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NEW KINEMATIC RELATIONS FOR THE LARGE ANGLE PROBLEM IN RIGID BODY ATTITUDE DYNAMICS†

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Abstract—For problems involving rotating rigid bodies (e.g. spin-stabilized satellites in space) one usually linearizes the equations of motion for the Eulerian angles to obtain tractable analytic solutions. However, these methods—based on a small angle assumption—fail to provide a comprehensive treatment of the behavior of a rigid body during large angular motions that occur, for example, during despinning. For such cases the nonlinear effects dictate a more sophisticated analysis. In this paper we discuss three different approaches for this class of problems. The first approach develops a quadratic equation based on a new formulation of the Eulerian angle kinematics. The second method uses a Riccati equation, derived from stereographic projection of the classical direction cosine formulation for the kinematics. The third method, based on a reformulation of the quaternion equations along with a known result from the theory of Lie algebras, derives an explicit, closed-form solution for the associated system of linear, time-varying differential equations. Numerical simulations demonstrate the validity of all three kinematic formulations.

1. INTRODUCTION

Rockets and spacecraft are often spun up to provide stability. When for some reason the spin rate decreases (e.g. though a spin-down maneuver), this stabilizing effect diminishes and, in the presence of transverse torques, the vehicle is subject to large angular displacements. In general, similar behavior is observed when large body-fixed torques, including transverse torques, are present. In such cases, attitude solutions based on small angle assumptions are no longer valid. For large angular displacements the nonlinearities play the predominant role, and a more sophisticated theory needs to be developed to handle such cases. In this paper three possible methods are presented to tackle the problem of large angles in rigid body attitude dynamics.

The first method, based on the Eulerian angle formulation, uses a nonlinear transformation that enables one to reformulate the kinematics of two of the Eulerian angles into a convenient, complex-valued differential equation, which we refer to as the *quadratic kinematic equation*. The transformation is not restricted to small angle assumptions and yields a system of two nonlinear differential equations, for its real and imaginary parts, that gives the *exact* solution to the attitude problem. The advantage of the transformation in this setting lies in the fact that the nonlinearities of the transformed equation are polynomial in nature, whereas the original equations involve trigonometric nonlinearities. As a result, the transformed equation is more tractable using classical series, perturbation or successive approximation techniques.

Secondly, an old but relatively unknown method due to Darboux [1] is used to reformulate the attitude problem as the solution of a single but complex *Riccati equation* that governs the attitude of a rotating body in space. Although equations of this form are often encountered in classical differential geometry to describe the orientation of a moving trihedral along a rectifiable curve (in terms of the Frenet formulas and direction cosines), nevertheless, its use in the rotating rigid body problem has been extremely limited. It is shown how this formulation is related to the previous method by a simple transformation, although the derivations of the two are completely independent of one another. A procedure for the solution of both formulations, based on the method of successive approximations, is briefly discussed. This procedure of solution gives very acurate results, at least for the class of problems where the linearized version of the equation gives a reasonably good first order approximation of the solution.

The third method uses an approach based on *quaternions* and their counterpart, the *Euler parameters*. It is well known that the quaternion formulation leads to a set of linear differential equations, but because of the time-varying nature of the coefficient matrix, analytic procedures do not fair well. A semi-analytic solution based on Picard's method of the *product integral* (also called the *time-ordered exponential*) is presented, that allows one to find approximate solutions to this system of linear time-varying differential equations. The methodology for the solution

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works for every linear time-varying system of differential equations, but is especially suited for problems of rotational kinematics, where the special structure of the state matrix allows the computation of its exponential in *closed form*. (This is not necessarily true for general time-varying state equation matrices.) Due to the iterative nature of the solution however, this approach has short term validity. The applicability of the Baker–Campbell–Hausdorff formula—appearing in the theory of infinitesimal generators of one-parameter subgroups of Lie groups—is discussed as an approach to overcome this limitation and to extend the validity of the approximation over longer time intervals.

2. PARAMETRIZATIONS OF THE ROTATION GROUP

The set of matrices that relate two arbitrary reference frames form what is commonly known as the (three-dimensional) rotation group. This group consists of all the matrices that are orthogonal and have determinant $+1$. This group is also known as the (three-dimensional) *special orthogonal group* and it is commonly denoted by SO(3). We therefore write that $SO(3) = \{M \in GL(3): M^T M = I \text{ and } \det(M) = +1\}$, where GL(3) denotes the general linear group of order 3, i.e. the group of all nonsingular 3×3 matrices. In this section we will concentrate on rotation matrices that describe the orientation of the body-fixed reference frame described by the unit vectors $\{\hat{\mathbf{b}}\} \triangleq \{\hat{b}_1, \hat{b}_2, \hat{b}_3\}$ with respect to the inertial reference frame described by the unit vectors $\{\hat{\mathbf{n}}\} \triangleq \{\hat{n}_1, \hat{n}_2, \hat{n}_3\}$, i.e. for some $R \in SO(3)$ we have that $\{\hat{\mathbf{n}}\} = R\{\hat{\mathbf{b}}\}$. The matrix R therefore describes the relative orientation between the reference frames $\{\hat{\mathbf{b}}\}$ and $\{\hat{\mathbf{n}}\}$ and is varying with time since it depends on the angular velocity vector between the two reference frames. Each possible orientation corresponds to an element of the rotation group SO(3), which we may view as the configuration space for all non-trivial rotational motions of the body. Henceforth, we will refer to SO(3) simply as *the rotation group*. In fact it is well known [2] that SO(3) is more than simply a group, but carries an inherent smooth manifold structure, and thus forms a (continuous) Lie group. We will not exploit the Lie group structure of the rotation group until later on, when we discuss the applicability of the Baker–Campbell–Hausdorff formula for approximating elements of a Lie group by the exponential map.

There is more than one way to parametrize the rotation group, i.e. to specify a set of parameters such that an element in SO(3) is uniquely and unambiguously determined. Different parametrizations of the rotation group correspond to the well-known alternatives of solving for the relative attitude history between two reference frames: direction cosines, Euler parameters, Eulerian angles, etc. Although Hopf [3] showed that five is the minimum number of parameters which suffice to represent the rotation group in a 1–1 global manner, the so-called "quaternion method" (to be discussed later in Section 5) of parametrizing the group in a 1–2 way, using four parameters, is sufficient for practical purposes. This four-dimensional parametrization is the lowest order singularity-free parametrization of SO(3). It is well known that the commonly used three-dimensional parametrization of the Eulerian angles leads to singular points for the rotation group, i.e. equations that exhibit singularities for certain orientations. Nevertheless, the use of Eulerian angles has survived until today, mainly because they represent physical quantities that are amenable to engineering insight. That is, the Euler angles themselves provide a useful output, whereas with the quaternion method it is necessary to transform the solution after integrating. In this work we are interested in solving the kinematic equations associated with the three-dimensional Eulerian angle, and the four-dimensional quaternion parametrizations of the rotation group SO(3). For an exposition on the complete parametrization of SO(3) one may consult [4].

3. TRANSFORMATION TECHNIQUES FOR EULERIAN ANGLES

For many problems involving rotating rigid bodies (e.g. spin-stabilized satellites in space), one often makes the assumption that the body spin axis does not deviate much from its original direction. In such cases, and for an appropriately chosen set of three Euler angles, to be defined shortly, one can simplify the kinematic equations relating the three Eulerian angles with the components of the angular velocity vector. One thus obtains a simplified system of differential equations that can be used for analytic studies [5–7]. For example, if one wants to analyze the motion of a spin-stabilized body about its z-axis, then for a 3–1–2 Eulerian angle sequence, the angles β_x and β_y describe the attitude deviation of the spin axis from its initial orientation (assumed to be the inertial Z-axis). These angles represent unwanted deviation of the spin axis caused by application of disturbances and are typically small (see Fig. 1). In fact the complex angle $\beta \triangleq \beta_x + i\beta_y$ represents a measure of the total deviation of the spin axis and is often referred to as the attitude "error angle". According to the previous discussion, a small angle assumption for β_x and β_y is quite reasonable for this particular problem and, thus, can be used to simplify the equations.

Recall that there are 12 different sets of angles that can be used to describe the orientation of a rigid body. Not all of the choices are equivalent for analytic representations of solutions, and the choice of the particular set of angles should be decided according to the relevance to the problem at hand. For a spin-stabilized vehicle for instance, the 3–1–2 system is different from the 3–1–3 system of Eulerian angles

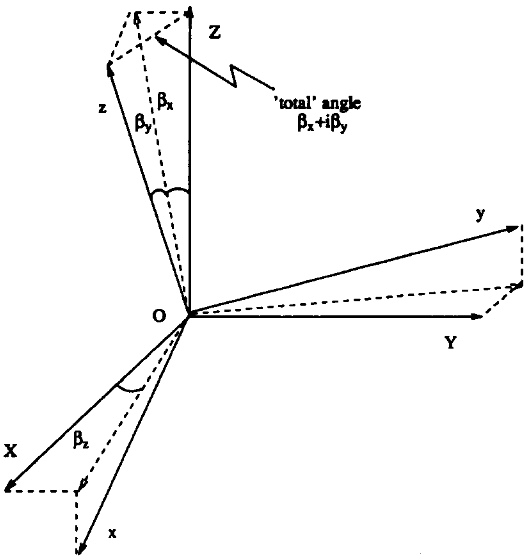

Fig. 1. Eulerian angle sequence 3–1–2 and attitude error components.

in the sense that the first set is composed of two small and one large rotation, whereas the second set is composed of two large rotations and a small one (Fig. 2). Therefore, if one approximates the true motion by linearization, as is often the case, the resulting equations derived are far more simple using the 3–1–2 system than using the 3–1–3 system, for this particular problem. This does not imply however that a description of the kinematic equations by an alternative set of Eulerian angles is fruitless. In fact, as we will show, one can use the interplay between different sets of Euler angles to directly derive alternative formulations for the kinematics that can be very helpful in the development of analytic solutions.

3.1. *Kinematic equations in terms of Eulerian angles*

The kinematic equations for a 3–1–2 Eulerian angle sequence that relate the Eulerian angles and their

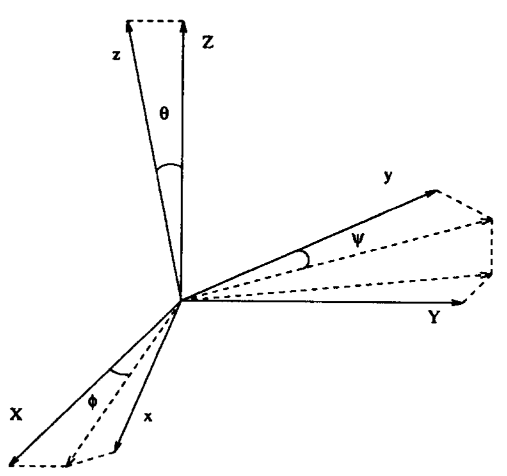

Fig. 2. Eulerian angle sequence 3–1–3.

rates to the components of the angular velocity vector, expressed in a body-fixed frame, are given by [8]

$$\dot\beta_x = \omega_x \cos\beta_y + \omega_z \sin\beta_y \tag{1a}$$

$$\dot\beta_y = \omega_y - (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\tan\beta_x \tag{1b}$$

$$\dot\beta_z = (\omega_z \cos\beta_y - \omega_x \sin\beta_y)\sec\beta_x. \tag{1c}$$

Any attempt to solve these equations directly, for arbitrary ω_x, ω_y and ω_z is futile. It is clear however from eqn (1) that β_z is an ignorable variable. The decoupling of β_z from β_x and β_y means that if one knows the solution for the latter two, one can immediately compute the solution for β_z by a simple quadrature. Therefore, one can merely concentrate on solving for the Eulerian angles β_x and β_y from eqn (1a) and (1b). According to the previous discussion, a small angle approximation of β_x, β_y is quite reasonable if eqn (1) describe the attitude evolution of a spin-stabilized (about its z-axis) rigid body, and therefore, together with the assumption that the product $\beta_y\omega_x$ in eqn (1c) is small compared to ω_z (as is usually the case for spin-stabilized bodies), the system of eqn (1) reduces to

$$\dot{\hat\beta}_x = \omega_x + \hat\beta_y\omega_z \tag{2a}$$

$$\dot{\hat\beta}_y = \omega_y - \hat\beta_x\omega_z \tag{2b}$$

$$\dot{\hat\beta}_z = \omega_z. \tag{2c}$$

The caret denotes the solution to the linear problem (2), in order to distinguish from the exact solution given by the system of eqn (1). Again, because of the decoupling of $\hat\beta_z$ from $\hat\beta_x$ and $\hat\beta_y$, one can concentrate on solving (2a) and (2b). Using the complex notation introduced by Tsiotras and Longuski [7], one writes these two equations in the following single complex equation for the linearized transverse Eulerian angles $\hat\beta_x$ and $\hat\beta_y$

$$\dot{\hat\beta} + i\omega_z\hat\beta = \omega \tag{3}$$

where $\hat\beta \triangleq \hat\beta_x + i\hat\beta_y$ and $\omega \triangleq \omega_x + i\omega_y$. Notice that (3) is a linear differential equation, the solution of which can be written immediately in terms of quadratures. The error between the linearized and original solutions $\beta_e \triangleq \beta - \hat\beta$ will be of course relatively small, as long as the angles β_x and β_y remain within the realm of the linear approximation. As mentioned earlier, this error is surely small for the case of a spin-stabilized body, thus justifying its terminology. It is not necessarily so, however, when for some reason the stabilizing effect of the axial spin ceases to exist (during a spin-down maneuver, for example), and as a result the body z-axis tends to depart from its initial orientation, giving rise to large values for the angles $\hat\beta_x$ and $\hat\beta_y$. The problem has entered the region of nonlinearity, as is vividly demonstrated in Fig. 3, and a more comprehensive method is needed to solve for the true attitude motion of the body. Figure 3 shows the result of a spin-down maneuver through zero spin rate, in the presence of constant transverse body-fixed torques, for a typical spacecraft [7]. As a first step to

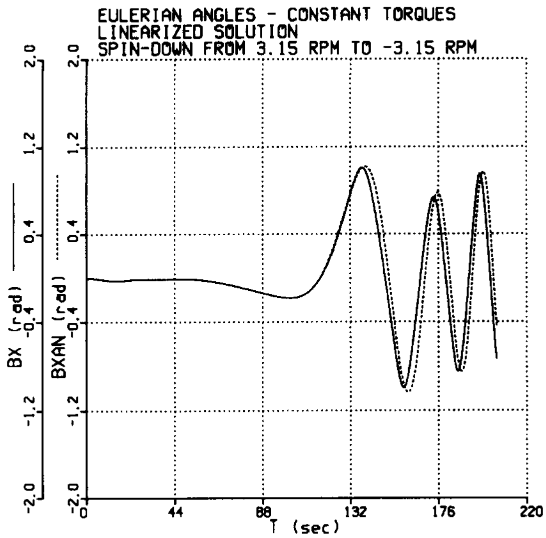

Fig. 3. Exact and analytic solution for β_x.

alleviate this problem we look to connections between different sets of Eulerian angle parametrizations of the rotation group. By looking into different sets of angles one gains valuable insight into the problem. Moreover, as often happens, the problem solution depends on the particular set used to formulate the kinematics. We exploit these ideas below.

where, as before, $\omega = \omega_x + i\omega_y$. Eliminating $\dot{\phi}$ from (6) with the help of (5c), the previous equation reduces to

$$\omega = [\dot{\theta} + i(\omega_z - \dot{\psi})\tan\theta]\exp(-i\psi). \tag{7}$$

If we now define the complex quantity

$$\alpha \triangleq \tan\theta \exp(-i\psi) \tag{8}$$

it is not difficult to verify that α obeys the new *quadratic kinematic equation*

$$\dot{\alpha} + i\omega_z\alpha = \omega + \mathrm{Re}(\bar{\omega}\alpha)\alpha \tag{9}$$

where the bar denotes complex conjugate. In terms of the real and imaginary parts of $\alpha = \alpha_x + i\alpha_y$ one has equivalently that

$$\dot{\alpha}_x = \omega_z\alpha_y + \omega_x + \omega_x\alpha_x^2 + \omega_y\alpha_x\alpha_y \tag{10a}$$

$$\dot{\alpha}_y = -\omega_z\alpha_x + \omega_y + \omega_y\alpha_y^2 + \omega_x\alpha_x\alpha_y. \tag{10b}$$

Solution of eqn (9) for α determines the two angles θ and ψ from (8) or equivalently from $\alpha_x = \tan\theta\cos\psi$ and $\alpha_y = -\tan\theta\sin\psi$. The corresponding angles in the 3–1–2 set, i.e. β_x and β_y, are given by the natural identification of the corresponding parametrizations of the rotation group. Using the parametrizations of the rotation group for the two sets of Eulerian angles, one has the following expressions for a typical element of SO(3):

$$R(\beta_z, \beta_x, \beta_y) = \begin{bmatrix} -s\beta_z s\beta_x s\beta_y + c\beta_z c\beta_y & c\beta_z s\beta_x s\beta_y + s\beta_z c\beta_y & -c\beta_x s\beta_y \\ -s\beta_z c\beta_x & c\beta_z c\beta_x & s\beta_x \\ s\beta_z s\beta_x c\beta_y + c\beta_z s\beta_y & -c\beta_z s\beta_x c\beta_y + s\beta_z s\beta_y & c\beta_x c\beta_y \end{bmatrix} \tag{11a}$$

$$R(\phi, \theta, \psi) = \begin{bmatrix} c\psi c\phi - s\phi c\theta s\psi & c\psi s\phi + c\phi c\theta s\psi & s\theta s\psi \\ -s\psi c\phi - s\phi c\theta c\psi & -s\psi s\phi + c\phi c\theta c\psi & s\theta c\psi \\ s\phi s\theta & -c\phi s\theta & c\theta \end{bmatrix} \tag{11b}$$

3.2. *A new quadratic kinematic equation*

Let us consider the 3–1–3 set of Eulerian angles which obeys the following set of differential equations [8]:

$$\dot{\theta} = \omega_x\cos\psi - \omega_y\sin\psi \tag{4a}$$

$$\dot{\psi} = \omega_z - (\omega_x\sin\psi + \omega_y\cos\psi)/\tan\theta \tag{4b}$$

$$\dot{\phi} = (\omega_x\sin\psi + \omega_y\cos\psi)/\sin\theta. \tag{4c}$$

Upon inverting these equations for ω_x, ω_y, ω_z one obtains

$$\omega_x = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{5a}$$

$$\omega_y = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{5b}$$

$$\omega_z = \dot{\psi} + \dot{\phi}\cos\theta. \tag{5c}$$

From (5a) and (5b) we get that

$$\omega = (\dot{\theta} + i\dot{\phi}\sin\theta)\exp(-i\psi) \tag{6}$$

for the 3–1–2 and 3–1–3 sets, respectively with c denoting cos and s denoting sin. Since (11a) and (11b) are different parametrizations of the same element of SO(3) we have, by comparing corresponding entries of the matrices, that

$$\sin\beta_x = \sin\theta\cos\psi, \quad \tan\beta_y = -\sin\psi\tan\theta$$

$$\tan\beta_z = \frac{\sin\phi\cos\theta\cos\psi + \sin\psi\cos\phi}{\cos\phi\cos\theta\cos\psi - \sin\psi\sin\phi}. \tag{12}$$

The previous equations are the standard relationships that provide the exact transformations between the 3–1–2 and 3–1–3 sets of Eulerian angles. One therefore also easily establishes the following relations between α_x, α_y and the Eulerian angles β_x and β_y:

$$\tan\beta_y = \alpha_y, \quad \tan\beta_x = \alpha_x\cos\beta_y. \tag{13}$$

The last two equations along with eqn (12) can be used to transform back and forth between the different sets of parameters.

Equation (10) is a system of *exact* differential equations that describes the kinematics of the two Eulerian angles β_x and β_y (in the 3–1–2 set), or θ and ψ (in the 3–1–3 set); the angles β_z or ϕ are ignorable variables in both sets and can be computed by quadrature once the other two angles are known. That is, the solution of (10) along with (8) or (13) give the exact solution to the differential eqns (1a) and (1b) or (4a) and (4b).

At first glance, it seems that no great improvement has been achieved by transforming to the new set of differential eqn (10). However this is not so, because the system of eqn (10) contains (up to quadratic) polynomial nonlinearities, whereas the original system of eqn (1) contains trigonometric nonlinearities. As such, the system of differential eqn (10), or equivalently (9), is suited for analytic treatments using series expansions, perturbation techniques or successive approximations, whereas the original system of equations, in terms of the Eulerian angles, is not directly amenable to such techniques. In some sense, we have traded the two nonlinear differential eqns (1a) and (1b) or (4a) and (4b) for the scalar, but complex, differential eqn (9), which has, nevertheless, a more suitable form for analytical studies. If we drop the nonlinear term in eqn (9) we get

$$\dot{\alpha} + i\omega_z\alpha = \omega \tag{14}$$

which is essentially eqn (3), the linearized system of eqn (1a) and (1b). Solution of this equation along with (12) can be used for direct reinterpretation of the solution of (3) without the need to solve any new equations. For a more detailed discussion on this approach see [9].

3.3. *Method of solution of the quadratic kinematic equation*

We now briefly discuss a procedure that will allow for approximate solutions of the quadratic kinematic eqn (9). Let the linearized solution of (9), be denoted by $\alpha_0 \triangleq \alpha_{x0} + i\alpha_{y0}$, i.e. let

$$\dot{\alpha}_0 + i\omega_z\alpha_0 = \omega. \tag{15}$$

The solution to this equation, or equivalently to eqn (3), has been computed in [7]. Using this solution for $\alpha_0(t)$, one can then obtain the first-order approximation to (9) by solving

$$\dot{\alpha} + i\omega_z\alpha = \omega + \mathrm{Re}(\bar{\omega}\alpha_0)\alpha_0. \tag{16}$$

Equation (16) is a linear differential equation that can be solved in terms of quadratures. Similarly, one can solve the zero-order solution and substitute into the first-order solution as follows

$$\dot{\alpha} + i[\omega_z + i\ \mathrm{Re}(\bar{\omega}\alpha_0)]\alpha = \omega. \tag{17}$$

In (17) the first-order solution has the effect of replacing the time-varying coefficient (frequency) of the zero-order (linear) equation, with the complex quantity $\tilde{\omega}_z \triangleq \omega_z + i\ \mathrm{Re}(\bar{\omega}\alpha_0)$, whereas in (16) the zero-order solution alters the forcing term of the equation. Preliminary simulations of the above two procedures indicate that these methods yield very accurate solutions of (9) and, in fact, capture the phase-shift error created by the zero-order (linearized) solution, an error which dictated the development of a large angle theory in the first place. Figure 4 shows that the result obtained for the solution for β_x using eqns (16) and (13), for spin-down under constant torques, is a dramatic improvement over Fig. 3. The numerical values for the simulations were taken from [7].

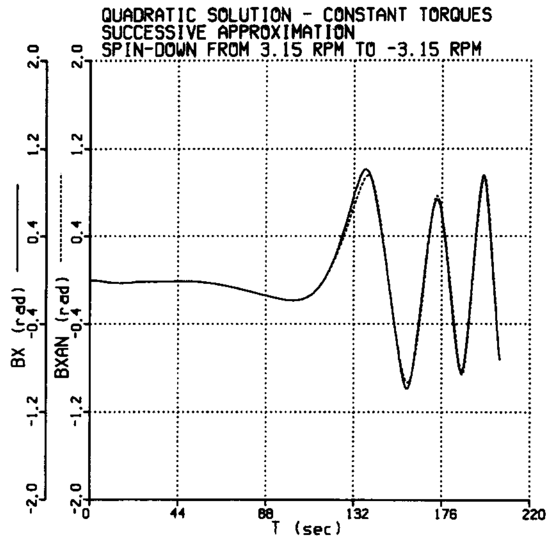

Fig. 4. Exact and analytic solution, β_x.

4. DIRECTION COSINES

4.1. *Kinematic equations from stereographic projection*

As already mentioned, each element of the rotation group SO(3) describes the orientation of two given sets of mutually orthogonal unit vectors (frames), the first of which is attached and moving with the rotating body, while the other remains constant. Both frames coincide at time zero. The attitude history of the moving reference frame with respect to the constant (inertial) reference frame can then be described by a curve traced by the corresponding rotation $R(t)$ matrix in SO(3). The differential equation satisfied while $R(t)$ is moving along this trajectory is given by

$$\dot{R} = S(\omega_x, \omega_y, \omega_z)R \tag{18}$$

where

$$S(\omega_x, \omega_y, \omega_z) \triangleq \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix}. \tag{19}$$

This matrix differential equation involves nine parameters (the direction cosines of the corresponding frames), however because of the constraint $RR^T = I$

imposed on the elements of SO(3), there are actually only *three* free parameters involved in the system of eqn (18). Now let $[a, b, c]^T$ denote a column vector of the matrix representation of R having entries r_{ij}, for $i, j = 1, 2, 3$. That is, $[a, b, c]^T = [r_{1j}, r_{2j}, r_{3j}]^T$ for some $j = 1, 2, 3$. Clearly,

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & \omega_z & -\omega_y \\ -\omega_z & 0 & \omega_x \\ \omega_y & -\omega_x & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}. \tag{20}$$

Because of the constraint $a^2 + b^2 + c^2 = 1$ we can eliminate one of the three parameters a, b, c, to get a system of *two* first order differential equations. The most natural and elegant way to reduce the third order system (20) to a second order system is by the use of *stereographic projection* [10]. That is, if we let a, b, and c represent the coordinates on the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, then for $(a, b, c) \in S^2$, the stereographic projection $(a, b, c) \rightarrow w$, with $w \in \mathbb{C}$, is given by

$$w \triangleq \frac{b - ia}{1 + c} = \frac{1 - c}{b + ia}. \tag{21}$$

In terms of the complex quantity w, the system of differential eqn (20) can be combined in the single differential equation

$$\dot{w} + i\omega_z w = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2. \tag{22}$$

The inverse transformation $w \rightarrow (a, b, c)$ is given by

$$a = \frac{i(w - \bar{w})}{|w|^2 + 1}, \quad b = \frac{w + \bar{w}}{|w|^2 + 1},$$

$$c = -\frac{|w|^2 - 1}{|w|^2 + 1} \tag{23}$$

and can be used to find a, b, c once w is known. In (23) $|\cdot|$ denotes the magnitude of a complex number, i.e. $z\bar{z} = |z|^2$, $z \in \mathbb{C}$. The real and imaginary parts of $w \triangleq w_1 + iw_2$ satisfy the differential equations

$$\dot{w}_1 = \omega_z w_2 + \omega_y w_1 w_2 + \frac{\omega_x}{2}(1 + w_1^2 - w_2^2) \tag{24a}$$

$$\dot{w}_2 = -\omega_z w_1 + \omega_x w_1 w_2 + \frac{\omega_y}{2}(1 + w_2^2 - w_1^2). \tag{24b}$$

4.2. *Method of solution for the Riccati equation*

Equation (22) is a Riccati equation for w with *time-varying* coefficients, the solution of which is very hard to establish. However, an approximate solution can be obtained, along the same lines as for α, as follows. One can obtain the zero-order (linear) solution of (22) by solving the equation

$$\dot{w}_0 + i\omega_z w_0 = \frac{\omega}{2}. \tag{25}$$

The solution is given by

$$w_0(t) = w_0(0)\exp\left[-i\int_0^t \omega_z(u)\,du\right] + \frac{1}{2}\exp\left[-i\int_0^t \omega_z(u)\,du\right] \times \int_0^t \omega(u)\exp\left[i\int_0^u \omega_z(v)\,dv\right]du. \tag{26}$$

The first-order approximation of the solution of (22) can then be obtained by solving the linear equation

$$\dot{w} + i\omega_z w = \frac{\omega}{2} + \frac{\bar{\omega}}{2} w_0^2 \tag{27}$$

the solution of which is given in terms of quadratures. In fact, the first-order approximation to the solution of (22) is given by

$$w(t) = w(0)\exp\left[-i\int_0^t \omega_z(u)\,du\right] + \frac{1}{2}\exp\left[-i\int_0^t \omega_z(u)\,du\right] \times \int_0^t \tilde{\omega}(u)\exp\left[i\int_0^u \omega_z(v)\,dv\right]du \tag{28}$$

where $\tilde{\omega} \triangleq \omega + \bar{\omega} w_0^2$. Alternatively, one can also solve the linear equation

$$\dot{w} + i\left(\omega_z + i\frac{\bar{\omega}}{2} w_0\right)w = \frac{\omega}{2}. \tag{29}$$

The difference between these two methods of solution lies in the fact that in eqn (27) the zero-order solution acts in such a way as to change the forcing term, whereas in eqn (29) it acts in such a way as to change the time-varying coefficient. Both equations retain the same form as the zero-order equation. The result of the solution of (22), using eqns (25) and (27), is shown in Fig. 5. Explicit formulas for the integrals that appear in (28) will be reported in a future work. We

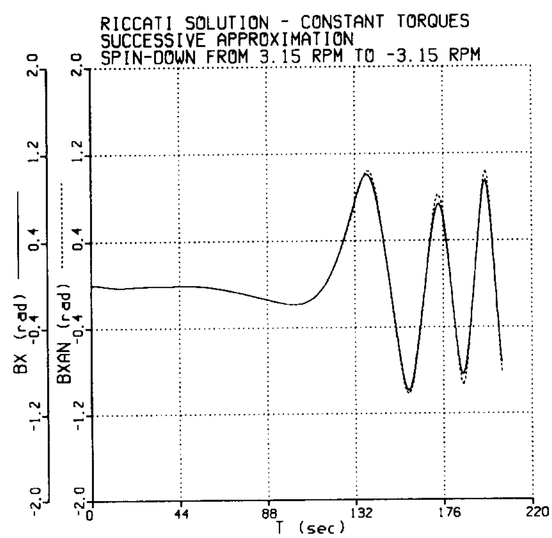

Fig. 5. Exact and analytic solution, β_x.

note in passing that the solution to the linear eqn (25), or the linear eqn (15), is easy to establish. In fact, by simple comparison, one sees that eqns (25) and (15) are of the same form as eqn (3), of the linearized 3–1–2 Eulerian angle problem, which has been solved in [7].

4.3. Relation between w and α

The resemblance of eqns (22), (25) and (27) to eqns (9), (15) and (16), respectively, prompts one to investigate relations between w and α. The purpose of this section is to show, that although these new kinematic equations are derived independently using two completely different procedures, they are nevertheless intimately related. As a first step, we establish the relationship between w and the Eulerian angles. Notice that we can in principle identify the column vector $[a, b, c]^T$ in (20) with any column vector of the rotation matrix R, where R can be expressed in terms of any of the parametrizations of SO(3). For a three-dimensional 3–1–3 Eulerian angle parametrization, the matrix $R(\phi, \theta, \psi)$ is given in (11b), however, any other parametrization is equally valid. Identifying $[a, b, c,]^T$ with the third column of R, establishes a one-to-one correspondence $(w_1, w_2) \leftrightarrow (\theta, \psi)$, as follows.

Let $a = \sin\theta\sin\psi$, $b = \sin\theta\cos\psi$ and $c = \cos\theta$. Then from (21) the following correspondence between w and ψ, θ is easily established

$$w = \frac{\sin\theta\sin\psi + i\sin\theta\cos\psi}{1+\cos\theta} = \frac{\sin\theta}{1+\cos\theta}\exp(i\psi) \tag{30}$$

or in terms of real and imaginary parts of w,

$$w_1 = \frac{\cos\theta\sin\psi}{1+\cos\theta}, \quad w_2 = \frac{\sin\theta\cos\psi}{1+\cos\theta}. \tag{31}$$

Recall now that α is related with θ and ψ through the relationship (8). From (8) and (30) one then immediately has that

$$\cos\theta = \frac{w}{\alpha - w}. \tag{32}$$

Since $\bar{\alpha} = \tan\theta\exp(i\psi)$ one obtains that $w\bar{\alpha} = (1 - \cos^2\theta)/(1+\cos\theta)\cos\theta$ and using (32) one finally gets the following relation between w and α

$$w^2\bar{\alpha} = \alpha - 2w. \tag{33}$$

For small θ, $\cos\theta \approx 1$ in (32) and hence $w = \alpha - w$, or $w = \alpha/2$. Comparison of the linearized eqns (15) and (25) shows that for small θ (small deviations of the spin axis) these two equations are in fact identical.

5. QUATERNION FORMULATION

As mentioned earlier, the parametrization of the rotation group with three Eulerian angles, in addition to the nonlinearity that it introduces in the kinematical equations, also has the disadvantage that it introduces singularities, i.e. points at which the parametrization is not defined. If one needs to avoid the singular points, one has to switch to another set of Eulerian angles. It is possible to circumvent this difficulty and introduce a parametrization that is globally valid, but this will imply, necessarily, the introduction of redundant parameters. The most often used global parametrization of the rotation group involves the introduction of *one* additional redundant parameter, and is called the *quaternion method* [8], first introduced by Lord Hamilton. The parameters are then called Euler parameters and in fact, when the kinematic equations are expressed in terms of these parameters, the result is a system of *linear* (although time-varying) differential equations. The linear nature of the kinematic equations is considered the most significant advantage of the four-dimensional parametrization of SO(3), limited however by the fact that, in general, no explicit formula for the solution of a system of linear *time-varying* differential equations is known to exist. It is true of course, that the solution to a system of linear differential equations is given in terms of the fundamental (or state transition) matrix, but for the time-varying case, no general method exists for computing this matrix, and one often has to resort to numerical simulations.

Next, we will show how one can apply a method, initially due to Picard [11], to approximate the solution to a linear, time-varying system of differential equations, as accurately as one desires, using the notion of the *product integral*. The methodology in essence seeks to approximate the state transition matrix and is *semi-analytic* in nature, since it is confined to small time steps. Picard's method can, in principle, be applied to all time-varying linear systems, but is especially convenient for the systems with the special form of skew-symmetric state matrices that appear in the kinematics of rotating bodies, since then the matrix exponentials can be computed in *closed form*.

5.1. Kinematic equations from the quaternion formulation

Recall that the quaternion vector $q \triangleq q_0 + q_1\hat{\imath} + q_2\hat{\jmath} + q_3\hat{k}$, evolves in time by the linear system of differential equations

$$\begin{bmatrix}\dot{q}_0\\ \dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & -\omega_x & -\omega_y & -\omega_z\\ \omega_x & 0 & \omega_z & -\omega_y\\ \omega_y & -\omega_z & 0 & \omega_x\\ \omega_z & \omega_y & -\omega_x & 0\end{bmatrix}\begin{bmatrix}q_0\\ q_1\\ q_2\\ q_3\end{bmatrix} \tag{34}$$

where q_0, q_1, q_2, q_3 are the *Euler parameters*. Although the kinematic equations in this form are linear, closed-form solutions are extremely difficult to obtain, due to the time-varying nature of the differential equations. Analytic solutions of (34) have been constructed for the special case of a torque-free rotating body [12]. Kane [13] has also obtained approximate solutions to (34) for an axisymmetric rigid body subject to body-fixed transverse torques of constant magnitude, employing an averaging technique. Similar approximate solutions have also been reported by Kane and Levinson [14]. As often occurs in practice, rotating rigid bodies have an axis of symmetry, which is also usually the spin axis. If this is the case, then it is advantageous to introduce, in place of the quaternions, the parameters

$$\rho \triangleq q_0 + iq_3 \quad \text{and} \quad \sigma \triangleq q_1 + iq_2 \tag{35}$$

because then, using (35), (34) can be reduced to the compact form

$$\begin{bmatrix} \dot{\rho} \\ \dot{\sigma} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i\omega_z & -\bar{\omega} \\ \omega & -i\omega_z \end{bmatrix}\begin{bmatrix} \rho \\ \sigma \end{bmatrix}. \tag{36}$$

The reason we prefer to work with (36) rather than with (34) is that for the case of spinning (usually near-symmetric) rigid bodies, one often takes advantage of the special skew-symmetry of the equations to obtain the solution of the transverse components of the angular velocity ω_x and ω_y in the compact complex form of $\omega = \omega_x + i\omega_y$. Van der Ha [15] attempted to obtain approximate solutions to equations of the form (36) using a perturbation scheme. Perturbation methods have also been used to obtain approximate solutions to the original form of eqn (34) by Kraige and Junkins [16].

Notice that eqn (36) is of the form

$$\dot{\xi} = A(t)\xi \tag{37}$$

which is a linear, time-varying differential equation in vector format. The solution of the previous differential equation is given by

$$\xi(t) = \Phi(t, 0)\xi(0) \tag{38}$$

where $\Phi(t, 0)$ is the *product integral* (state transition matrix) satisfying the matrix differential equation

$$\dot{\Phi}(t, 0) = A(t)\Phi(t, 0), \quad \Phi(0, 0) \triangleq I. \tag{39}$$

Following Nelson [11], the solution of the previous equation is approximated by

$$\Phi(t, 0) = \exp(A_n \Delta t_n) \cdots \exp(A_1 \Delta t_1) \tag{40a}$$

$$A_j = A(t) \quad \text{for} \quad t_{j-1} < t < t_j$$

and

$$\Delta t_j = t_j - t_{j-1}, \quad 0 = t_0 < t_1 < \cdots < t_n = t \tag{40b}$$

Notice that in (40) operators with the smallest value of the time parameter operate first. This is very important, because commutativity does not hold in general between matrix exponentials. For $\Delta t_j \to 0$ $(j = 1, 2, \ldots, n)$, eqn (40) gives the exact solution to the differential equation for $\Phi(t, 0)$.

The closed-form calculation of the matrix exponential $\exp[A(t)]$ for a time-varying $A(t)$ is, in general, a formidable task. However, for the special structure of the matrices that appear in eqn (34), or equivalently in eqn (36), one can immediately verify [13], that

$$\exp(A) = I\cos(v) + \frac{A}{v}\sin(v) \tag{41}$$

where $v^2 \triangleq \det(A) = \omega\bar{\omega} + \omega_z^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$ and I is the 2×2 identity matrix. This formula holds for all skew-hermitian matrices A, for which $A^2 = -\det(A)I$. One can easily verify this property for the matrix in (36). Use of formula (41) allows the (approximate) evaluation of the exponentials in eqn (40a) for $\Phi(t, 0)$. However, an accurate calculation for $\Phi(t, 0)$ will require very small time intervals Δt_j. We can circumvent this difficulty, and extend the solution to larger time steps, but we first require some results from the theory of Lie groups and their associated Lie algebras.

5.2. Generalized Baker–Campbell–Hausdorff formula

Because of the special structure of the state matrix A in (37), it is known that the state transition matrix $\Phi(t, 0)$ is a unitary matrix, and as such, it is given by the exponential of some skew-hermitian matrix $W(t)$, i.e. $\Phi(t, 0) = \exp[W(t)]$ for all t. We want to find the matrix $W(t)$, starting from eqn (40a), namely, to combine the product of exponentials into a single exponential (that of the matrix W). Recall that if X, Y are $n \times n$ matrices, then $\exp(X)\exp(Y) \neq \exp(X + Y)$, in general, unless $XY = YX$, i.e. unless the matrices X and Y commute. However, the following result from the Lie group theory [17] states that if X and Y are sufficiently near the zero matrix, there exists a matrix Z in the Lie algebra generated by X and Y that satisfies

$$\exp(X)\exp(Y) = \exp(Z). \tag{42a}$$

Specifically, Z is given by the expansion

$$Z = X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}[X, [X, Y]] + \tfrac{1}{12}[[X, Y], Y] + \cdots \tag{42b}$$

where $[\cdot\,, \cdot]$ denotes the Lie bracket (commutator), defined by $[X, Y] \triangleq XY - YX$. Equation (42) is called the Baker–Campbell–Hausdorff formula. Applying this formula to eqn (40a), starting from the left, and keeping terms only up to $O(\Delta t^2)$, we get that

$$\begin{aligned}\Phi(t, 0) &= \exp(A_n \Delta t_n) \cdots \exp(A_2 \Delta t_2)\exp(A_1 \Delta t_1) \\ &= \exp(A_n \Delta t_n) \cdots \exp(A_3 \Delta t_3)\exp(\hat{A}_2)\end{aligned} \tag{43a}$$

where

$$\hat{A}_2 = A_2 \Delta t_2 + A_1 \Delta t_1 + \tfrac{1}{2}[A_2 \Delta t_2, A_1 \Delta t_1] + O(\Delta t^3). \tag{43b}$$

The next application of the BCH formula to the exponentials $\exp(A_3 \Delta t_3)\exp(\hat{A}_2)$, keeping again only terms up to $O(\Delta t^2)$, gives

$$\Phi(t, 0) = \exp(A_n \Delta t_n) \cdots \exp(A_4 \Delta t_4)\exp(\hat{A}_3) \tag{44a}$$

where

$$\begin{aligned}\hat{A}_3 = {} & A_3 \Delta t_3 + A_2 \Delta t_2 + A_1 \Delta t_1 \\ & + \tfrac{1}{2}[A_2 \Delta t_2, A_1 \Delta t_1] \\ & + \tfrac{1}{2}[A_3 \Delta t_3, A_2 \Delta t_2 + A_1 \Delta t_1] + O(\Delta t^3).\end{aligned} \tag{44b}$$

Continuing the same way, one obtains the following approximation of $\Phi(t,0)$ to order $O(\Delta t^3)$:

$$\Phi(t,0)=\exp\left\{\sum_{j=1}^{n}A_j\Delta t_j+\frac{1}{2}\sum_{j=1}^{n}\left[A_j\Delta t_j,\sum_{i=1}^{n-j}A_i\Delta t_i\right]+O(\Delta t^3)\right\}. \quad (45)$$

Taking limits for $n\to\infty$, or $\Delta t\to 0$, one easily gets from the Riemann sums of (45) that

$$\Phi(t,0)=\exp[W(t)]$$

$$W(t)\triangleq\int_0^t A(\tau)\,d\tau+\frac{1}{2}\int_0^t\left[A(\tau)\,d\tau,\int_0^\tau A(\sigma)\,d\sigma\right]+\cdots. \quad (46)$$

Equation (46) gives the expression for the state transition matrix $\Phi(t,0)$ required for the solution of (37). It can be easily verified that the matrix W is skew-hermitian with $W^2=-\det(W)I$ so that it has the form required, in order to compute its exponential from eqn (41), for all t. The calculation of $W(t)$ from (46) can be performed easily, by direct integrations. From (36)

$$\int_0^t A(\tau)\,d\tau=\frac{1}{2}\begin{bmatrix} i\int_0^t\omega_z(\tau)\,d\tau & -\int_0^t\bar{\omega}(\tau)\,d\tau \\ \int_0^t\omega(\tau)\,d\tau & -i\int_0^t\omega_z(\tau)\,d\tau\end{bmatrix}. \quad (47)$$

Because of the skew-hermitian structure of the matrix A, one needs to calculate only two of the above integrals. The second term of $W(t)$ requires the evaluation of

$$\hat{A}(t)\triangleq\int_0^t\left[A(\tau),\int_0^\tau A(\sigma)\,d\sigma\right]d\tau. \quad (48)$$

Carrying out the algebra, it can be immediately shown that $[A(\tau),\int_0^\tau A(\sigma)\,d\sigma]$ takes the form

$$\begin{bmatrix} \omega\int_0^\tau\bar{\omega}\,d\tau-\bar{\omega}\int_0^\tau\omega\,d\tau & 2i\left[\bar{\omega}\int_0^\tau\omega_z\,d\tau-\omega_z\int_0^\tau\bar{\omega}\,d\tau\right] \\ 2i\left[\omega\int_0^\tau\omega_z\,d\tau-\omega_z\int_0^\tau\omega\,d\tau\right] & \bar{\omega}\int_0^\tau\omega\,d\tau-\omega\int_0^\tau\bar{\omega}\,d\tau\end{bmatrix}. \quad (49)$$

Again, because of the special skew-hermitian structure of the matrix $\hat{A}$, we need to evaluate the integrals of only two of the entries of $\hat{A}$, say

$$\hat{A}_{11}(t)=2i\int_0^t\operatorname{Im}\left[\omega(\tau)\int_0^\tau\bar{\omega}(\sigma)\,d\sigma\right]d\tau \quad (50a)$$

$$\hat{A}_{21}(t)=2i\int_0^t\left[\omega(\tau)\int_0^\tau\omega_z(\sigma)\,d\sigma-\omega_z(\tau)\int_0^\tau\omega(\sigma)\,d\sigma\right]d\tau. \quad (50b)$$

Of course, the calculation of the integrals of these quantities becomes very involved. For simple enough expressions for the angular velocities ω_x, ω_y, and ω_z, symbolic language manipulation routines can be used to alleviate the effort.

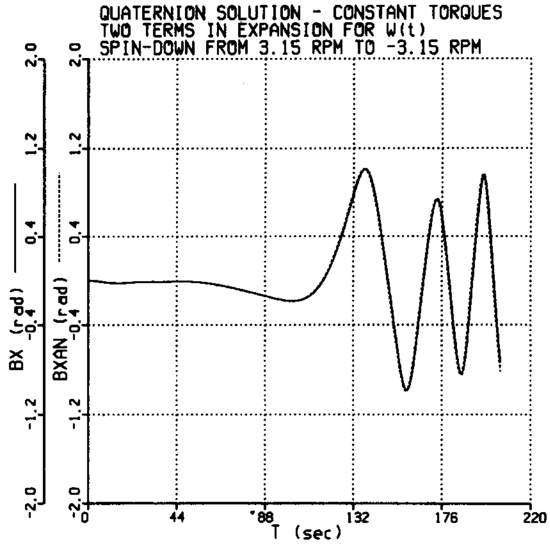

Fig. 6. Exact and analytic solution, β_x.

The BCH Theorem states that (42) holds for some matrices X and Y "close" to the zero matrix. That is, X and Y should be "small" with respect to a norm $\|\cdot\|$ that is compatible with the Lie bracket operation, i.e. a norm such that $\|[X,Y]\|\leqslant\|X\|\,\|Y\|$. Thus, the expansion (42b) is only locally convergent, so (42a) can be used to determine the existence of Z in the Lie algebra generated by X and Y when the norm of $[X,Y]$ is sufficiently small. The applicability of the BCH formula is thus limited inside a ball of unit radius (with respect to a compatible norm). This local convergence of the BCH formula restricts the validity of (46) to the neighborhood of $t=0\triangleq T_0$. One can circumvent this problem, by redefininig the initial condition in regular time intervals as follows: choose a time T_1 such that the series in (46) converges. Then the solution is given by

$$\xi(t)=\Phi(t,T_0)\xi(T_0),\quad\text{for}\quad T_0\leqslant t<T_1. \quad (51)$$

Next choose a time T_2 such that the series expansion starting from T_1 converges. Then the solution is given by

$$\xi(t)=\Phi(t,T_1)\xi(T_1),$$

$$T_1\leqslant t<T_2,\quad\text{and}\quad\xi(T_1)=\Phi(T_1,T_0)\xi(T_0). \quad (52)$$

In practice one usually chooses $T_{j+1}-T_j=T$, $(j=0,1,2,\ldots,n-1)$. Thus, redefining the initial condition every T seconds, one can keep the norm of the matrices small and keep the convergence of the BCH formula under control. The result for the solution of (36) using $\Phi(t, 0)$ from (46) with only the first two terms, and with reinitialization every 10 s, is shown in Fig. 6. Once one knows q_0, q_1, q_2, q_3, the Eulerian angles β_x, β_y, β_z are given by comparing (11a) with the corresponding typical element of SO(3), when expressed in terms of the Euler parameters. For such a parametrization we have [8]

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2+q_0q_3) & 2(q_1q_3-q_0q_2) \\ 2(q_1q_2-q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3+q_0q_1) \\ 2(q_1q_3+q_0q_2) & 2(q_2q_3-q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}. \tag{53}$$

One easily sees then that β_x, β_y, β_z are given by

$$\sin\beta_x = 2(q_2q_3+q_0q_1),$$

$$\tan\beta_y = -\frac{2(q_1q_3-q_0q_2)}{q_0^2-q_1^2-q_2^2+q_3^2}$$

$$\tan\beta_z = -\frac{2(q_1q_2-q_0q_3)}{q_0^2-q_1^2+q_2^2-q_3^2}. \tag{54}$$

6. CONCLUSIONS

Solution techniques for the kinematic equations of a rotating rigid body, that are *not* limited to small angular displacements, have been developed using three different approaches. A typical example of practical interest, for which such approaches have proved to be very useful, is the despinning of an (initially) spin-stabilized vehicle in the presence of large transverse torques. Two of the methodologies developed, one using a new formulation of the Eulerian angle kinematics and the other using direction cosines, can provide very accurate results, but require that the linearized (zero-order) solution be reasonably good. In fact, it was shown that these two methodologies are very similar and the solutions obtained by the two formulations are related by a simple transformation. The third method, based on an Eulerian parameter formulation, does not have this limitation, and is therefore globally valid. This method employs a generalization of the Baker–Campbell–Hausdorff formula, encountered in the theory of Lie groups, in order to find explicitly the state transition matrix as the exponential of a known matrix. As such, it is applicable to any linear time-varying system of differential equations, but its applicability is especially suited for rotational dynamics problems, because then one can evaluate the arising matrix exponentials in closed form.

REFERENCES

1. G. Darboux, *Lecons sur la Théorie Générale des Surfaces*, Vol. 1. Gauthier-Villars, Paris (1887).
2. P. Olver, *Applications of Lie Groups to Differential Equations*. Springer, New York (1986).
3. H. Hopf, *Systeme Symmetrischer Bilinearformen und Euklidische Modelle der Projektiven Räume*. Vierteljahr, Naturforsch. Ges. Zürick (1940).
4. J. Stuelpnagel, On the parametrization of the three-dimensional rotation group. *SIAM Rev.* **6**, 422–430 (1964).
5. P. R. Kurzhals, An approximate solution of the equations of motion for arbitrary rotating spacecraft. NASA Technical Report TR R-269 (1967).
6. J. M. Longuski, Real solutions for the attitude motion of a self-excited rigid body. *Acta Astronautica* **25**, 131–140 (1991).
7. P. Tsiotras and J. M. Longuski, Complex analytic solution for the attitude motion of a near-symmetric rigid body under body-fixed torques. *Celest. Mech.* **51**, 281–301 (1991).
8. T. R. Kane, P. W. Likins and P. A. Levinson, *Spacecraft Dynamics*. McGraw–Hill, New York (1983).
9. P. Tsiotras and J. M. Longuski, On the large angle problem in rigid body attitude dynamics. *43rd IAF Congress*, Washington, D.C., Paper 92-034 (1992).
10. J. B. Conway, *Functions of One Complex Variable*. Springer, New York (1978).
11. E. Nelson, *Topics in Dynamics. I: Flows. Mathematical Notes*. Princeton University Press (1969).
12. H. S. Morton, J. L. Junkins and J. N. Blanton, Analytical solutions for Euler parameters. *Celest. Mech.* **10**, 287–301 (1974).
13. T. R. Kane, Solution of kinematical differential equations for a rigid body. *J. appl. Mech.* **40**, 109–113 (1973).
14. T. R. Kane and D. A. Levinson, Approximate solution of differential equations governing the orientation of a rigid body in a reference frame. *J. appl. Mech.* **54**, 232–234 (1987).
15. J. F. Van der Ha, Perturbation solution of attitude motion under body-fixed torques. *35th IAF Congress*, Lausanne, Switzerland, Paper 84-357 (1984).
16. L. G. Kraige and J. L. Junkins, Perturbation formulations for satellite attitude dynamics. *Celest. Mech.* **13**, 39–64 (1976).
17. V. S. Varadarajan, *Lie Groups, Lie Algebras, and their Representations*. Springer, New York (1984).