# A Unified Approach to Time-Delay System Stability via Scaled Small Gain *

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## Abstract

This paper introduces a unified approach to the analysis of time-delay systems through the application of scaled small gain concepts to a comparison system that is free of delay. This provides a new interpretation of several Lyapunov-Krasovskii and Lyapunov-Razumikhin based results.

## 1 Introduction

The analysis of time-delay systems has attracted much interest in the literature over a half century, especially in the last decade. Two types of stability conditions, delay-independent and delay-dependent, have been studied [4]. Some frequency domain-based stability criteria are also provided in the literature [4]. Some frequency interpretations on the delay-independent stability conditions can be found in [7, 8], and some similar ideas for delay-dependent case in [1]. In this paper, we demonstrate that several different conditions for stability of time-delay systems all arise from scaled small gain analysis of a single comparison system which is linear, uncertain, and free of delay.

## 2 Comparison System

Consider the linear time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \tag{1}$$

where $0 \le \tau \le \bar{\tau}$ is constant but unknown. For analysis, we will use the following preliminary results.

**Lemma 1** *Let $M \in R^{n\times n}$ be any constant matrix. The system (1) is asymptotically stable for all $0 \le \tau \le \bar{\tau}$, if the following comparison system*

$$\begin{array}{rcl} sX & = & (A+MA_d)X + \Delta_1(I_n - M)A_d X \\ & & +\Delta_2\bar{\tau}MA_dAX + \Delta_1\Delta_2\bar{\tau}MA_dA_dX \end{array} \tag{2}$$

*where $\Delta_i = \delta_i(s)I_n$, $\|\delta_i(s)\|_\infty \le 1$, $i = 1, 2$, is robustly stable.*

**Lemma 2** ***(Scaled Small Gain LMI)*** *Consider a system with plant $G(s)$ and uncertainty $\Delta$. Let $(A, B, C, D)$ be a minimal realization of $G(s)$ with*

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

*Suppose $\Delta = diag\{\delta_1 I_{n_1},\ \delta_2 I_{n_2},\ \cdots,\ \delta_r I_{n_r}\}$, where $\delta_1$, $\delta_2$, $\cdots$,$\delta_r$ are uncertain scalar dynamic systems. Then the closed loop system is robustly stable for all $\|\delta_i(s)\|_\infty \le 1$, $i = 1, 2, \cdots, r$,if there exist matrices $X > 0$ and $Q = diag\{Q_1,\ Q_2,\ \cdots,\ Q_r\} > 0$, $Q_i \in R^{n_i\times n_i}$, $i = 1, 2, \cdots, r$, satisfying the following LMI:*

$$\left[\begin{array}{ccc} A^TX+XA & XB & C^TQ \\ B^TX & -Q & D^TQ \\ QC & QD & -Q \end{array}\right] < 0. \tag{3}$$

**Definition 1** *If a system satisfies (3), we will refer to it as meeting the scaled small gain sufficiency (SSGS) condition for robust stability.*

## 3 Several Stability Analysis Results

### 3.1 Delay-Independent Stability

**Theorem 1** *The delay-independent condition of [7, 8]:*

$$\begin{array}{c} A^TX + XA + XA_dQ^{-1}A_d^TX + Q < 0 \\ X > 0, Q > 0 \end{array} \tag{4}$$

*is equivalent to the SSGS condition for comparison system (2) with $M = 0$.*

**Proof.** With $M = 0$, the comparison system (2) becomes

$$\begin{array}{c} sX(s) = AX(s) + \Delta_1(s)A_dX(s) \\ \Delta_1 = \delta_1(s)I_n, \quad \|\delta_1(s)\|_\infty \le 1 \end{array}$$

which can be described as

$$\begin{array}{l} \dot{x} = Ax + A_du \\ y = x \\ u = \delta_1 y \end{array}$$

Applying Lemma 2 with $G(s) = \left[\begin{array}{c|c} A & A_d \\ \hline I_n & 0 \end{array}\right]$, the conclusion follows straightforwardly. ■

### 3.2 Delay-Dependent Stability

**Theorem 2** *The delay-dependent condition of [2]:*

$$\begin{array}{c} \left[\begin{array}{ccc} H & \bar{\tau}PA^T & \bar{\tau}PA_d^T \\ \bar{\tau}AP & -\bar{\tau}P_1 & 0 \\ \bar{\tau}A_dP & 0 & -\bar{\tau}P_2 \end{array}\right] < 0 \\ P>0, P_1>0, P_2>0 \end{array} \tag{5}$$

*is equivalent to the SSGS condition for comparison system (2) with $M = I_n$.*

**Proof.** Let $M = I_n$ and $\delta_3(s) = \delta_1(s)\delta_2(s)$. Then (2) becomes

$$\begin{array}{c} sX(s) = (A + A_d)X(s) + \Delta_2(s)\bar{\tau}A_dAX(s) \\ +\Delta_3(s)\bar{\tau}A_dA_dX(s) \\ \Delta_i = \delta_i(s)I_n, \quad \|\delta_i(s)\|_\infty \le 1, i = 2, 3 \end{array}$$

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By applying Lemma 2 with

$$G(s) = \left[ \begin{array}{c|c} A + A_d & \begin{bmatrix} \bar{\tau} A_d & \bar{\tau} A_d \end{bmatrix} \\ \hline \begin{bmatrix} A \\ A_d \end{bmatrix} & 0 \end{array} \right],$$

and letting $P = X^{-1}$, $P_1 = \bar{\tau} Q_1^{-1}$ and $P_2 = \bar{\tau} Q_2^{-1}$,it is straightforward to show that conditions (5) and (3) are equivalent. ■

In addition, applying Lemma 1 and 2 with

$$G(s) = \left[ \begin{array}{c|c} A + A_d & \begin{bmatrix} \bar{\tau} A_d A & \bar{\tau} A_d A_d \end{bmatrix} \\ \hline \begin{bmatrix} I \\ I \end{bmatrix} & 0 \end{array} \right],$$

and letting $Q_1 = \bar{\tau}\beta_1^{-1} X$ and $Q_2 = \bar{\tau}\beta_2^{-1} X$, where constants $\beta_1$ and $\beta_2 > 0$, we recover the result of [3].

**Theorem 3** *The delay-dependent condition of [6]:*

$$\begin{bmatrix} H_1 & H_2 & H_3 & H_4 \\ H_2^T & -U & -A_d^T Y & 0 \\ H_3^T & -Y A_d & -Y & 0 \\ H_4^T & 0 & 0 & -V \end{bmatrix} < 0 \qquad (6)$$
$$Y \geq A_d^T V A_d$$
$$X > 0,\ U > 0,\ V > 0,\ Y > 0$$

*where $H_1 = (A + A_d)^T X + X(A + A_d)$, $H_2 = W A_d + U$, $H_3 = (A + A_d)^T X$, and $H_4 = \bar{\tau}(W + X)$, is a sufficient condition for the SSGS condition of (2) with $M$ as a free matrix variable.*

**Proof.** For the general case of (2), applying Lemma 2 with

$$G(s) = \left[ \begin{array}{c|c} A + M A_d & \begin{bmatrix} \bar{\tau} M & (I - M) A_d \end{bmatrix} \\ \hline \begin{bmatrix} A_d A \\ I \end{bmatrix} & \begin{bmatrix} 0 & A_d A_d \\ 0 & 0 \end{bmatrix} \end{array} \right]$$

and letting $W = X(M - I)$, and $Q = diag(V, U)$, it is straightforward but tedious to verify that (6) ensures the condition (3). ■

**Remark 1** *SSGS condition (3) for (2) with $M$ as a free matrix variable is equivalent to*

$$\begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & A^T A_d^T V \\ \Omega_2^T & -V & 0 & 0 \\ \Omega_3^T & 0 & -U & A_d^T A_d^T V \\ V A_d A & 0 & V A_d A_d & -V \end{bmatrix} < 0 \qquad (7)$$
$$X > 0,\ U > 0,\ V > 0$$

*where $\Omega_1 = (A + A_d)^T X + X(A + A_d) + W A_d + A_d^T W^T + U$, $\Omega_2 = \bar{\tau}(X + W)$, and $\Omega_3 = X A_d$. (7) may be used directly for analysis with fewer matrix variables than the formulation of (6); the variable $Y$ and constraint $Y \geq A_d^T V A_d$ are not needed.*

**Remark 2** *$M = 0$ leads to the delay-independent condition (4), and $M = I_n$ leads to the delay-dependent criterion (5). The above proof indicates that (6) generalizes and encompasses (4) and (5), and thus it is less conservative.*

**Remark 3** *Condition (3) is equivalent to applying the small $\mu$ theorem with $\mu$ upper-bound found via D-scale optimization. This upper-bound is equal to $\mu$ only when $2S + F \leq 3$ [5], where $S$ and $F$ are the number of repeated complex scalar blocks and number of full complex blocks, respectively. Hence, (4) is a necessary and sufficient condition for robust stability of (2) with $M = 0$, while (5) and (7) are only sufficient conditions.*

## 4 Discussion

A $\mu$ framework may be used for analysis (and, more importantly, controller synthesis) of uncertain time-delay systems *without incurring any penalty vis-à-vis known Lyapunov-based approaches.* It offers several advantages:

1. Robustness analysis with respect to LTI dynamic or parametric uncertainties in the time-delay system can be accomplished via the introduction of these uncertainties into the model description.

2. The treatment of multiple delays is trivial involving only the creation of more fictitious inputs and outputs for the additional "uncertainties".

3. The commensurable delay case can also be easily handled through the augmentation of the generalized plant with "pass-throughs" and repeating the delay "uncertainty" blocks.

## 5 Conclusions

It has been demonstrated that several recent results in the analysis of the stability of linear time-delay systems are, in fact, equivalent to robust stability analysis of a linear uncertain delay-free comparison system via the scaled small gain LMI. This result unifies several previous criteria, all of which were originally derived via Lyapunov's Second Method. Furthermore, this result indicates that $\mu-$analysis and synthesis can be applied to uncertain time-delay systems without incurring any penalty with respect to known Lyapunov-based results.

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