An $\mathcal{L}_2$ Disturbance Attenuation Approach to the Nonlinear Benchmark Problem

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Abstract

In this paper, we use the theory of $\mathcal{L}_2$ disturbance attenuation for linear ($\mathcal{H}_\infty$) and nonlinear systems to obtain solutions to the Nonlinear Benchmark Problem (NLBP) proposed in the companion paper by Bupp *et. al.* [2]. By considering a series expansion solution to the Hamilton-Jacobi-Isaacs Equation associated with the nonlinear disturbance attenuation problem, we obtain a series expansion solution for a nonlinear controller. Numerical simulations compare the performance of the third order approximation of the nonlinear controller with its first order approximation (which is the same as a linear $\mathcal{H}_\infty$ controller obtained from the linearized problem.)

1 Introduction

The control of nonlinear systems has received much attention in recent years and many nonlinear control design methodologies have been developed. It is important to determine the advantages and limitations of the different nonlinear control design methodologies. The Nonlinear Benchmark Problem (NLBP) proposed by Bupp *et. al.* [2] is an initial attempt to achieve this objective.

The NLBP involves a cart of mass M whose mass center is constrained to move along a straight horizontal line; see Figure 1. Attached to the cart is a "proof body" actuator of mass m and moment of inertia I. Relative to the cart, the proof body rotates about a vertical line passing through the cart mass center. The nonlinearity of the problem comes from the interaction between the translational motion of the cart and the rotational motion of the eccentric proof mass.

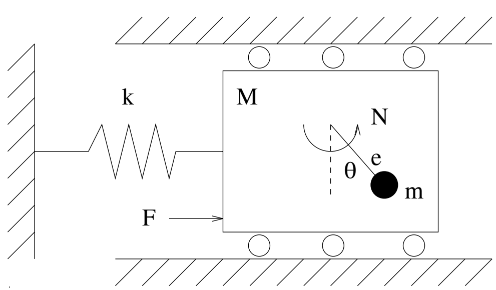

Figure 1: Nonlinear Benchmark Problem

After suitable normalization [2], the equations of motion for this nonlinear system are

$$\ddot{\xi} + \xi = \varepsilon\,(\dot{\theta}^2 \sin\theta - \ddot{\theta}\cos\theta) + w \tag{1a}$$

$$\ddot{\theta} = -\varepsilon\,\ddot{\xi}\cos\theta + u \tag{1b}$$

where ξ is the (nondimensionalized) translational position of the cart and θ is the angular position of the rotational proof body. In equations (1), w and u are the (nondimensionalized) exogenous disturbance and the control torque, respectively. The coupling between the translational and rotational motions is captured by the parameter ε which is defined by

$$\varepsilon := \frac{me}{\sqrt{(I + me^2)(M+m)}} \tag{2}$$

where e is eccentricity of the rotor. Clearly, $0 \leq \varepsilon < 1$ and $e = 0$ if and only if $\varepsilon = 0$; in this case the translational and rotational motions decouple and equations (1) reduce to

$$\ddot{\xi} + \xi = w \tag{3a}$$

$$\ddot{\theta} = u \tag{3b}$$

For this latter system, the effect of w is completely decoupled from the effect of u. Also, to control translational motion using u, the eccentricity must be nonzero.

Letting $x := [x_1, x_2, x_3, x_4]^T = [\xi, \dot{\xi}, \theta, \dot{\theta}]^T$, the system (1) can be written compactly in state-space form as

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{-x_1 + \varepsilon x_4^2 \sin x_3}{1-\varepsilon^2\cos^2 x_3} \\ x_4 \\ \frac{\varepsilon\cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3} \\ 0 \\ \frac{1}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{1}{1-\varepsilon^2\cos^2 x_3} \\ 0 \\ \frac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix} w \tag{4}$$

2 Problem Definition

In this paper, we propose the following control design problem to address the qualitative design guidelines given in [2]:

Disturbance Attenuation Problem (DAP): For the system (4) find a memoryless state-feedback controller

$$u = k(x) \tag{5}$$

such that the following conditions hold:

1. When $w(t) = 0$ for all $t \geq 0$,
$$\lim_{t\to\infty} x(t) = 0$$
for all initial states $x(0)$ in some neighborhood $\mathcal{D}$ of the origin.

2. Let z denote a performance output defined by

$$z = \begin{bmatrix} Cx \\ u \end{bmatrix} \quad (6)$$

where the matrix C is a design parameter. Given any disturbance $w \in \mathcal{L}_2(0,\infty)$ and zero initial state ($x(0) = 0$), the closed loop system satisfies

$$\int_0^\infty \{||z(t)||^2 - \gamma^2 ||w(t)||^2\}\, dt \leq 0 \quad (7)$$

where γ is a design parameter.

Note that the second requirement implies that the $\mathcal{L}_2$-gain of the closed loop system from the disturbance input w to the performance output z is less than or equal to γ.

The Disturbance Attenuation Problem has been treated in [1, 3, 5]. In these references it has been shown that, under mild conditions, the DAP can be solved, provided one has a positive definite solution to the so-called Hamilton-Jacobi-Isaacs Equation. The original idea behind this approach was to formulate the DAP as a differential game where u and w are the two opposing players. The next section reviews the basic results of [3, 5] which will be used in the sequel.

3 The Hamilton-Jacobi-Isaacs Equation (HJIE)

System (4), along with its performance output, is described by

$$\begin{aligned} \dot{x} &= F(x) + G_1(x)u + G_2(x)w && \text{(8a)} \\ z &= \begin{bmatrix} Cx \\ u \end{bmatrix} && \text{(8b)} \end{aligned}$$

where the functions F, G_1, G_2 are obtained from (4). Also

$$F(0) = 0 \quad (9)$$

and we assume that the system

$$\begin{aligned} \dot{x} &= F(x) \\ z &= Cx \end{aligned}$$

is observable in the sense that the zero solution is the only solution for which $z(t)$ is zero for all t.

One can readily show [3, 5, 6] that if there is a positive definite function V which satisfies the following Hamilton-Jacobi-Isaacs Equation

$$DV(x)F(x) - \frac{1}{4}DV(x)(\, G_1(x)G_1^T(x) - \gamma^{-2}G_2(x)G_2^T(x)\,)DV^T(x) + x^TC^TCx = 0 \quad (10)$$

where DV is the derivative of V, i.e,

$$DV = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix}$$

then the feedback controller

$$u_*(x) = -\tfrac{1}{2}G_1^T(x)DV^T(x) \quad (11)$$

yields a closed loop system with the following property: For every initial condition $x(0) = x_0$ and for every disturbance input w one has

$$\int_0^\infty \{||z(t)||^2 \; - \gamma^2||w||^2\}dt \quad \leq \quad V(x_0) \quad (12)$$

Also, the "worst case disturbance" is given by

$$w_*(x) = \frac{1}{2\gamma^2}G_2^T(x)DV^T(x) \quad (13)$$

Using V as a Lyapunov function one can show that the undisturbed ($w = 0$) closed loop system corresponding to controller (11) is globally asymptotically stable. Hence, a solution to the DAP is given by controller (11).

The main stumbling block in using the above result is that only rarely is one able to compute a function V satisfying (10) in *closed-form*. So, instead of insisting on closed form solutions, we solve (10) in an iterative fashion based on series expansions. This is the methodology proposed in [4, 7] for the solution of Hamilton-Jacobi equations arising in optimal control problems. We demonstrate here that the same procedure can be applied to nonlinear $\mathcal{L}_2$ disturbance attenuation problems, provided that the linearized version of the problem has a solution.

First we rewrite system (8) in the form

$$\begin{aligned} \dot{x} &= F(x) + G(x)v && \text{(14a)} \\ z &= \begin{bmatrix} Cx \\ u \end{bmatrix} && \text{(14b)} \end{aligned}$$

where

$$G(x) := [G_1(x)\ G_2(x)] \qquad \text{and} \qquad v := \begin{bmatrix} u \\ w \end{bmatrix} \quad (15)$$

Letting

$$Q(x) := x^TC^TCx, \qquad R := \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix} \quad (16)$$

HJIE (10) can be rewritten as

$$DVF - \frac{1}{4}DVGR^{-1}G^TDV^T + Q = 0 \quad (17)$$

and letting

$$v_* := \begin{bmatrix} u_* \\ w_* \end{bmatrix}$$

we have

$$v_* = -\tfrac{1}{2}R^{-1}G^TDV^T \quad (18)$$

4 Problem Solution

4.1 Linearized Problem

In the next section, it will be shown that the first term in the series expansion for the controller (11) is the solution to the corresponding linearized problem. Thus, we first consider the linearized problem, which amounts to solving a linear state-feedback $\mathcal{H}_\infty$ problem.

The linearization of system (14) about $x = 0$ is given by

$$\begin{aligned} \dot{x} &= Ax + Bv && \text{(19a)} \\ z &= \begin{bmatrix} Cx \\ u \end{bmatrix} && \text{(19b)} \end{aligned}$$

with

$$A = DF(0)\,, \quad B = G(0)$$

Considering a quadratic form

$$V(x) = x^TPx$$

as a candidate solution to the HJIE associated with the linear DAP we obtain

$$x^T[PA + A^TP - PBR^{-1}B^TP + C^TC]x = 0$$

This is satisfied for all x iff the matrix P solves the following Algebraic Riccati Equation (ARE):

$$PA + A^T P - PBR^{-1}B^T P + C^T C = 0 \tag{20}$$

In this case, v_* is given by

$$v_*(x) = -R^{-1}B^T P x \tag{21}$$

Also, the corresponding controller u_* is a suboptimal (in terms of the achievable $\mathcal{H}_\infty$ norm from w to z) $\mathcal{H}_\infty$ state feedback controller.

According to standard linear $\mathcal{H}_\infty$ theory, if the pair (C, A) is observable and the pair (A, B_1) $(B_1 = G_1(0))$ is stabilizable, a necessary and sufficient condition for the linear system (19) to have $\mathcal{L}_2$-gain less than γ is that the above Algebraic Riccati Equation (ARE) has a positive definite solution P with the matrix

$$A_* := A - BR^{-1}B^T P \tag{22}$$

Hurwitz.

4.2 Nonlinear Problem

First note that the HJIE (17) can be written as

$$\begin{aligned} DVF - v_*^T R v_* + Q &= 0 && (23a)\\ v_* + \tfrac{1}{2}R^{-1}G^T DV^T &= 0 && (23b)\end{aligned}$$

Suppose

$$\begin{aligned} F &= F^{[1]} + F^{[2]} + F^{[3]} + \cdots \\ G &= G^{[0]} + G^{[1]} + G^{[2]} + \cdots \end{aligned}$$

where $F^{[k]}$, $G^{[k]}$ are homogeneous functions of order k. A homogeneous function of order k in n scalar variables $x_1, x_2, \ldots, x_n$ is a linear combination of

$$N_k^n := \left(\begin{array}{c} n+k-1 \\ k \end{array} \right)$$

terms of the form $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$, where i_j is a nonnegative integer for $j = 1, ..., n$ and $i_1 + i_2 + \cdots + i_n = k$. The vector whose elements consist of these terms is denoted by $x^{[k]}$; for example, with four scalar variables one has

$$\begin{aligned} x^{[1]} &= [x_1, x_2, x_3, x_4]^T \\ x^{[2]} &= [{x_1}^2, x_1x_2, {x_2}^2, x_1x_3, x_2x_3, {x_3}^2, x_1x_4, x_2x_4, x_3x_4, {x_4}^2]^T \end{aligned}$$

Therefore, a scalar homogeneous function $\psi^{[k]}$ of order k can be written as

$$\psi^{[k]}(x) = \Psi x^{[k]} \tag{24}$$

where $\Psi \in \mathbb{R}^{1 \times N_k^n}$. Note that

$$F^{[1]}(x) = Ax, \qquad G^{[0]}(x) = B$$

We consider a series expansion for V of the form

$$V = V^{[2]} + V^{[3]} + V^{[4]} + \cdots \tag{25}$$

where $V^{[k]}$ is a homogeneous function of order k. Substituting (25) in equation (23b) one obtains that

$$v_* = v_*^{[1]} + v_*^{[2]} + v_*^{[3]} + \cdots \tag{26}$$

where $v_*^{[k]}$ is the homogeneous function of order k given by

$$v_*^{[k]} = -\tfrac{1}{2}R^{-1} \sum_{j=0}^{k-1} G^{[j]\,T} DV^{[k+1-j]\,T} \tag{27}$$

Substitution of the expansions in (25) and (26) into (23a) and equating terms of order $m \geq 2$ to zero yields

$$\sum_{k=0}^{m-2} DV^{[m-k]} F^{[k+1]} - \sum_{k=1}^{m-1} v_*^{[m-k]\,T} R v_*^{[k]} + Q^{[m]} = 0 \tag{28}$$

For $m = 2$ equation (28) simplifies to

$$DV^{[2]}F^{[1]} - v_*^{[1]} R v_*^{[1]} + Q^{[2]} = 0$$

Since $F^{[1]}(x) = Ax$,

$$v_*^{[1]} = -\tfrac{1}{2}R^{-1}B^T DV^{[2]\,T}$$

and

$$Q^{[2]}(x) = x^T C^T C x$$

we obtain

$$DV^{[2]}(x)Ax - \frac{1}{4}DV^{[2]\,T}(x)BR^{-1}B^T DV^{[2]}(x) + x^T C^T C x = 0$$

which is the HJIE for the linearized problem. Hence

$$V^{[2]}(x) = x^T P x$$

where $P^T = P > 0$ solves the ARE with $A_* := A - BR^{-1}B^T P$ Hurwitz; also,

$$v_*^{[1]}(x) = Kx\,, \qquad K = -R^{-1}B^T P \tag{29}$$

Consider now any $m \geq 3$ and rewrite (28) as

$$\sum_{k=0}^{m-2} DV^{[m-k]}F^{[k+1]} - 2v_*^{[m-1]\,T} R v_*^{[1]} - \sum_{k=2}^{m-2} v_*^{[m-k]\,T} R v_*^{[k]} = 0$$

Note that the last term in the above expression does not depend on $V^{[m]}$. Using

$$v_*^{[m-1]\,T} = -\tfrac{1}{2} \sum_{k=0}^{m-2} DV^{[m-k]} G^{[k]} R^{-1}$$

and defining

$$f := F + Gv_*^{[1]} \tag{30}$$

the first two terms can be written as

$$\begin{aligned} & \sum_{k=0}^{m-2} DV^{[m-k]}F^{[k+1]} + \sum_{k=0}^{m-2} DV^{[m-k]}G^{[k]}v_*^{[1]} \\ = \; & \sum_{k=0}^{m-2} DV^{[m-k]}f^{[k+1]} \\ = \; & DV^{[m]}f^{[1]} + \sum_{k=1}^{m-2} DV^{[m-k]}f^{[k+1]} \end{aligned}$$

where

$$f^{[1]}(x) = A_* x \tag{31}$$

For $m \geq 3$, equation (28) can now be written as

$$DV^{[m]}f^{[1]} = -\sum_{k=1}^{m-2} DV^{[m-k]}f^{[k+1]} + \sum_{k=2}^{m-2} v_*^{[m-k]\,T} R v_*^{[k]} \tag{32}$$

Equation (32) can be solved for $V^{[m]}$ as follows. Consider an expression for $V^{[m]}(x)$ of the form $V^{[m]}(x) = V_m x^{[m]}$, with $V_m \in \mathbb{R}^{1 \times N_m^n}$. Substitute this expression for $V^{[m]}(x)$ into (32) and solve the resulting linear system of N_m^n equations for the unknown N_m^n elements of the coefficient vector V_m.

Thus, starting with $V^{[2]}(x) = x^T P x$ and $v_*^{[1]}(x) = Kx$ one can use equations (32) and (27) to compute consecutively the sequence of terms

$$V^{[3]}, v_*^{[2]}, V^{[4]}, v_*^{[3]}, \dots \tag{33}$$

and construct iteratively the solution V of HJIE and the associated v_*. Notice that this procedure generates not only the feedback control strategy $u_*(x)$ defined in (11) for disturbance attenuation, but also the worst case disturbance $w_*(x)$ given in (13).

5 The Nonlinear Benchmark Problem

For the performance output z, we chose $C = \sqrt{0.1}\,\mathrm{I}$ where I is the 4×4 identity matrix. We also chose the eccentricity parameter $\varepsilon = 0.5$.

In order to apply the proposed methodology to the NLBP, we first expand the F and G functions corresponding to the right hand side of (4) in a multivariable series expansion about $x = 0$. Noting that $1 - \varepsilon^2 \cos^2 x_3 \neq 0$, these expansions can be readily computed as

$$F(x) = \begin{bmatrix} x_2 \\ -\frac{4}{3}x_1 + \frac{2}{3}{x_4}^2 x_3 + \frac{4}{9}x_1 x_3^2 + \cdots \\ x_4 \\ \frac{2}{3}x_1 - \frac{1}{3}x_4^2 x_3 - \frac{5}{9}x_1 x_3^2 + \cdots \end{bmatrix} \tag{34}$$

$$G(x) = \begin{bmatrix} 0 & 0 \\ -\frac{2}{3} + \frac{5}{9}x_3^2 + \cdots & \frac{4}{3} - \frac{4}{9}x_3^2 + \cdots \\ 0 & 0 \\ \frac{4}{3} - \frac{4}{9}x_3^2 + \cdots & -\frac{2}{3} + \frac{5}{9}x_3^2 + \cdots \end{bmatrix} \tag{35}$$

5.1 Linear terms

The linearized system is given by (19) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{4}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{2}{3} & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 \\ \frac{4}{3} & -\frac{2}{3} \end{bmatrix} \tag{36}$$

Using this data one can show that the linearized problem has a solution if and only if $\gamma > \gamma_* \approx 2.2$. Choosing $\gamma = 3$ one can solve (20) for $P > 0$ and compute the linear term of $v_*(x)$ as

$$v_*^{[1]}(x) = \begin{bmatrix} 0.3568x_1 + 0.0409x_2 - 0.3184x_3 - 0.9275x_4 \\ -0.0015x_1 + 0.1782x_2 + 0.0126x_3 + 0.0330x_4 \end{bmatrix}$$

where the first row is the control $u_*^{[1]}(x)$ and the second row the disturbance $w_*^{[1]}(x)$.

5.2 Higher order terms

The higher order terms are calculated using the procedure described in Section 4.2. The calculations are simplified for the NLBP because, as it is evident from (34) and (35),

$$F^{[2k]}(x) = 0\,, \qquad G^{[2k-1]}(x) = 0, \qquad k = 1, 2, \dots \tag{37}$$

As a result, $V^{[3]}(x) = 0$ and $v_*^{[2]}(x) = 0$. The first nonzero higher order term for the controller is third order and can be computed from

$$v_*^{[3]}(x) = -\frac{1}{2}R^{-1}\left(B^T DV^{[4]\,T}(x) + G^{[2]T}(x) DV^{[2]\,T}(x)\right)$$

$$DV^{[4]}(x) A_* x = -DV^{[2]}(x)\left(F^{[3]}(x) + G^{[2]}(x) v_*^{[1]}(x)\right)$$

Specifically, both control (first row) and disturbance (second row) are given by

$$v_*^{[3]}(x) = \begin{bmatrix} \begin{array}{l} +0.0471{x_4}^2 x_3 - 0.0018 x_1 {x_3}^2 + 0.4305 {x_4}^2 x_1 \\ -0.0341{x_4}^3 + 0.7114 x_4 x_2 x_1 - 0.3779 x_4 {x_2}^2 \\ +0.2721 x_4 x_3 x_1 + 0.2080 x_4 x_3 x_2 - 0.3980 x_3 x_1^2 \\ +0.3076 x_3 x_2 x_1 - 0.1482 x_3 {x_2}^2 + 8\cdot 10^{-5} x_3^3 \\ +0.3239 {x_2}^2 x_1 + 0.2585 {x_3}^2 x_2 - 0.69301 x_4 x_1^2 \\ +0.0217 x_4 x_3^2 - 0.1116 {x_4}^2 x_2 - 0.43100 x_2 x_1^2 \\ -0.1937 {x_2}^3 + 0.1306 {x_1}^3 \end{array} \\ \begin{array}{l} +0.0268 {x_4}^2 x_3 - 0.0137 x_1 {x_3}^2 - 0.0325 {x_4}^2 x_1 \\ +0.0237 {x_4}^3 - 0.0678 x_4 x_2 x_1 + 0.1860 x_4 {x_2}^2 \\ -0.0231 x_4 x_3 x_1 + 0.1432 x_4 x_3 x_2 + 0.0203 x_3 {x_1}^2 \\ -0.0081 x_3 x_2 x_1 + 0.0987 x_3 {x_2}^2 - 0.0002 {x_3}^3 \\ -0.0354 x_2^2 x_1 + 0.0499 {x_3}^2 x_2 + 0.0450 x_4 {x_1}^2 \\ -0.0213 x_4 {x_3}^2 + 0.1288 {x_4}^2 x_2 + 0.0713 x_2 x_1^2 \\ +0.0949 {x_2}^3 - 0.0052 {x_1}^3 \end{array} \end{bmatrix}$$

In fact, because of (37), one can show that all the nontrivial terms of the series expansion for $v_*(x)$ are odd, that is, $v_*^{[2k]}(x) = 0$ for $k = 1, 2, 3, \dots$.

6 Simulations

Here, we compare the closed loop system with the linear controller $u_*^{[1]}$ with the closed loop system with the nonlinear controller $u_*^{[1]} + u_*^{[3]}$. All the symbolic calculations for the gains of the nonlinear controller were performed using MAPLE. The numerical simulations were performed using MATLAB.

Two simulations were performed. The first simulation compared the two controllers on the issue of asymptotic stability. The results are shown in Figures 2-3 which contain the "phase portraits" of the variables ξ and θ. The solid lines denote the response due to the nonlinear controller, and the dashed lines denote the response due to the linear controller. The initial conditions for this simulation were chosen as $x(0) = [5, 5, -2, 2]^T$. ¿From this simulation, it seems that the region of attraction due to the nonlinear controller is larger than that due to the linear controller: for the chosen initial state, the state trajectory resulting from nonlinear control tends asymptotically to the origin, whereas the trajectory resulting from linear control tends to a limit cycle. The control histories are shown in Figure 4.

The second simulation compares the disturbance attenuation properties of the two controllers. Since the computation of the $\mathcal{L}_2$-gain of a nonlinear system is not an easy task, we carried out the following procedure. We simulated the nonlinear closed loop system, with both the linear and the nonlinear controller, with zero initial state and with a disturbance w that approximates in some sense the worst possible disturbance for this problem. More specifically, we took the disturbance $w = \widehat{w}$, where $\widehat{w}$ is the solution to the problem of maximizing

$$\int_0^\infty \{||z(t)||^2 - \gamma^2 ||w(t)||^2\}\, dt$$

subject to the linearized closed loop dynamics (which is the same for both controllers) and a given initial condition.

A measure of the disturbance attenuation level, at any time $T > 0$, of the closed loop system is given by

$$ratio(T) := \frac{\int_0^T ||z(t)||^2\, dt}{\gamma^2 \int_0^T ||\widehat{w}(t)||^2\, dt} \qquad (38)$$

Note that if the $\mathcal{L}_2$-gain of a nonlinear closed loop system is less than or equal to γ, then $ratio(T) \leq 1$ for all $T > 0$.

Simulations were performed using several initial conditions to generate the "worst disturbance" $\widehat{w}$. In all cases, the nonlinear controller outperformed the linear one although only by a small margin. Figure 5 contains results when the disturbance $\widehat{w}$ is generated with initial condition $x(0) = [5, 5, -2, 2]^T$. In this case, it turns out that $\lim_{T\to\infty} ratio(T) = 0.74$ for the linear controller, while $\lim_{T\to\infty} ratio(T) = 0.65$ for the nonlinear controller.

7 Conclusion

We have applied the theory of $\mathcal{L}_2$ disturbance attenuation for nonlinear systems to the recently proposed nonlinear benchmark problem. A nonlinear state-feedback controller is computed recursively by considering a series expansion solution to the associated Hamilton-Jacobi-Isaacs Equation. The procedure is straightforward and can be readily automated in a computer. Numerical simulations indicate that the performance of the third order approximation of the nonlinear controller provides an improvement over its first order approximation (which is the same as a linear $\mathcal{H}_\infty$ controller obtained from the linearized problem). This improvement is however not very significant, thus indicating that higher order terms may be necessary to extend the region of attraction of the closed-loop system, or to increase the level of disturbance attenuation.

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References

[1] Ball, J.A., and Helton, W., "$\mathcal{H}_\infty$ Control for Nonlinear Plants: Connections with Differential Games," *Proc. 28th IEEE Conf. Decision and Control,* Tampa, FL, pp. 956-962, 1989.

[2] Bupp, R.T., Bernstein, D.S., and Coppola, V.T., "Benchmark Problem for Nonlinear Control Design," *Proc. Amer. Control Conf.,* Seattle, WA, 1995.

[3] Isidori, A., "Feedback Control of Nonlinear Systems," *Int. J. Nonl. Robust Control,* Vol. 2, pp. 291-311, 1992.

[4] Lukes, D.L., "Optimal Regulation of Nonlinear Dynamical Systems," *SIAM J. Control Optim.,* Vol. 7, pp. 75-100, 1969.

[5] van der Schaft, A.J., "$\mathcal{L}_2$-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback $\mathcal{H}_\infty$ Control," *IEEE Trans. Automat. Control,* Vol. AC-37, pp. 770-784, 1992.

[6] van der Schaft, A.J., "Nonlinear State Space $\mathcal{H}_\infty$ Control Theory," Essays on Control (Eds. H.L. Trentelman, J.C. Willems), PSCT 14, pp. 153-190, Birkhaüser, Boston, 1993.

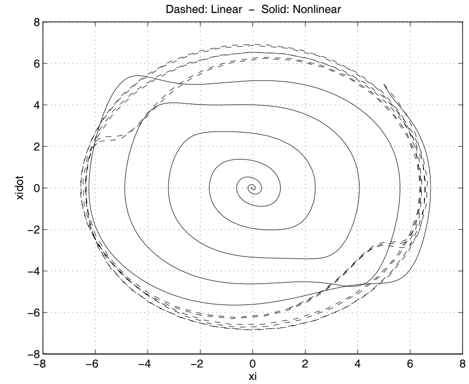

Figure 2: Phase Portrait of ξ

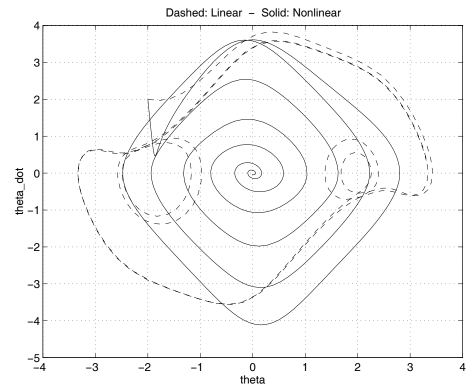

Figure 3: Phase Portrait of θ

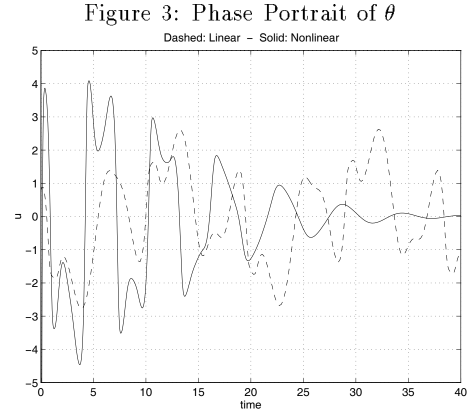

Figure 4: Control Histories

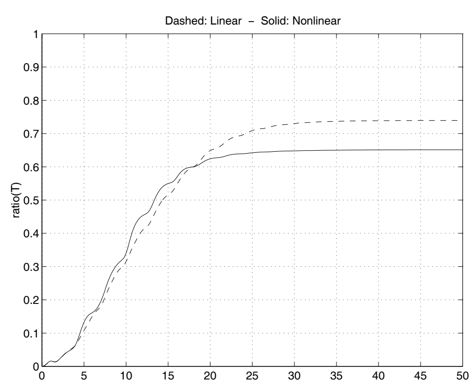

Figure 5: Disturbance Attenuation

[7] Yoshida, T., and Loparo, K.A., "Quadratic Regulatory Theory for Analytic Non-linear Systems with Additive Controls," *Automatica,* Vo. 25, pp. 531-544, 1989.