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Exponentially Convergent Controllers for n-Dimensional Nonholonomic Systems in Power Form*

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Abstract

This paper introduces a new method for constructing exponentially convergent control laws for n-dimensional nonholonomic systems in power form. The methodology is based on the construction of a series of invariant manifolds for the closed-loop system under a linear control law. A recursive algorithm is presented to derive a feedback controller which drives the system exponentially to the origin. A numerical example illustrates the proposed theoretical developments.

1. Introduction

Nonholonomic control systems commonly arise from mechanical systems when non-integrable constraints are imposed on the motion, e.g., velocity constraints, which can not be integrated to generate constraints on the configuration space. Examples include a rolling disk [1], mobile robots [2] and underactuated symmetric rigid spacecraft [3, 4]. One of the main reasons these systems have attracted much attention in the past few years is that they represent "inherently nonlinear" systems in a certain sense. For example, these systems are controllable but not stabilizable by a smooth static or dynamic state feedback control laws [5].

A number of approaches have been proposed to solve the stabilization problem for nonholonomic systems. These methodologies can be broadly classified as discontinuous, time-invariant stabilization and time-varying (usually smooth) stabilization. The non-smoothness of time-invariant feedback controls is a consequence of the structural properties of the system [5]. Stabilization results using non-smooth, time-invariant control laws have been proposed in [3, 4, 6, 7, 8]. References [3, 4] deal with the attitude stabilization of underactuated spacecraft by developing non-smooth, time-invariant control laws. Piecewise continuous stabilization controller have been reported in [6, 7]. A nonsmooth transformation was used to develop time-invariant, exponential convergent controller in [8]. Samson in [9] showed how to asymptotically stabilize a mobile robot to a point using time-varying, smooth state feedback. Coron in [10] proved that all controllable driftless systems could be stabilized to an equilibrium point using smooth, periodic, time-varying feedback. References [11, 12] and [13] deal with the construction of time-varying control laws for several nonholonomic systems. Hybrid feedback time-varying control laws are constructed for a class of cascade nonlinear systems in [14], which could also be used for stabilizing a class of nonholonomic systems, as well as for solving tracking problems. References [15] and [16] develop time-varying control laws of exponential convergence with respect to homogeneous norms. Finally, [17] develops nonsmooth, time-varying feedback control laws which guarantee global, asymptotic stability with exponential convergence about an arbitrary configuration. For a more comprehensive review of all the recent advances in the control of nonholonomic systems the interested reader may consult [18].

The analysis of dynamic systems is often simplified by the introduction of *canonical* or *normal* forms, that is, systems of equations which all systems in a given family are "equivalent" to. For nonholonomic systems there are two normal forms which have been used extensively in the past, namely, the *chained form* and the *power form*. The mathematical model of an n-dimensional nonholonomic system in power form with two inputs can be described as [7, 18]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_j &= \frac{1}{(j-2)!}\, x_1^{j-2} u_2, \qquad j = 2, 3, \ldots, n \end{aligned} \tag{1}$$

Although not all nonholonomic systems can be transformed into chained or power form, a large number of mechanical systems encountered in practice can be converted into these forms. Wheeled robots, multi-trailers, underactuated symmetric rigid spacecraft are only but a few examples.

In this paper we derive feedback control laws for nonholonomic systems in power form with two inputs. We first show that a set of invariant manifolds can be constructed for n-dimensional nonholonomic systems in power form. These manifolds are derived from the exact closed-loop system solution subject to a linear feedback law. The derivation of these manifolds for systems in power form first appeared in [7]; no controllers were derived for the general case, however. Here we use these manifolds to introduce state and input transformations. The transformed system is still in power form but of reduced dimension. By repeating the process we end up with a 3-dimensional system in power form which is easy to stabilize. We show that the stabilization of this system implies the stabilization of the original n-dimensional system.

The resulting controllers are similar in form to the ones proposed in [8], where discontinuous controllers for chained systems were constructed by a nonlinear transformation (σ process). The approach proposed here, on the other hand, generates controllers with multi-time scale convergent properties, as a result of the invariable manifold method, which is not present in [8].

The paper is organized as follows. Section 2 derives the invariant manifolds and discusses their properties. In Section 3 we present a recursive algorithm for power form systems based on the derived invariant manifolds. We also provide an ex-

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ponentially convergent controller for a 3-dimensional nonholonomic system in power form. The main result of the paper is given in Section 4 (Theorem 4.1). We basically show that the feedback control law for the 3-dimensional generated system in power form can be used to make the original n-dimensional system exponentially converge to the origin with a proper choice of the control gains. A numerical example in Section 5 illustrates the theoretical developments.

2. Invariant Manifolds and Their Properties

Consider the system in Eqs. (1) and the following linear feedback

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -kx_1 \\ -kx_2 \end{bmatrix}, \qquad k > 0 \tag{2}$$

With this linear control law, the closed-loop equations are

$$\begin{aligned} \dot{x}_1 &= -kx_1 \\ \dot{x}_j &= -\frac{1}{(j-2)!} k x_1^{j-2} x_2, \qquad j = 2, 3, \dots, n \end{aligned} \tag{3}$$

Equations (3) can be explicitly integrated to obtain

$$\begin{aligned} x_1(t) &= x_{10}\, e^{-kt} \\ x_2(t) &= x_{20}\, e^{-kt} \\ x_j(t) &= s_{1,j-2}(x_0) + \frac{1}{(j-1)!} x_{10}^{j-2} x_{20}\, e^{-(j-1)kt} \end{aligned} \tag{4}$$

where $j = 3, 4, \dots, n$, $x_0 = [x_{10}, x_{20}, x_{30}, \dots, x_{n0}]^T \in \Re^n$ is the initial state of the system, and where

$$s_{1,j} = x_{j+2} - \frac{1}{(j+1)!} x_1^j x_2, \qquad j = 1, 2, \dots, n-2 \tag{5}$$

Each equation in Eqs. (5) defines a smooth function in terms of $x_1, x_2, \dots, x_n$. Therefore, Eqs.(5) define a series of smooth manifolds by

$$\Pi_j = \{x \in \Re^n : s_{1,j}(x) = 0\}, \qquad j = 1, 2, \dots, n-2 \tag{6}$$

each of dimension $n-1$. Consider the following smooth manifold

$$\Pi = \bigcap_{j=1}^{n-2} \Pi_j = \{x \in \Re^n : s_{1,j}(x) = 0, \;\; j = 1, 2, \dots, n-2\}$$

Since

$$rank \left[\frac{\partial s_{1,j}}{\partial x_i}\right] = n - 2 \tag{7}$$

Π is a two-dimensional smooth manifold [19].

Lemma 2.1 *Consider the system in Eqs. (1) under the feedback control in Eqs. (2) and the manifold Π. Then, for all initial conditions $x_0 \in \Pi$ the closed-loop trajectories of the system will tend to the origin exponentially, with rate of decay k.*

Proof: First we show that each manifold Π_j is invariant for the closed-loop system. Indeed,

$$\begin{aligned} \dot{s}_{1,j} &= \dot{x}_{j+2} - \frac{1}{(j+1)!} (j\, x_1^{j-1} \dot{x}_1 x_2 + x_1^j \dot{x}_2) \\ &= -k \frac{1}{j!} x_1^j x_2 + \frac{1}{(j+1)!} \left(k\,(j+1)\, x_1^j x_2\right) = 0 \end{aligned}$$

Subsequently, the manifold Π is invariant for Eqs. (3). For $x_0 \in \Pi$ the solutions of the closed-loop system are given by

$$\begin{aligned} x_1(t) &= x_{10}\, e^{-kt} \\ x_j(t) &= \frac{1}{(j-1)!} x_{10}^{j-2} x_{20}\, e^{-(j-1)kt}, \quad j = 2, 3, \dots, n \end{aligned}$$

The assertion of the lemma follows immediately. ■

The idea of constructing invariant manifolds by directly integrating a closed-loop system subject to linear feedback has been initially used in [4] to derive controllers for underactuated symmetric spacecraft. This idea was subsequently generalized to nonholonomic systems in power form in [7].

3. A Recursive Algorithm for Systems in Power Form

In this section we present a recursive algorithm to create a series of systems which will be used to construct convergent feedback controllers for the system in Eqs. (1). All the systems generated by this recursive process (herein called the *generated systems*) can be put into power form through a linear transformation. These generated systems are, however, of reduced dimension. The methodology is based on the idea that by constructing a set of $(n-2)$ manifolds for the n-dimensional system, the problem of constructing convergent controllers for the initial system becomes one of constructing convergent controllers for a similar system in power form but of dimension $(n-1)$. By repeating this process, we end up with a system in power form of 3-dimension.

3.1. The Recursive Process

Consider the n-dimensional system as given in Eqs. (1), and construct a set of $(n-2)$ invariant manifolds under the linear feedback $u_1 = -kx_1$, $u_2 = -kx_2$ as in Eqs. (5). Define the following linear transformation

$$\begin{aligned} x_{2,1} &= x_1 \\ x_{2,j} &= j\, s_{1,j-1}, \qquad j = 2, 3, \dots, n-1 \end{aligned} \tag{8}$$

Then one obtains the following system in terms of $x_{2,j}$, for $1 \le j \le n-1$.

$$\dot{x}_{2,1} = u_{2,1} \tag{9a}$$

$$\dot{x}_{2,j} = \frac{1}{(j-2)!} x_{2,1}^{j-2} u_{2,2}, \qquad j = 2, 3, \dots, n-1 \tag{9b}$$

where

$$u_{2,1} = u_1 \tag{10a}$$

$$u_{2,2} = x_1 u_2 - x_2 u_1 \tag{10b}$$

The system in Eqs. (9) will be called the *generated system* of second order and we use the first index in the subscript of the state elements to denote this. For consistency, we define the first generated system to be simply the original system in Eqs. (1), that is, we let $x_{1,j} = x_j$ for $1 \le j \le n$. Notice that the system in Eqs. (9) is a system in power form of dimension $(n-1)$. The same process can be therefore repeated.

After repeating this process $(i-1)$ times one obtains the ith generated system (of dimension $(n-i+1)$)

$$\dot{x}_{i,1} = u_{i,1} \tag{11a}$$

$$\dot{x}_{i,j} = \frac{1}{(j-2)!} x_{i,1}^{j-2} u_{i,2}, \quad j = 2, 3, \dots, n-i+1 \tag{11b}$$

For the ith generated system, one can construct $(n-i-1)$ invariant manifolds using the linear control law

$$u_{i,1} = -kx_{i,1} \tag{12a}$$
$$u_{i,2} = -2^{i-1}\,k\,x_{i,2} \tag{12b}$$

and the methodology described earlier. The corresponding manifolds are defined by

$$\Pi_{i,j} = \{x \in \Re^{n-i+1} : s_{i,j}(x) = 0,\ 1 \le j \le n-i-1\} \tag{13}$$

where

$$s_{i,j} = x_{i,j+2} - \frac{2^{i-1}}{j!\,(2^{i-1}+j)}\,x_{i,1}^{j}\,x_{i,2} \tag{14}$$

where $j = 1, 2, \ldots, n-i-1$. Defining now

$$\begin{aligned} x_{i+1,1} &= x_{i,1} \\ x_{i+1,j} &= (2^{i-1}+j)\,s_{i,j-1}, \qquad j = 2, 3, \ldots, n-i \end{aligned}$$

the $(i+1)$th generated system can be described as follows

$$\begin{aligned} \dot{x}_{i+1,1} &= u_{i+1,1} \\ \dot{x}_{i+1,j} &= \frac{1}{(j-2)!}\,x_{i+1,1}^{j-2}\,u_{i+1,2}, \qquad j = 2, 3, \ldots, n-i \end{aligned}$$

where

$$u_{i+1,1} = u_{i,1} \tag{15a}$$
$$u_{i+1,2} = x_{i,1}u_{i,2} - 2^{i-1}\,u_{i,1}\,x_{i,2} \tag{15b}$$

This process can be continued until the $(n-2)$th generated system, which is the 3-dimensional system

$$\dot{x}_{n-2,1} = u_{n-2,1} \tag{16a}$$
$$\dot{x}_{n-2,2} = u_{n-2,2} \tag{16b}$$
$$\dot{x}_{n-2,3} = x_{n-2,1}\,u_{n-2,2} \tag{16c}$$

By construction, it is immediate that if $x_{i+1,1} = x_{i+1,2} = x_{i+1,3} = \cdots = x_{i+1,n-i} = 0$ for the $(i+1)$th generated system, then for the ith generated system we have that $x_{i,1} = x_{i,3} = \cdots = x_{i,n-i+1} = 0$. Thus, any convergent feedback controller about the origin for the $(i+1)$th generated system, will also make the ith generated system converge to the one-dimensional manifold $\mathcal{M}_i = \{x \in \Re^{n-i+1} : x_{i,1} = x_{i,3} = \cdots = x_{i,n-i+1} = 0\}$. In particular, if the controller for $(i+1)$th generated system is chosen such that, in addition, it satisfies the property $\lim_{t\to\infty} x_{i,2} = 0$ then the same control law will drive the ith generated system converge to the *origin*.

3.2. The 3-dimensional System

The first step in the proposed derivation of the feedback controller is to construct a static, state-feedback controller for the $(n-2)$th generated system in Eqs. (16). From the discussion in Section 1, this controller has to be necessarily non-smooth. For notational convenience, let us redefine $z_i = x_{n-2,i}$ $(i = 1, 2, 3)$ and $v_i = u_{n-2,i}$ $(i = 1, 2)$. Then the system in Eqs. (16) can be rewritten as

$$\dot{z}_1 = v_1 \tag{17a}$$
$$\dot{z}_2 = v_2 \tag{17b}$$
$$\dot{z}_3 = z_1 v_2 \tag{17c}$$

The following lemma will be useful in the sequel.

Lemma 3.1 *Consider the scalar, linear differential equation*

$$\dot{y} = -\alpha\,y + \sum_{j=1}^{N} h_j\,e^{-\beta_j t}, \quad \alpha > 0, \ \ \beta_j > 0 \tag{18}$$

where h_j, $(j = 1, 2, \ldots, N)$ are constants. Then, the solution $y(t)$ of Eq. (18) decays exponentially to zero. If, in addition, $\beta_j > \alpha$ for $j = 1, 2, \ldots, N$, then the solution decays exponentially with rate of decay α.

The proof of this lemma can be easily established by direct integration of Eq. (18). The following theorem provides an exponentially convergent controller for the system in Eqs. (17).

Theorem 3.1 *Consider the system in Eqs. (17) and the feedback control*

$$v_1 = -k\,z_1 \tag{19a}$$
$$v_2 = -m\,k\,z_2 - (m+1)\,\mu\,\frac{s}{z_1} \tag{19b}$$

with $k > 0$, $m > 0$ and $\mu > (m+1)\,k$, and where

$$s = z_3 - \frac{m}{m+1}\,z_1 z_2 \tag{20}$$

This control law is bounded along the trajectories of the system and has the property that, for the closed-loop system, $\lim_{t\to\infty}(z_1(t), z_2(t), z_3(t)) = 0$ with exponential rate of convergence, for all initial conditions such that $z_1(0) \neq 0$.

Proof: The proof is quite straightforward. First, notice that $z_1 = z_1(0)\,e^{-kt}$, and z_1 decreases exponentially with rate of decay k. The variable s in Eq. (20) is the invariant manifold for the system in Eq. (17) under the linear feedback

$$v_1 = -k\,z_1 \tag{21a}$$
$$v_2 = -m\,k\,z_2 \tag{21b}$$

Using Eqs. (19), the differential equation for s is

$$\dot{s} = \dot{z}_3 - \frac{m}{m+1}(v_1 z_2 + z_1 v_2) = -\mu\,s \tag{22}$$

and s decreases exponentially with rate of decay μ.

By definition, $\lim_{t\to\infty} s(t) = 0$ implies that $\lim_{t\to\infty} z_3(t) = 0$. The differential equation for z_2 can be written as follows

$$\dot{z}_2 = -\,m\,k\,z_2 - (m+1)\,\mu\,\gamma(t) \tag{23}$$

where the function γ is an exponentially decaying function with rate of decay $\mu - k$, since

$$\gamma(t) = \frac{s(t)}{z_1(t)} = \frac{s_0}{z_{10}}\,e^{-(\mu-k)t} \tag{24}$$

From Lemma 3.1 and the fact that $\mu > (m+1)\,k$, one has $\lim_{t\to\infty} z_2(t) = 0$. Moreover, the rate of decay of z_2 is equal to mk. Therefore the closed-loop trajectories of the system in Eqs. (17) with the control law in Eqs. (21) have the property that $\lim_{t\to\infty}(z_1(t), z_2(t), z_3(t)) = 0$.

The claim that the control law (19) is bounded follows immediately by virtue of Eq. (24). ■

4. The Feedback Controller

The following theorem contains the main result of this paper. It shows that the convergent control law in Eqs. (19), can also be used to drive the original system in Eqs. (1) to the origin.

Theorem 4.1 *Consider the system in Eqs. (1) ($n \geq 3$) and the feedback controller*

$$\begin{aligned} u_1 &= u_{n-2,1} && (25a) \\ u_2 &= -\sum_{\ell=0}^{n-4} 2^{\ell}\, k\, \frac{x_{1+\ell,2}}{x_1^{\ell}} + \frac{u_{n-2,2}}{x_1^{n-3}} && (25b) \end{aligned}$$

where

$$\begin{aligned} u_{n-2,1} &= -k\, x_1 && (26a) \\ u_{n-2,2} &= -m\, k\, x_{n-2,2} - (m+1)\, \mu\, \frac{s}{x_{n-2,1}} && (26b) \end{aligned}$$

where $k > 0$, $\mu > (m+1)\, k$, $m = 2^{n-3}$, and where

$$s = x_{n-2,3} - \frac{m}{m+1}\, x_{n-2,2}\, x_1 \qquad (27)$$

and $x_{n-2,2}$, $x_{n-2,3}$ and $x_{\ell,2}$ ($\ell = 1, 2, \ldots, n-3$) are derived through the recursive process described in Section 3. Then this control law is bounded along the trajectories of the system and has the property that $\lim_{t\to\infty}(x_1(t), x_2(t), \ldots, x_n(t)) = 0$ with exponential rate of convergence, for all initial conditions such that $x_1(0) \neq 0$.

Proof: The proof of this theorem requires repeated use of Lemma 3.1. We assume here that $n \geq 4$ since the case when $n = 3$ has been addressed in Theorem 3.1.

From Theorem 3.1 we have that the control law in Eqs. (26) achieves $\lim_{t\to\infty} x_{n-2,j}(t) = 0$ for $j = 1, 2, 3$. In addition, from the same theorem we have that the function

$$\gamma_1 = \frac{s}{x_1} \qquad (28)$$

decays exponentially with rate $\mu - k$.

The rest of the proof is shown by induction. To this end, let us assume that for the $(i+1)$th order generated system we have that $\lim_{t\to\infty} x_{i+1,j}(t) = 0$, for $j = 1, 2, 3, \ldots, n-i$, which implies that $\lim_{t\to\infty} x_{i,j}(t) = 0$, for $j = 1, 3, \ldots, n-i+1$. It has been shown previously that with the control law in Eqs. (26) $x_{n-2,2}$ decays exponentially with rate $2^{n-3}\, k$ and $x_{n-3,2}$ decays exponentially with rate $2^{n-4}\, k$. Assume now that the functions $x_{i+\ell,2}$ ($\ell = n-2-i, n-3-i, \ldots, 1$) decay exponentially, each with corresponding rate $2^{i+\ell-1}\, k$. We will show that also $\lim_{t\to\infty} x_{i,2}(t) = 0$ with rate $2^{i-1}\, k$.

The differential equation for $x_{i,2}$ is given by

$$\begin{aligned} \dot{x}_{i,2} &= -2^{i-1}\, k\, x_{i,2} - \sum_{\ell=1}^{n-2-i} 2^{i-1+\ell}\, k\, \frac{x_{i+\ell,2}}{x_1^{\ell}} \\ &\quad -(m+1)\, \mu\, \frac{s}{x_1^{n-1-i}}, \qquad i = 1, 2, \ldots, n-3 && (29) \end{aligned}$$

A straightforward calculation shows that ($n \geq 4$) $2^{n-3} + 2 - n + i \geq 2^{i-1}$ for all $i = 1, 2, \ldots, n-3$ and $2^{i+\ell-1} - \ell > 2^{i-1}$ for all $i = 2, 3, \ldots, n-3$ and $\ell = n-2-i, n-3-i, \ldots, 1$. Since $\mu > (2^{n-3}+1)\, k$, the functions

$$\gamma_{n-1-i} = \frac{s}{x_1^{n-1-i}}, \qquad i = 1, 2, \ldots, n-3 \qquad (30)$$

decay exponentially with rates $\mu - (n-1-i)\, k > 2^{i-1}\, k$, where $i = 1, \ldots, n-3$. Moreover, since by assumption $x_{i+\ell,2}$ decays exponentially with rate $2^{i+\ell-1}\, k$ and x_1 decays with rate k, one has that the functions

$$\rho_{\ell} = \frac{x_{i+\ell,2}}{x_1^{\ell}}, \qquad \ell = n-2-i, n-3-i, \ldots, 1 \qquad (31)$$

decay exponentially with rate $(2^{i+\ell-1} - \ell)\, k > 2^{i-1}\, k$ for $i = 2, 3, \ldots, n-3$ and $\ell = 1, \ldots, n-2-i$. Use of Lemma 3.1 indicates that

$$\lim_{t\to\infty} x_{i,2}(t) = 0, \qquad i = 1, 2, \ldots, n-3 \qquad (32)$$

with rate of decay at least $2^{i-1}\, k$ and the proof is complete.

The fact the the control law in Eqs. (25) is bounded follows immediately from the fact that x_1 reaches the origin only asymptotically (not in finite time), and the fact that the the functions $\rho_{\ell}(t)$ in Eq. (31) and $\gamma_{\nu}(t)$ in Eq. (30) are bounded. ■

The control law in Eqs. (25) makes the constructed invariant manifolds in each step attractive. It is clear that for this procedure to work the attraction of the trajectories to the corresponding manifolds at each step should take place on different times scales. For the sake of simplicity, this is achieved by taking the gains of the control laws $u_{i,2}$ twice as the one in the previous step. This can be relaxed at the expense of more complicated control expressions.

Remark 4.1 The control law in Eqs. (25) will work as long as $x_1(0) \neq 0$. If initially $x_1(0) = 0$ one can use any control law such that x_1 becomes nonzero. One possible choice is to use $u_1 = u_{10}$, $u_2 = 0$, where u_{10} is some nonzero constant.

5. Numerical Example

Simulation results show that the recursive algorithm provides an effective way to drive an n-dimensional nonholonomic control system in power form to the origin with exponential rates of convergence. Figure 1 shows the trajectories of a 5-dimensional system with initial state $x(0) = (4, 1, -1, -2, 3)$ subject to the feedback control in Eqs. (25). The gains were chosen as $k = 1$ and $\mu = 6$. Notice that the control law drives the states to the origin at different time scales. Figure 2 shows the time history of the logarithm of the euclidean norm of the state. The linear slopes indicate the exponential rate of convergence. The control effort is shown in Fig. 3.

6. Conclusions

We present a new technique for constructing exponentially convergent controllers for nonholonomic systems in power form. The construction of the proposed control laws is based on a recursive algorithm which uses a series of invariant manifolds in order to construct a sequence of generated systems in power form of reduced dimension. Using this process, one ends up with a 3-dimensional system in power form. The proposed control law for the n-dimensional system is the one that stabilizes this 3-dimensional system by proper choice of the gains. Finally, because of the equivalence between chained and power form systems, the control laws proposed here can also be used for nonholonomic systems in chained form as well.

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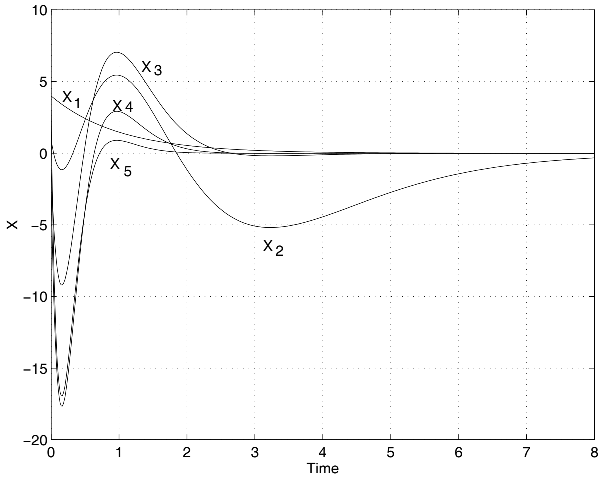

Figure 1: Trajectories of closed-loop system.

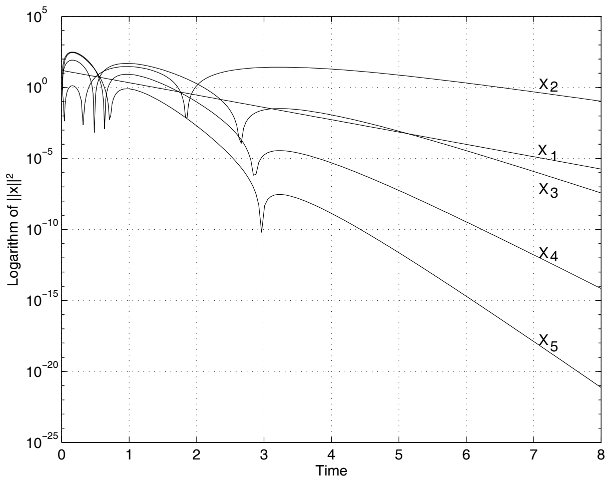

Figure 2: Logarithmic plot of $\|x\|^2$ vs. time.

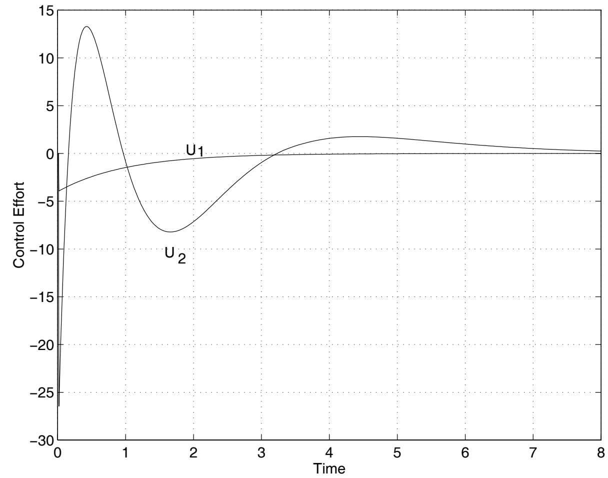

Figure 3: Control history.

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