstruction to smooth stabilizability due to Brockett's condition.[15] These time-varying control laws are, in fact, periodic and may result to highly oscillatory control (angular velocity) commands. In addition, as it was shown in Ref. 16, any smooth, time-varying control provides only polynomial (not exponential) rates of convergence. Typically, non-smooth, continuous (or even discontinuous) controllers must be used to achieve exponential convergence rates and avoid oscillations. Also, to this date, there does not seem to exist feedback control laws that achieve global asymptotic stability. The global stabilization problem of a non-symmetric spacecraft using, preferably, time-invariant controllers still remains open. Some recent advances have been presented in Ref. 17.

The aim of this paper is to continue the avenue of research started in Refs. 9, 10, 18. These references assumed an axi-symmetric body subject to certain restrictions on the initial angular velocity. Here, these restrictions are removed. The body can be completely non-symmetric, although our controllers work better for bodies with small asymmetries. This is typically the case for actual spacecraft. In particular, we seek *time-invariant* control laws that can stabilize the angular velocity and the attitude of the spacecraft about a certain axis. This may be of interest for the case of a space telescope or a spacecraft carrying a communications antenna. In those cases, three-axis stabilization may not be as important as stabilization of the telescope or the antenna axis in inertial space. For these cases, the control laws in this paper will suffice. In addition, as was recently indicated in Ref. 17 the control laws of Ref. 9 work well even for non-symmetric bodies if the initial condition of the underactuated axis angular velocity is very small (or zero). The control law presented in this paper, if necessary, can be used to achieve this initial elimination of the angular velocity component of the unactuated spacecraft axis. At the same time, the proposed control law reduces the angular displacement of the unactuated axis as well.

The paper is structured as follows. The next section deals with some mathematical preliminaries. In particular, the concept of homogeneity of a function and a vector field with respect to a dilation operator is introduced and some important properties of differential systems with homogeneous rhs are reviewed and discussed. Next, the equations of motion for a rigid spacecraft are presented. The kine-

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matic equations make use of a recent attitude parameterization[19] that allows for an intuitive design of control laws for underactuated spacecraft.[9] This is because this kinematic description leads to a natural separation of the motion of the unactuated axis from the rest of the spacecraft motion. The derivation of the feedback control laws and the proofs of stability are given next. Numerical simulations show that the proposed control laws guarantee asymptotic stability for the closed-loop system. Issues left for future investigation are discussed in the Conclusions.

Mathematical Preliminaries

In this section the concept of homogeneity and some of its properties will be reviewed in order to build the mathematical background for the stability proofs in the subsequent sections.

Definition 1 *Let $\lambda > 0$ and any set of positive scalars $r_i > 0$, $i = 1,\ldots,n$. Then the* dilation operator δ_λ *is defined by*

$$\delta_\lambda(x_1,\ldots,x_n) = (\lambda^{r_1}x_1,\ldots,\lambda^{r_n}x_n)$$

The scalars r_i are called the weights *of the dilation.*

Using the dilation operator we can define the concept of homogeneity.

Definition 2 *A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be* positively homogeneous *of degree k with respect to a given dilation δ_λ if*

$$h(\delta_\lambda(x_1,\ldots,x_n)) = \lambda^k h(x_1,\ldots,x_n)$$

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be homogeneous of degree k *with respect to a given dilation δ_λ if its i^{th} coordinate is a homogeneous function of degree $r_i + k$, i.e.*

$$f^i(\delta_\lambda(x_1,\ldots,x_n)) = \lambda^{r_i+k} f^i(x_1,\ldots,x_n)$$

where f^i denotes the i^{th} component of the vector field f.

Having introduced the concept of homogeneity, it is now possible to state some important properties of homogeneous functions.

Theorem 1 (20) *Let f be a homogeneous vector field of degree k with respect to a given dilation δ_λ and let g be a continuous vector field, both defined on $\mathbb{R}^n$, such that for all $i = 1,\ldots,n$,*

$$\frac{g_i(\delta_\lambda(x_1,\ldots,x_n))}{\lambda^{k+r_i}} \to 0$$

uniformly as $\lambda \to 0$. Then if the trivial solution $x = 0$ of $\dot{x} = f(x)$ is locally asymptotically stable, the same is true for the trivial solution of the perturbed system $\dot{x} = f(x) + g(x)$

Homogeneous systems, defined by homogeneous vector fields have certain appealing properties. The following fact, taken from Ref. 21, along with Theorem 1, justifies our interest in homogeneous systems of degree zero.

Theorem 2 *Let f be a homogeneous vector field of degree zero. Then local asymptotic stability of the origin of $\dot{x} = f(x)$ is equivalent to global exponential stability with respect to the homogeneous norm ρ, defined by $\rho(x) = |x_1^{c/r_1} + x_2^{c/r_2} + \cdots + x_n^{c/r_n}|^{1/c}$ where c is a positive integer evenly divisible by r_i.*

Equations of Motion for Rigid Spacecraft

The rotational motion of a rigid body can be described by Euler's equations of motion. With the assumption that the rigid body has a body-fixed reference frame along its principal axes of inertia with the origin at the center of mass, Euler's equations of motion take the following form

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 + \frac{M_1}{I_1} \tag{1}$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 + \frac{M_2}{I_2} \tag{2}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + \frac{M_3}{I_3} \tag{3}$$

where ω_1, ω_2, ω_3 denote the components of the angular velocity vector with respect to the principal axes, M_1, M_2, M_3 denote the external torques and I_1, I_2, I_3 represent the principle moments of inertia of the rigid body. By assumption, there is no external torque about the 3-axis, i.e., $M_3 = 0$ due to, say, a thruster failure. Define the control torques $u_1 = M_1/I_1$ and $u_2 = M_2/I_2$. For simplicity, let also

$$a = \frac{I_2 - I_3}{I_1}, \qquad \varepsilon = \frac{I_1 - I_2}{I_3}$$

Solving the above equations for I_1 and I_2 leads to an expression in terms of only a, ε and I_3

$$I_1 = \frac{\varepsilon + 1}{1 - a} I_3, \qquad I_2 = \frac{1 + a\varepsilon}{1 - a} I_3$$

Substituting these expressions into equations (1)-(3) leads to following dynamic equations

$$\dot{\omega}_1 = a\omega_2\omega_3 + u_1 \tag{4a}$$

$$\dot{\omega}_2 = -\frac{a + \varepsilon}{1 + a\varepsilon}\omega_2\omega_3 + u_2 \tag{4b}$$

$$\dot{\omega}_3 = \varepsilon\omega_1\omega_2 \tag{4c}$$

For a general rigid body which is almost axisymmetric about the 3-axis, it is $|\varepsilon| \ll 1$. Since $|a| < 1$ and $|\varepsilon| \ll 1$, the term $a\varepsilon$ can be omitted and $1 + a\varepsilon \approx 1$. For the sake of symmetry of the equations another transformation may be performed to yield the dynamic equations in the form

$$\dot{\omega}_1 = (\bar{\varepsilon} + \bar{a})\omega_2\omega_3 + u_1 \tag{5}$$

$$\dot{\omega}_2 = (\bar{\varepsilon} - \bar{a})\omega_3\omega_1 + u_2 \tag{6}$$

$$\dot{\omega}_3 = -2\bar{\varepsilon}\omega_1\omega_2 \tag{7}$$

where $\bar{a} = a + \varepsilon/2$ and $\bar{\varepsilon} = -\varepsilon/2$.

Remark 1 There exists also an "exact" transformation for obtaining equations (5)-(6). However, the resulting expressions for $\bar{a}$ and $\bar{\varepsilon}$ and $\dot{\omega}_3$ are slightly more complicated. They are given by

$$\bar{a} = \frac{1}{2}\left(a + \frac{a+\varepsilon}{1+a\varepsilon}\right), \qquad \bar{\varepsilon} = \frac{1}{2}\left(a - \frac{a+\varepsilon}{1+a\varepsilon}\right) \tag{8}$$

leading to the following expression for $\dot{\omega}_3$

$$\dot{\omega}_3 = -2\frac{\bar{\varepsilon}}{1+\bar{\varepsilon}^2 - \bar{a}^2}\,\omega_1\omega_2$$

A trivial redefinition of the control inputs in Eqs. (4) finally yields the dynamic equations

$$\dot{\omega}_1 = u_1 \tag{9a}$$
$$\dot{\omega}_2 = u_2 \tag{9b}$$
$$\dot{\omega}_3 = \varepsilon\,\omega_1\omega_2 \tag{9c}$$

The variable ε gives a measure of the body asymmetry about its unactuated axis. In case the spacecraft is nearly symmetric about the unactuated axis, $|\varepsilon| \ll 1$. The axisymmetric case corresponds to $\varepsilon = 0$. In this case Eq. (9c) reduces to $\dot{\omega}_3 = 0$ and, without additional assumptions, the system is not controllable at the origin. The symmetric case has been addressed, for example, in Refs. 8, 9, 22. In this paper will assume that $\varepsilon \neq 0$.

For the kinematics we use the following equations

$$\dot{w}_1 = w_2(\omega_2 w_1 + \omega_3) + \frac{1}{2}\omega_1\left(1 + w_1^2 - w_2^2\right) \tag{10a}$$
$$\dot{w}_2 = w_1(\omega_1 w_2 - \omega_3) + \frac{1}{2}\omega_2\left(1 - w_1^2 + w_2^2\right) \tag{10b}$$
$$\dot{z} = \omega_2 w_1 - \omega_1 w_2 + \omega_3 \tag{10c}$$

The reader may refer to Ref. 19 for the physical significance of the kinematic variables w_1, w_2, z and the derivation of these kinematic equations. Suffice it to say that the parameters w_1, w_2 and z offer a parameterization of the group of 3-dimensional rotation matrices and thus, can uniquely describe the attitude of a rigid body.[19,23]

A Feedback Control Law

The dynamical system that has to be controlled can be described by the following differential equations

$$\dot{\omega}_1 = u_1 \tag{11}$$
$$\dot{\omega}_2 = u_2 \tag{12}$$
$$\dot{\omega}_3 = \varepsilon\,\omega_1\omega_2 \tag{13}$$
$$\dot{w}_1 = w_2(\omega_2 w_1 + \omega_3) + \frac{1}{2}\omega_1\left(1 + w_1^2 - w_2^2\right) \tag{14}$$
$$\dot{w}_2 = w_1(\omega_1 w_2 - \omega_3) + \frac{1}{2}\omega_2\left(1 - w_1^2 + w_2^2\right) \tag{15}$$
$$\dot{z} = \omega_2 w_1 - \omega_1 w_2 + \omega_3 \tag{16}$$

One can verify that a standard linearization of this system is not stabilizable. Thus, control design methods based on the linearization of the previous equations are doomed to fail. In the sequel, we use an appropriate dilation operator to derive an alternative local approximation of (11)-(16) that can be used to design asymptotically stabilizing controllers for this system.

Control Law for a Reduced System

Equations (11)-(16) indicate that the angular velocity components of the actuated axes ω_1 and ω_2 can be used as intermediate control inputs for the system (13)-(16). This idea has been used in several papers[9,12,14,18] and will be used here as well. A more rigorous justification of this approach is given later on.

We therefore concentrate on the subsystem of the four differential equations (13)-(16) where ω_1 and ω_2 are the corresponding control inputs. Tsiotras et al.[9] introduced a control law for this subsystem for a rigid body spacecraft that has an axis of symmetry along its body 3-axis. In this case $\varepsilon = 0$ and equation (13) reduces to $\dot{\omega}_3 = 0$. It is easy to see that in this case ω_3 is uncontrollable. Therefore, the authors in Ref. 9 dropped equation (13) and focused their attention on the subsystem given by equations (14)-(16). With the additional assumption that the initial spin-rate is $\omega_3(0) = 0$, they were able to derive a control law that guarantees global asymptotic stability for the (w_1, w_2, z) system.

In contrast to the results of Ref. 9, this paper will focus on the subsystem given by equations (13)-(15), thus omitting the $\dot{z}$-equation. Such a control law can be used for detumbling and stabilization about a certain spacecraft axis (here the body 3-axis), such as for the case of a spacecraft carrying an optical telescope, a laser beam or a dish antenna.

The system under consideration is therefore given by

$$\dot{\omega}_3 = \varepsilon\,\omega_1\omega_2 \tag{17}$$
$$\dot{w}_1 = w_2(\omega_2 w_1 + \omega_3) + \frac{1}{2}\omega_1\left(1 + w_1^2 - w_2^2\right) \tag{18}$$
$$\dot{w}_2 = w_1(\omega_1 w_2 - \omega_3) + \frac{1}{2}\omega_2\left(1 - w_1^2 + w_2^2\right) \tag{19}$$

which can be written in the following compact form

$$\begin{aligned}\begin{bmatrix}\dot{\omega}_3\\ \dot{w}_1\\ \dot{w}_2\end{bmatrix} &= \begin{bmatrix}\varepsilon\omega_1\omega_2\\ \frac{1}{2}\omega_1\\ \frac{1}{2}\omega_2\end{bmatrix} \\ &\quad + \begin{bmatrix}0\\ w_1 w_2\omega_2 + w_2\omega_3 + \frac{1}{2}\omega_1\left(w_1^2 - w_2^2\right)\\ w_1 w_2\omega_1 - w_1\omega_3 + \frac{1}{2}\omega_2\left(w_2^2 - w_1^2\right)\end{bmatrix} \\ &= f(\omega_1,\omega_2) + g(\omega_3, w_1, w_2, \omega_1, \omega_2)\end{aligned} \tag{20}$$

To this end, introduce the following dilation operator

$$\delta_\lambda(\omega_3, w_1, w_2, \omega_1, \omega_2) = \left(\lambda^2\omega_3, \lambda w_1, \lambda w_2, \lambda\omega_1, \lambda\omega_2\right) \tag{21}$$

From Definition 2 it can be easily seen that f is homogeneous of degree zero with respect to the previous dilation operator, since

$$
\begin{aligned}
f_1(\delta_\lambda(\cdot)) = \lambda^2(\varepsilon\omega_1\omega_2) \ &= \ \lambda^2 f_1(\cdot) \\
f_2(\delta_\lambda(\cdot)) = \lambda\left(\tfrac{1}{2}\omega_1\right) \ &= \ \lambda f_2(\cdot) \\
f_3(\delta_\lambda(\cdot)) = \lambda\left(\tfrac{1}{2}\omega_2\right) \ &= \ \lambda f_3(\cdot)
\end{aligned}
$$

where f_i is the i^{th} component of the vector field f. To make the appropriate use of Theorem 1, the homogeneity properties of the vector-valued function g need to be investigated first. To this end, notice that

$$
\begin{aligned}
\lim_{\lambda\to 0}\frac{g_1(\delta_\lambda(\cdot))}{\lambda^2} &= 0 \\
\lim_{\lambda\to 0}\frac{g_2(\delta_\lambda(\cdot))}{\lambda} &= 0 \\
\lim_{\lambda\to 0}\frac{g_3(\delta_\lambda(\cdot))}{\lambda} &= 0
\end{aligned}
$$

where g_i is the i^{th} component of the vector field g. According to Theorem 1, it will be sufficient to find a homogeneous control law of degree at least one, such that the trivial solution for the system

$$
\begin{aligned}
\dot{\omega}_3 &= \varepsilon\,\omega_1\omega_2 & (22a)\\
\dot{w}_1 &= \frac{1}{2}\omega_1 & (22b)\\
\dot{w}_2 &= \frac{1}{2}\omega_2 & (22c)
\end{aligned}
$$

is asymptotically stable. By Theorem 1 asymptotic stability of the trivial solution for the perturbed system (20) follows immediately.

For the sake of simplicity, and without loss of generality, assume that $\varepsilon > 0$. Then the transformation

$$
\begin{gathered}
\omega_1^n = \sqrt{\varepsilon}\,\omega_1, \quad \omega_2^n = \sqrt{\varepsilon}\,\omega_2, \quad \omega_3^n = \omega_3, \\
w_1^n = \sqrt{4\varepsilon}\,w_1, \quad w_2^n = \sqrt{4\varepsilon}\,w_2
\end{gathered}
$$

results in the system

$$
\begin{aligned}
\dot{\omega}_3^n &= \omega_1^n\omega_2^n & (23a)\\
\dot{w}_1^n &= \omega_1^n & (23b)\\
\dot{w}_2^n &= \omega_2^n & (23c)
\end{aligned}
$$

Henceforth, unless otherwise stated, the superscript n will be dropped, bearing in mind that all the subsequent developments actually refer to the transformed system (23).

Control Design

Initially we consider the effect of the linear feedback control

$$
\omega_1 = -w_1, \qquad \omega_2 = -w_2 \qquad (24)
$$

With this control law, one can explicitly integrate (23) to obtain $w_1(t) = w_{10}e^{-t}, w_2(t) = w_{20}e^{-t}$ and $\omega_3(t) = \omega_{30} + \frac{1}{2}w_{10}w_{20}(1-e^{-2t})$. For $t\to\infty$ the trajectories of the closed-loop system yield $\lim_{t\to\infty} w_1(t) = 0, \lim_{t\to\infty} w_2(t) = 0$, and $\lim_{t\to\infty}\omega_3(t) = \omega_{30} + \frac{1}{2}w_{10}w_{20}$. Now let

$$
s = \omega_3 + \frac{1}{2}w_1w_2
$$

and introduce the new tentative control

$$
\begin{aligned}
\omega_1 &= -w_1 + s\,w_2\phi & (25a)\\
\omega_2 &= -w_2 + s\,w_1\phi & (25b)
\end{aligned}
$$

where $\phi = \phi(w_1, w_2, s)$ is a function still to be determined, but positive for all values of w_1,w_2 and s.

The differential equation for s is given by

$$
\begin{aligned}
\dot{s} &= \dot{\omega}_3 + \frac{1}{2}(\dot{w}_1w_2 + w_1\dot{w}_2) \\
&= \omega_1\omega_2 + \frac{1}{2}(\omega_1w_2 + \omega_2w_1) \\
&= -\frac{1}{2}\left(w_1^2 + w_2^2\right)s\phi + w_1w_2s^2\phi^2
\end{aligned}
$$

Proposition 1 *Let the control for the subsystem (23) be given by*

$$
\begin{aligned}
\omega_1 &= -w_1 + s\,w_2\phi \\
\omega_2 &= -w_2 + s\,w_1\phi
\end{aligned}
$$

where $\phi > 0$ and $1 - |s|\phi \geq 0$. This control law ensures that the trajectories of the closed-loop system

$$
\begin{aligned}
\dot{w}_1 &= -w_1 + s\,w_2\phi & (26)\\
\dot{w}_2 &= -w_2 + s\,w_1\phi & (27)\\
\dot{s} &= -\frac{1}{2}\left(w_1^2 + w_2^2\right)s\phi + w_1w_2s^2\phi^2 & (28)
\end{aligned}
$$

remain bounded for all $t \geq 0$.

Proof. Consider the following positive definite and radially unbounded function

$$
V(w_1, w_2, s) = \frac{1}{2}\left(w_1^2 + w_2^2\right) + s^2
$$

The time derivative of V is

$$
\begin{aligned}
\dot{V} &= w_1\dot{w}_1 + w_2\dot{w}_2 + 2s\dot{s} \\
&= w_1(-w_1 + sw_2\phi) + w_2(-w_2 + sw_1\phi) \\
&\quad +2\left[-\frac{1}{2}\left(w_1^2 + w_2^2\right)s\phi + w_1w_2s^2\phi^2\right]s \\
&= -\left(w_1^2 + w_2^2\right) + 2w_1w_2s\phi \\
&\quad -\left(w_1^2 + w_2^2\right)s^2\phi + 2w_1w_2s^3\phi^2
\end{aligned}
$$

Using the following inequalities

$$
\begin{aligned}
2w_1w_2s\phi &\leq \left(w_1^2 + w_2^2\right)|s|\phi \\
2w_1w_2s^3\phi^2 &\leq \left(w_1^2 + w_2^2\right)|s|^3\phi^2
\end{aligned}
$$

it is easy to see that

$$\begin{aligned}\dot V &\le -\left(w_1^2+w_2^2\right)\left(1-|s|\phi+s^2\phi-|s|^3\phi^2\right)\\ &= -\left(w_1^2+w_2^2\right)\left(1+s^2\phi\right)\left(1-|s|\phi\right)\end{aligned}$$

By definition $\phi > 0$, hence $1+s^2\phi > 0$. Since $1-|s|\phi \ge 0$ it follows that the trajectories are bounded. ∎

The previous result shows that the control design hinges upon our ability to find a positive function ϕ such that $1-|s|\phi \ge 0$ for all s, w_1, w_2. We know turn to the choice of this function ϕ.

Proposition 2 *Let the function ϕ defined by*

$$\phi = \frac{\mu}{\sqrt{w_1^4+w_2^4+(\mu s)^2}} \tag{29}$$

where $\mu > 0$. This function will ensure that all the trajectories of the closed-loop system given by equations (26)-(28) are bounded. Moreover, $w_1 \to 0$ and $w_2 \to 0$ as $t \to \infty$.

Proof. Recall that from Proposition 1 all trajectories are bounded if $\phi > 0$ and $1-|s|\phi \ge 0$. Since by definition $\mu > 0$, it is easy to see that the first condition for ϕ is immediately satisfied. To show that the second condition is satisfied, consider the following sequence of inequalities

$$(\mu|s|)^2 \le w_1^4+w_2^4+(\mu s)^2 \tag{30}$$

$$\mu|s| \le \sqrt{w_1^4+w_2^4+(\mu s)^2} \tag{31}$$

$$\frac{\mu|s|}{\sqrt{w_1^4+w_2^4+(\mu s)^2}} \le 1 \tag{32}$$

The last inequality implies that $1-|s|\phi \ge 0$. Thus, for the given function ϕ all trajectories of the closed-loop system will be bounded. Moreover, notice that (30) is a strict inequality unless both w_1 and w_2 are equal to zero. Therefore $\dot V < 0$ unless $w_1 = w_2 = 0$. It follows that $w_1 \to 0$ and $w_2 \to 0$ as $t \to \infty$. ∎

To complete the proof of asymptotic stability with the specific function ϕ given in Proposition 2, only the convergence of s still remains to be shown. Looking again at inequality (30), the only time (30) is not strict is when both w_1 and w_2 are equal to zero. Therefore, the system will be locally asymptotically stable as long as s goes to zero faster than w_1 and w_2. To prove that this is indeed true, we introduce the following ratio

$$\eta = \frac{s}{w_1^2+w_2^2}$$

The derivative of η along the closed-loop trajectories is given by

$$\begin{aligned}\dot\eta &= \frac{\dot s}{w_1^2+w_2^2} - \frac{s}{w_1^2+w_2^2}\frac{1}{w_1^2+w_2^2}\frac{\mathrm{d}}{\mathrm{d}t}(w_1^2+w_2^2)\\ &= -\frac{1}{2}s\phi + \frac{w_1w_2}{w_1^2+w_2^2}s^2\phi^2\\ &\quad -2\eta\frac{1}{w_1^2+w_2^2}(w_1\dot w_1+w_2\dot w_2)\\ &= -\frac{1}{2}s\phi\left(1-\frac{2w_1w_2}{w_1^2+w_2^2}s\phi\right)\\ &\quad +2\eta\left(1-\frac{2w_1w_2}{w_1^2+w_2^2}s\phi\right)\\ &= -\frac{1}{2}(s\phi-4\eta)\left(1-\frac{2w_1w_2}{w_1^2+w_2^2}s\phi\right)\end{aligned} \tag{33}$$

Since $1-|s|\phi \ge 0$ and $|2w_1w_2| \le w_1^2+w_2^2$ we have that

$$\left(1-\frac{2w_1w_2}{w_1^2+w_2^2}s\phi\right) \ge 0 \tag{34}$$

Notice now that

$$s\phi - 4\eta = \eta\left(\mu\frac{w_1^2+w_2^2}{\sqrt{w_1^4+w_2^4+(\mu s)^2}} - 4\right)$$

Since $\sqrt{w_1^4+w_2^4} \le w_1^2+w_2^2$ one obtains that

$$\frac{w_1^2+w_2^2}{\sqrt{w_1^4+w_2^4+(\mu s)^2}} = \frac{1}{\sqrt{\frac{w_1^4+w_2^4}{(w_1^2+w_2^2)^2}+\mu^2\eta^2}} \ge \frac{1}{\sqrt{1+\mu^2\eta^2}}$$

A straightforward calculation then shows that if

$$|\eta(0)| \le \frac{\sqrt{\mu^2-16}}{4\mu} \tag{35}$$

then η remains bounded whenever $\mu > 4$. Since $w_1^2+w_2^2 \to 0$, it follows immediately that $s \to 0$ as $t \to \infty$.

The above derivation can now be stated formally as follows.

Proposition 3 *Let the subsystem (23) and the control law*

$$\omega_1 = -w_1 + sw_2\phi, \qquad \omega_2 = -w_2 + sw_1\phi \tag{36}$$

where $\phi = \mu/\sqrt{w_1^4+w_2^4+(\mu s)^2}$ and $\mu > 4$. This control law asymptotically stabilizes (23) for all initial conditions such that $|\eta(0)| \le \sqrt{\mu^2-16}/4\mu$.

Corollary 1 *With the control law (36), the trajectories of the closed-loop system (17)-(19) converge to the origin for initial conditions close to the origin.*

Proof. The proof follows directly from Theorem 1 and the fact that the proposed control law is homogeneous of degree one with respect to the dilation δ_λ. ∎

Remark 2 It should be clear that the control law in Eq. (36) requires that $w_1(0)^2 + w_2(0)^2 \neq 0$. In addition, the calculation of μ from condition (35) may be too restrictive. A better way to handle the case of small $w_1(0)$ and $w_2(0)$ is to use a preliminary control law such that at some later time (35) is satisfied. This can always be achieved using, for example, $\omega_1 = 0$ and $\omega_2 = w_2$ or $\omega_1 = w_1$ and $\omega_2 = 0$.

Remark 3 Since the initial conditions that satisfy (35) do not form an open neighborhood or the origin, our use of the term "stability" here is – strictly speaking – incorrect. Nonetheless, we appeal to the intuitive understanding of the reader in order to avoid the introduction of additional terminology (e.g., attractivity, convergence, etc.).

Dynamic Extension

Thus far, we have considered only the subsystem (13)-(15) and the control design was done at the kinematic level. The actual control inputs u_1 and u_2 can be constructed by noticing that the system in (11)-(15) falls into the general class of nonlinear systems of the following cascade form

$$\begin{aligned} \dot{y} &= v \\ \dot{x} &= f(x,y) \end{aligned}$$

The following Theorem, taken from Ref. 20, considers the stabilization of systems in the previous form.

Theorem 3 (20) *Let f be a continuous vector field, homogeneous of degree k with respect to a given dilation δ_λ, and assume that the system $\dot{x} = f(x,u)$ is locally asymptotically stabilizable with a continuous feedback $u : \mathbb{R}^n \to \mathbb{R}^m$ such that for all $\lambda > 0$ and some set of positive scalars $r_i > 0$, $i = 1, \ldots, n+1$*

$$u(\delta_\lambda(x_1, \ldots, x_n)) = \lambda^{r_{n+1}} u(x_1, \ldots, x_n)$$

then the system

$$\begin{aligned} \dot{y} &= v \\ \dot{x} &= f(x,y) \end{aligned}$$

is globally asymptotically stabilizable with a continuous feedback law $v : \mathbb{R}^{n+1} \to \mathbb{R}^m$ such that for all $\lambda > 0$

$$v(\delta_\lambda(x_1, \ldots, x_n), \lambda^{r_{n+1}} y) = \lambda^{k+r_{n+1}} v(x_1, \ldots, x_n, y)$$

According to Theorem 3, if the system $\dot{x} = f(x,y)$ can be locally asymptotically stabilized with a homogeneous feedback y, then the extended system will be asymptotically stabilizable using a homogeneous feedback v. Such a control law can be easily constructed using standard techniques; see, for instance, Refs. 12, 17, 18, 24–26.

Numerical Example

To illustrate the results developed previously, a numerical example is presented next. The following initial conditions are used $w_1(0) = 4$, $w_2(0) = 1$, $z(0) = 1$, $\omega_3(0) = -0.5$ r/s. The asymmetry parameter is chosen as $\varepsilon = 0.2$ and the controller gain is $\mu = 7$. The results are shown in Figs. 1-7. These simulations show that the proposed control law locally asymptotically stabilizes the system given by equations (17)-(19). In addition, the control commands ω_1 and ω_2 remain bounded and have bounded time derivative. The second requirement is important in order to implement the kinematic controllers through the integrators (9a)-(9b).

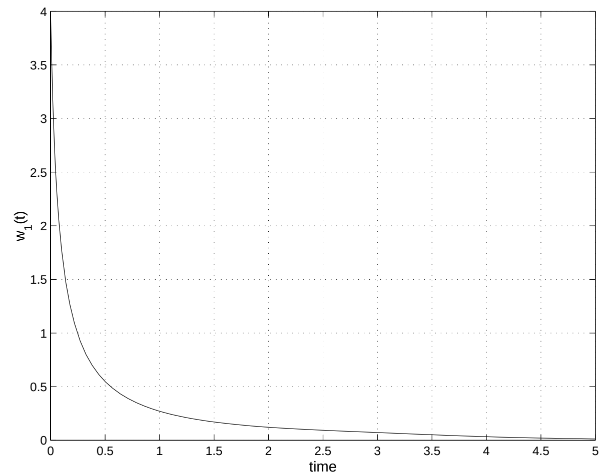

Fig. 1 Time history of w_1 ($\mu = 7$, $\varepsilon = 0.2$)

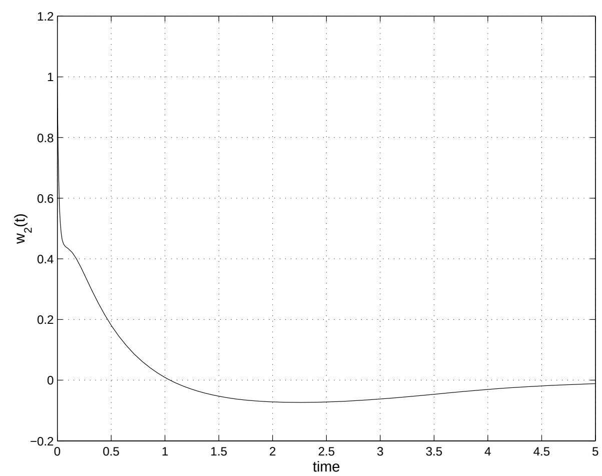

Fig. 2 Time history of w_2 ($\mu = 7$, $\varepsilon = 0.2$)

Conclusions

In this paper the problem of angular velocity and attitude stabilization of a nonsymmetric rigid spacecraft is addressed. Only a subset of the complete equations is stabilized. The design methodology uses the homogeneity properties of the original open loop system to obtain a suitable approximation of this system. Although the results prove local asymptotic stability, numerical simulations show that the region of attraction for the proposed control law can be potentially quite large. Future research in this area might provide an estimation of the region of attraction for this controller, as well as a stabilizing control law for the complete system. The possibility of combining the control law

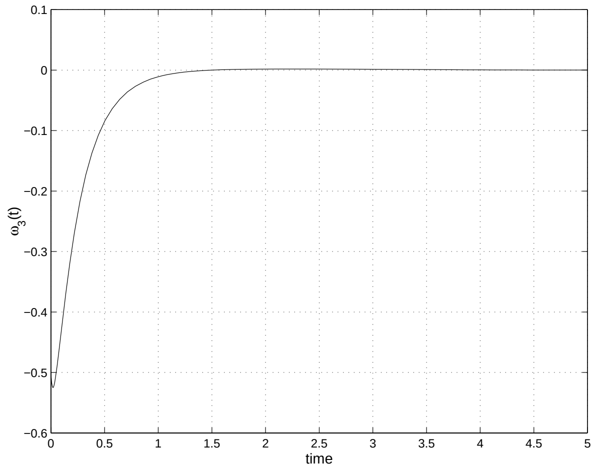

Fig. 3 Time history of ω_3 ($\mu = 7$, $\varepsilon = 0.2$)

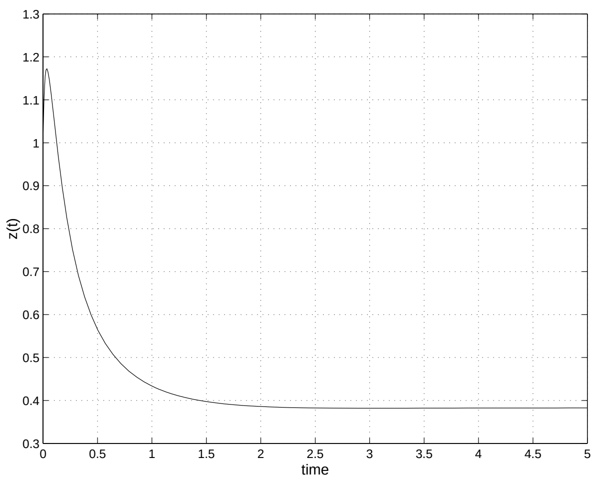

Fig. 4 Time history of z ($\mu = 7$, $\varepsilon = 0.2$)

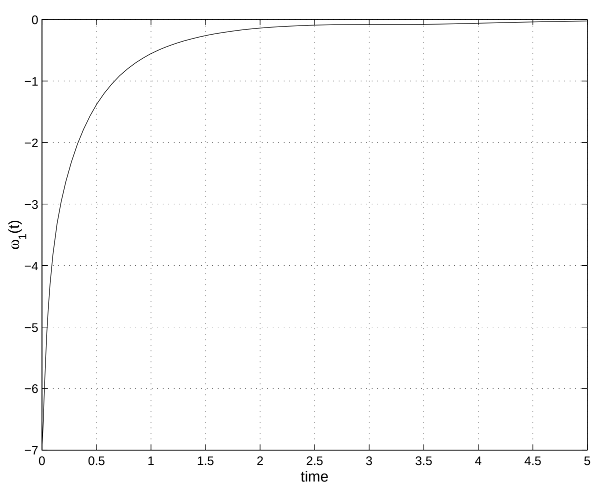

Fig. 5 Time history of control input ω_1 ($\mu = 7$, $\varepsilon = 0.2$)

derived in this paper with the controller derived by Tsiotras et al.[9] is appealing. For instance, the control law in Ref. 9 has been shown to be pretty robust to small asymmetries of the spacecraft as long as the initial angular velocity ω_3 is zero. The control law derived herein can be used to achieve this initial detumbling maneuver.

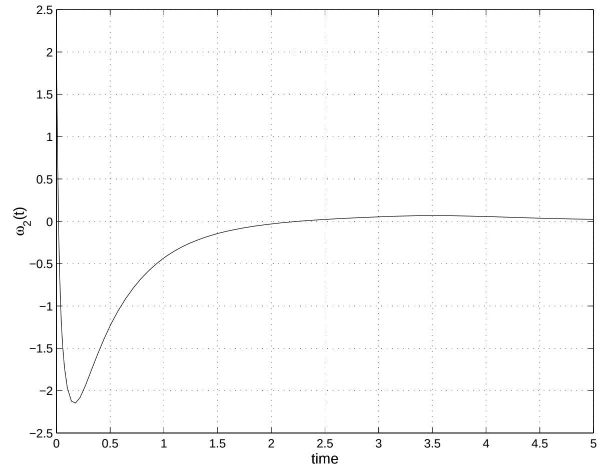

Fig. 6 Time history of control input ω_2 ($\mu = 7$, $\varepsilon = 0.2$)

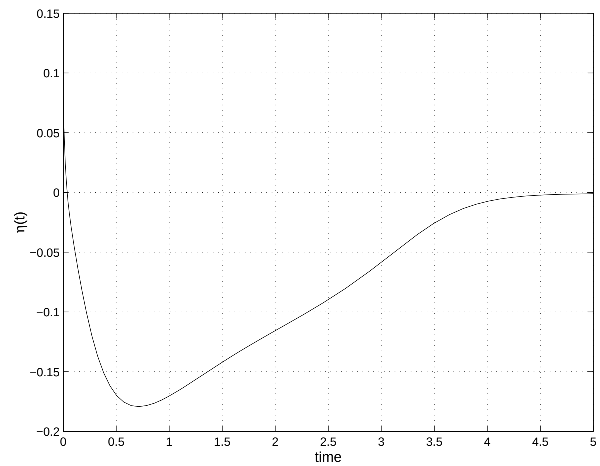

Fig. 7 Time history of the ratio $\eta = s/(w_1^2 + w_2^2)$ ($\mu = 7$, $\varepsilon = 0.2$)

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