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On Attitude Stabilization of Symmetric Spacecraft with Two Control Torques

Panagiotis Tsiotras and James M. Longuski

School of Aeronautics and Astronautics
Purdue University
West Lafayette, IN 47907-1282

Abstract

It is a well-known fact in the literature of spacecraft stabilization, that a symmetric spacecraft with two control torques supplied by gas jet actuators is not controllable, if the two control torques are along axes that span the two-dimensional plane which is orthogonal to the axis of symmetry. However, feedback control laws can be derived for a restricted problem corresponding to attitude stabilization about the symmetry axis. The final orientation angle about this axis is undetermined. The purpose of this paper is to present a new methodology for constructing feedback control laws for this restricted problem, based on a new formulation for the kinematics.

1 Introduction

The problem of attitude stabilization of a rigid body has received a lot of attention recently [1, 2, 3, 4, 5, 6, 7]. One of the earliest works on the subject is due to Mortensen [1], where he considered global asymptotic stabilization of the complete attitude motion using three independent gas jet actuators. A complete mathematical description of the problem however, was first given by Crouch [3], where he provided necessary and sufficient conditions for controllability of a rigid body in case of one, two, or three independent acting torques. The results of [3] can be summarized as follows. For *three* independent control torques the system is completely controllable, although for the case of momentum wheel actuators a certain minimum control effort is required. A necessary and sufficient condition for complete controllability of a *symmetric* rigid body with control torques supplied by *two* pairs of *gas jet* actuators, about axes spanning a two dimensional plane, is that the axis orthogonal to this plane must not be a principal axis of symmetry of the spacecraft. For the general case, the system is generically controllable, unless the inertia matrix is a multiple of the identity matrix and certain algebraic criteria also hold. These criteria impose certain conditions on the relative magnitude of the principal inertias, as in the case of stability considerations. For such a system, it is further shown that controllability is equivalent to local controllability at any equilibrium. When a spacecraft is controlled by less than three independent *momentum wheel* actuators, the system is not controllable, or even accessible at any equilibrium.

Many results are available in the literature for the case of three independent controls. For example [2, 4, 5] derive linear and nonlinear feedback stabilizing control laws for the attitude regulation of rigid spacecraft. On the contrary, the problem of attitude stabilization with less than three independent control torques has been only recently dealt with [6, 7]. In [6] it is shown that a rigid spacecraft controlled by two pairs (couples) of gas jet actuators cannot be asymptotically stabilized to an equilibrium using a continuously differentiable, i.e. C^1, feedback control law. In [7] the problem of attitude stabilization of a symmetric spacecraft is treated, using control torques supplied by two pairs of gas jet actuators about axes spanning a two dimensional plane orthogonal to the axis of symmetry. The complete dynamics of the spacecraft system fail to be controllable or even accessible in these cases thus, the methodologies of [3] and [6] are not applicable. However, the spacecraft dynamics is strongly accessible and small time locally controllable in a restricted sense, namely when the spin rate remains zero. It is shown that the restricted (non-spinning spacecraft) dynamics cannot be asymptotically stabilized using *smooth* C^1 feedback. A *nonsmooth* control strategy is developed for the restricted spacecraft dynamics which achieves an arbitrary reorientation of the spacecraft. This nonsmooth control law is based on previous results on stabilization of nonholonomic mechanical systems [8, 9].

In this paper the problem of attitude stabilization of a rigid body (spacecraft) is revisited. Specifically, we consider the attitude stabilization of an axially symmetric spacecraft using two control torques by a pair of gas jets about axes spanning a two-dimensional plane orthogonal to the axis of symmetry. Without loss of generality we can assume that the torques are acting along the principal axes. This problem is of particular theoretical and practical interest because, under these assumptions, as mentioned earlier, the system dynamics is not controllable or even accessible. Using a new formulation of the kinematic equations derived in [10], we derive asymptotically stabilizing feedback controls for the restricted problem of a non-spinning spacecraft. The results can be naturally extended to the case of non-zero spin rate, and in this case lead to spin axis stabilization, i.e., to a revolute motion about the axis of symmetry. This is of prime practical importance, since spin stabilization is often utilized during deployment and station-keeping of modern satellites in orbit.

2 System Dynamics and Kinematics

Euler's Equations of Motion

Let $\omega_1, \omega_2, \omega_3$ denote the angular velocity components along a body-fixed reference frame located at the center of mass, and aligned along the principal axes of a rotating rigid body. Then Euler's equations of motion describe the dynamics of the motion and, for a symmetric body ($I_1 = I_2$) subject to two control torques along principal axes, take the form

$$\dot{\omega}_1 = a_1\omega_2\omega_3 + u_1 \tag{1a}$$

$$\dot{\omega}_2 = a_2\omega_3\omega_1 + u_2 \tag{1b}$$

$$\dot{\omega}_3 = 0 \tag{1c}$$

where $a_1 \triangleq (I_2 - I_3)/I_1$, $a_2 \triangleq (I_3 - I_1)/I_2$, $u_1 \triangleq M_1/I_1$ and $u_2 \triangleq M_2/I_2$. Here M_1, M_2 are the acting torques and I_1, I_2, I_3 denote the principal moments of inertia. Introducing the complex variables $\omega \triangleq \omega_1 + i\omega_2$ and

$u \triangleq u_1 + i u_2$, one rewrites (1a-1b) in the compact form

$$\dot{\omega} = -ia_1\omega_{30}\omega + u \qquad (2)$$

where $\omega_{30} = \omega_3(0)$. A complete formulation of the attitude problem requires the description of the kinematics, in addition to the dynamics introduced here. In contrast to the dynamics formalism, there is more than one way to describe the kinematics of a rotating body.

Kinematics

The kinematic equations relate the components of the angular velocity vector with the rates of a set of parameters, that describe the relative orientation of two reference frames (commonly the inertial and the body-fixed frames). Any two reference frames are related by a rotation matrix R. The set of all such matrices form what is commonly known as the (three-dimensional) rotation group, consisting of all matrices which are orthogonal and have determinant +1, denoted by $SO(3)$. Henceforth, we will refer to $SO(3)$ simply as the rotation group. In fact, $SO(3)$ is more than a group, but carries an inherent smooth manifold structure, and thus, forms a (continuous) Lie group. The attitude history of the moving reference frame with respect to the constant (inertial) reference frame can therefore be described by a curve traced by the corresponding rotation $R(t) \in SO(3)$, with $SO(3)$ taken with its manifold structure. The differential equation satisfied while $R(t)$ is moving along this trajectory is given by

$$\dot{R}(t) = S(\omega_1, \omega_2, \omega_3)R(t) \qquad (3)$$

where $S(\omega_1, \omega_2, \omega_3)$ is the skew-symmetric matrix

$$S(\omega_1, \omega_2, \omega_3) \triangleq \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

There is more than one way to parametrize the rotation group, i.e., to specify a set of parameters such that every element R in $SO(3)$ is uniquely and unambiguously determined [11]. The commonly used three-dimensional parametrization of the rotation group leads to the familiar Eulerian angle formulation of the kinematical equations. Introducing, for example, the three-dimensional parametrization of $SO(3)$, based on a 3-2-1 Eulerian angle sequence [12], one has that the rotation matrix $R = R(\psi, \theta, \phi)$ is given by

$$R = \begin{bmatrix} c\psi c\theta & s\psi c\theta & -s\theta \\ -s\psi c\phi + c\psi s\theta s\phi & c\psi c\phi + s\psi s\theta s\phi & c\theta s\phi \\ s\psi s\phi + c\psi s\theta c\phi & -c\psi s\phi + s\psi s\theta c\phi & c\theta c\phi \end{bmatrix} \qquad (4)$$

where c and s denote *cos* and *sin*, respectively. The associated kinematic equations are

$$\dot{\phi} = \omega_1 + (\omega_2 \sin\phi + \omega_3 \cos\phi)\tan\theta \qquad (5a)$$

$$\dot{\theta} = \omega_2 \cos\phi - \omega_3 \sin\phi \qquad (5b)$$

$$\dot{\psi} = (\omega_2 \sin\phi + \omega_3 \cos\phi)\sec\theta \qquad (5c)$$

Using this parametrization of $SO(3)$, the orientation of the local body-fixed reference frame with respect to the inertial reference frame is found by first rotating the body about its 3-axis through an angle ψ, then rotating about its 2-axis by an angle θ and finally rotating about its 1-axis by an angle ϕ. Equations (5) exhibit a singularity at $\theta = \pm\pi/2$. For this reason one must restrict the subsequent discussion to $-\pi < \phi \leq \pi$, $-\pi/2 < \theta < \pi/2$ and $-\pi < \psi \leq \pi$. Let $\mathcal{M}$ denote the submanifold of $T^3 \triangleq S^1 \times S^1 \times S^1$ determined by the previous inequalities, where S^1 represents the usual mathematical notation for the unit circle. That is, let $\mathcal{M} = \{(\phi, \theta, \psi) \in T^3 : -\pi < \phi \leq \pi, -\pi/2 < \theta < \pi/2, -\pi < \psi \leq \pi\}$. On this submanifold, ϕ and θ determine the orientation of the local body-fixed 3-axis (the symmetry axis) with respect to the inertial 3-axis, and ψ determines the relative rotation about this axis [12]. Throughout the following discussion we will assume that the system trajectories are confined on $\mathcal{M}$.

3 Alternative Formulation of the Kinematics

Next we present a reformulation of the kinematics that will simplify the consequent analysis significantly. This new formulation is based on an idea by Darboux [13], and was initially applied to the problem of attitude dynamics in [10], although it appears that Leimanis [14] was aware of this possibility. From (3) one sees that this matrix differential equation involves nine parameters (the direction cosines of the corresponding frames), however because of the constraint $RR^t = I$ imposed on the elements of $SO(3)$, there are actually only *three* free parameters involved in the system of equations (3). These three parameters can be chosen as the direction cosines of one of the body-axes with respect to the inertial axes. Let $[a, b, c]^t$ denote any column vector of the matrix representation of R having entries r_{ij}, for $i, j = 1, 2, 3$. That is, $[a, b, c]^t = [r_{1j}, r_{2j}, r_{3j}]^t$, for some $j = 1, 2, 3$. Clearly,

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad (6)$$

Note that these three parameters do not provide another three-dimensional parametrization of the rotation group. (Check, for example, that the transformation $(\phi, \theta, \psi) \leftrightarrow (a, b, c)$ is singular.) Because of the constraint $a^2 + b^2 + c^2 = 1$ we can eliminate one of the three parameters a, b, c, to get a system of *two* first order differential equations. The most natural and elegant way to reduce the third order system (6) to a second order system is by the use of *stereographic projection* [15]. That is, if we let a, b, and c represent coordinates on the unit sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$, then, for $(a, b, c) \in S^2$, the stereographic projection $\sigma : S^2 \to \mathbf{C}$ defined by

$$w = \sigma(a, b, c) \triangleq \frac{b - ia}{1 + c} = \frac{1 - c}{b + ia}$$

induces the following differential equation for the complex quantity w

$$\dot{w} + i\,\omega_3 w = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \qquad (7)$$

where $\omega = \omega_1 + i\,\omega_2$ and the bar denotes complex conjugate. Equation (7) is a scalar Riccati equation with time-varying coefficients. The real and imaginary parts of $w = w_1 + i\,w_2$ satisfy the differential equations

$$\dot{w}_1 = \omega_3 w_2 + \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 + w_1^2 - w_2^2)$$

$$\dot{w}_2 = -\omega_3 w_1 + \omega_1 w_1 w_2 + \frac{\omega_2}{2}(1 + w_2^2 - w_1^2)$$

The stereographic projection σ establishes a one-to-one correspondence between the unit sphere S^2 and the complex plane $\mathbf{C}$. It can be easily verified that the inverse map $\sigma^{-1} : \mathbf{C} \to S^2$, $w \mapsto (a, b, c)$ is given by

$$a = \frac{i(w - \bar{w})}{|w|^2 + 1}, \quad b = \frac{w + \bar{w}}{|w|^2 + 1}, \quad c = -\frac{|w|^2 - 1}{|w|^2 + 1}$$

and can be used to find a, b, c once w is known. Here $|.|$ denotes the absolute value of a complex number, i.e., $z\bar{z} = |z|^2$, $z \in \mathbf{C}$.

In order to establish the relationship between w and the particular Eulerian angle set used, notice that we can in principle identify $[a, b, c]^t$ with any column vector of the rotation matrix R, where R can be expressed in terms of any of the parametrizations of $SO(3)$. This gives a great deal of flexibility in the analysis and design of control laws for attitude stabilization. For the three-dimensional 3-2-1 Eulerian angle parametrization, the matrix $R = R(\psi, \theta, \phi)$ is given by (4). Any other parametrization is equally valid, however. Identifying, for example, $[a, b, c]^t$ with the third column of R one establishes a one-to-one correspondence between w and (θ, ϕ) from

$$w = \frac{\sin\phi\cos\theta + i\sin\theta}{1 + \cos\phi\cos\theta}$$

or in terms of real and imaginary parts of w,

$$w_1 = \frac{\sin\phi\cos\theta}{1 + \cos\phi\cos\theta}, \qquad w_2 = \frac{\sin\theta}{1 + \cos\phi\cos\theta} \tag{8}$$

As can easily be checked, the determinant of the Jacobian of the transformation (8) is $\cos\theta/(1 + \cos\phi\cos\theta)^2$. Zeros occur for $\theta = \pm\pi/2$. Moreover, $1 + \cos\phi\cos\theta \neq 0$ as long as $\theta \neq \pm\pi/2$. Thus, the proposed transformation does not introduce any additional singularities, than the original ones due to the intrinsic singularity of the particular Eulerian angle formulation. In fact, (8) establishes a smooth change of coordinates (i.e., a diffeomorphism) between the (w_1, w_2) and (ϕ, θ).

Although not necessary at this point, for completeness we also give the counterpart of the differential equation (5c) for $\dot{\psi}$ in the (ω, w) space:

$$\dot{\psi} = \frac{i}{2}(\omega - \bar{\omega})\frac{(w + \bar{w})(1 + |w|^2)}{(1 + w^2)(1 + \bar{w}^2)}$$

4 Control Strategy

It is clear from equation (1c) that for a symmetric body, no control can affect the value of the component of the angular velocity ω_3 along the symmetry axis. In fact, the value of ω_3 remains constant for all times. Clearly, as already mentioned, this system is not controllable. Therefore, if the initial condition $\omega_3(0)$ is not zero, no control can drive the system to the origin ($\omega_1 = \omega_2 = \omega_3 = \phi = \theta = \psi = 0$). Of course, if $\omega_3(0) \neq 0$ then it is meaningless to require $\psi = 0$, but we may require a control law such that $\omega_1 = \omega_2 = \phi = \theta = 0$. This last control corresponds to spin axis stabilization for a spinning (symmetric) spacecraft and is of important practical interest. From equations (8) notice that $w = 0$ implies that $\sin\theta = 0$ and $\sin\phi = 0$, therefore $w = 0$ implies $\theta = 0$ and $\phi = 0$ on $\mathcal{M}$. We have therefore that $w = 0$ (with the previous identification of the third column of the rotation matrix) implies that the body-fixed 3-axis (the symmetry axis), is aligned with the inertial 3-axis (for the 3-2-1 set). However, we have no *a priori* information about the relative rotation of the body about its symmetry axis. That is, stabilization is achieved about a submanifold $\psi = const.$ of $\mathcal{M}$. On this submanifold, the angle ψ can have any value.

Zero Spin-Rate Case

We now turn our attention to the problem of zero spin rate, i.e., assume *a priori* that $\omega_3(0) = 0$. Following the terminology of [7] we refer to this problem as the *restricted* stabilization problem. For $\omega_3 \equiv 0$ the restricted spacecraft dynamics are given by the equations

$$\dot{\omega}_1 = u_1 \tag{9a}$$

$$\dot{\omega}_2 = u_2 \tag{9b}$$

$$\dot{\phi} = \omega_1 + \omega_2 \sin\phi\tan\theta \tag{10a}$$

$$\dot{\theta} = \omega_2\cos\phi \tag{10b}$$

$$\dot{\psi} = \omega_2\sin\phi\sec\theta \tag{10c}$$

In this section we present a methodology to construct feedback control laws for the system of equations (9) and (10a-10b), which depends on the alternative formulation of the kinematic equations presented in section 3. Asymptotic stability of the closed-loop system is easily demonstrated by Lyapunov's direct method. Recalling that ψ is an ignorable variable for the system (10), in the subsequent analysis we tacitly discard the equation for $\dot{\psi}$. The problem of also stabilizing $\psi = 0$ is more difficult. In fact, in [7] it was shown that any stabilizing feedback control law of the complete restricted system, i.e., for $(\omega_1, \omega_2, \phi, \theta, \psi)$, must be necessarily *nonsmooth*. In the same paper a methodology based on the theory of control of nonholonomic systems [8, 9] was used to construct such nonsmooth stabilizing control laws. The stabilization of the complete system (9)-(10) will be the subject of a forthcoming paper [16].

Introducing the complex control variable $u = u_1 + i\, u_2$ equations (9) and the kinematic equation (7) simplify to

$$\dot{\omega} = u \tag{11a}$$

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{11b}$$

where $(\omega, w) \in \mathbf{C}\times\mathbf{C}$. This system of differential equations is in one-to-one correspondence with the system of equations (9)-(10a-10b). The system (11) falls within the more general class of nonlinear systems of the form

$$\dot{y} = u \tag{12a}$$

$$\dot{x} = f(x, y) \tag{12b}$$

where $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is smooth, with $f(0,0) = 0$. System (12) is a system in cascade form and it is a well-known result [17] that for systems of this form, if the subsystem $\dot{x} = f(x, y)$ is smoothly stabilizable (regarding y as a control-like variable), then the extended system (12) is also smoothly stabilizable. In other words, if in (12) the subsystem (12b) is smoothly stabilizable, then adding an integrator does not change this property. We will use this result in order to derive asymptotically stabilizing control laws for the system (11). We have the following theorems concerning asymptotic stabilization of the system (11).

Theorem 4.1 *The choice of the linear feedback control*

$$\omega = -\kappa w \tag{13}$$

where $\kappa > 0$, globally asymptotically stabilizes (11b).

Proof. With this choice of feedback, the closed-loop system becomes

$$\dot{w} = -\frac{\kappa}{2}(1 + |w|^2)w \tag{14}$$

The positive definite function $V : \mathbf{C} \to \mathbf{R}$ defined by $V(w) = w\bar{w} = |w|^2$ is a Lyapunov function for (14). Indeed, differentiating along trajectories of (14) one obtains

$$\begin{aligned} \dot{V}(w) &= \dot{w}\bar{w} + w\dot{\bar{w}} \\ &= -\frac{\kappa}{2}(1 + |w|^2)w\bar{w} - \frac{\kappa}{2}(1 + |w|^2)\bar{w}w \\ &= -\kappa(1 + |w|^2)|w|^2 \leq 0 \end{aligned}$$

Since $\dot{V}(w) = 0$ if and only if $w = 0$, the closed-loop system (14) is asymptotically stable. Global asymptotic stability follows from the facts that these statements hold for all $w \in \mathbf{C}$ and V is radially unbounded, i.e., $V(w) \to \infty$, for $|w| \to \infty$. Notice that since $\dot{V} \le -\kappa V$ one, in fact, guarantees *exponential stability* for the system (14) with rate of decay $\kappa/2$. ■

Theorem 4.2 *The choice of the feedback control law*

$$u = -\frac{\kappa}{2}(\omega + \bar{\omega}w^2) - \alpha(\omega + \kappa w) \tag{15}$$

with $\kappa > 0$ and $\alpha > 0$, globally asymptotically stabilizes system (11).

Proof. With this choice of feedback, the closed-loop system becomes

$$\dot{\omega} = -\frac{\kappa}{2}(\omega + \bar{\omega}w^2) - \alpha(\omega + \kappa w) \tag{16a}$$

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{16b}$$

The set $\mathcal{E} = \{(\omega, w) \in \mathbf{C}\times\mathbf{C} : \omega + \kappa w = 0\}$ is a positively invariant set and a global asymptotic attractor for (16). To see this, let $z \triangleq \omega + \kappa w$. Then the system equations become

$$\dot{z} = -\alpha z \tag{17a}$$

$$\dot{w} = -\frac{\kappa}{2}w + \frac{z}{2} - \frac{\kappa}{2}w|w|^2 + \frac{\bar{z}}{2}w^2 \tag{17b}$$

La Salle's theorem guarantees the global asymptotic stability of (16), if the trajectories of (16), or equivalently of (17) remain bounded [18]. To this end, let V be the positive definite function of Theorem 4.1, i.e., let $V(w) = |w|^2$. We will show that V is nonincreasing outside a bounded set that contains the origin; in particular, we claim that $\dot{V}(w) \le 0$ on the set $\mathcal{D} = \{w \in \mathbf{C} : |w| \ge |z(0)|/\kappa\}$. This will imply boundedness of solutions of w, hence of (17). Differentiating along trajectories of (17b) one obtains

$$\begin{aligned} \dot{V}(w) &= -\kappa|w|^2 - \kappa|w|^4 + \frac{z}{2}\bar{w}(1 + |w|^2) + \frac{\bar{z}}{2}w(1 + |w|^2) \\ &= -\kappa|w|^2 - \kappa|w|^4 + Re(z\bar{w})(1 + |w|^2) \\ &\le -\kappa|w|^2 - \kappa|w|^4 + |z||\bar{w}|(1 + |w|^2) \end{aligned}$$

where $Re(.)$ denotes the real part of a complex number and where we made use of the fact that $Re(z) \le |z|$ for all $z \in \mathbf{C}$. From (17a) one has that $z(t) = z(0)e^{-\alpha t}$ and in particular $|z(t)| \le |z(0)|$. Thus,

$$\begin{aligned} \dot{V}(w) &\le -\kappa|w|^2 - \kappa|w|^4 + |z(0)||\bar{w}|(1 + |w|^2) \\ &= -(1 + |w|^2)|w|(\kappa|w| - |z(0)|) \end{aligned}$$

For $|w| \ge |z(0)|/\kappa$ one has $\dot{V}(w) \le 0$ as claimed. This completes the proof. ■

The previous control law is not the only choice of stabilizing feedback for the system (11). In fact, one has the following

Theorem 4.3 *The choice of the feedback control law*

$$u = -\frac{\kappa}{2}(\omega + \bar{\omega}w^2) - \alpha(\omega + \kappa w) - w(1 + |w|^2) \tag{18}$$

with $\kappa > 0$ and $\alpha > 0$, globally asymptotically stabilizes system (11).

Proof. With this choice of feedback, the closed-loop system becomes

$$\dot{\omega} = -\frac{\kappa}{2}(\omega + \bar{\omega}w^2) - \alpha(\omega + \kappa w) - w(1 + |w|^2) \tag{19a}$$

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{19b}$$

Indeed, the positive definite function $V : \mathbf{C}\times\mathbf{C} \to \mathbf{R}$ defined by $V(\omega, w) = |w|^2 + |\omega + \kappa w|^2/2$ is a Lyapunov function for the system (19). Differentiating along trajectories of the system (19), one can show that

$$\dot{V}(\omega, w) = -\alpha|\omega + \kappa w|^2 - \kappa|w|^2(1 + |w|^2) \le 0$$

Since $\dot{V}(\omega, w) = 0$ if and only if $w = 0$ and $\omega = 0$, the system (19) is asymptotically stable. Global asymptotic stability follows from the facts that the previous statements hold for all $(\omega, w) \in \mathbf{C}\times\mathbf{C}$ and V is radially unbounded, i.e., $V(\omega, w) \to \infty$, for $\|(\omega, w)\| \to \infty$. In fact, since $\dot{V} \le -\beta V$, where $\beta = \min\{2\alpha, \kappa\}$ the system (19) is *globally exponentially stable* with rate of decay $\beta/2$. ■

Non-zero Spin-Rate Case

We mention in passing that, surprisingly enough, the stabilizing control laws given above, can also be used to achieve stabilization about the symmetry axis, even when the spin rate ω_3 is *not* zero. In such a case the final state is a pure revolute motion about the symmetry axis. Using (2) and (7) the attitude equations for a symmetric body, with $\omega_3(0) \ne 0$, can be written as

$$\dot{\omega} = -i\, a_1 \omega_{30} \omega + u \tag{20a}$$

$$\dot{w} = -i\, \omega_{30} w + \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{20b}$$

Notice first that with the control (13) the subsystem (20b) is (locally) asymptotically stable; for its linearization has eigenvalue $-\kappa/2 - i\,\omega_{30}$ ($\kappa > 0$). In fact, one can easily verify the following two Theorems.

Theorem 4.4 *The choice of the feedback control law*

$$\omega = -\kappa w \tag{21}$$

with $\kappa > 0$ globally asymptotically stabilizes (20b).

Proof. Use the Lyapunov function of Theorem 4.1. In fact, with this Lyapunov function one can show *global exponential* stability of (20b) with rate of decay $\kappa/2$. ■

Theorem 4.5 *The choice of the feedback control law*

$$u = i\, a_1 \omega_{30} \omega + \kappa(i\omega_{30} w - \frac{\omega}{2} - \frac{\bar{\omega}}{2}w^2) - \alpha(\omega + \kappa w) \tag{22}$$

with $\kappa > 0$ and $\alpha > 0$, globally asymptotically stabilizes system (20).

The proof of Theorem 4.5 traces the steps of the proof of the Theorem 4.2, and will not be repeated here.

5 Numerical Example

We illustrate the previous ideas with a numerical example. The control law given in Theorem 4.2 is used to stabilize the system of equations (11) about the origin. The initial conditions are given by $\omega_1(0) = 0.75\ rad/sec$, $\omega_2(0) = -0.5\ rad/sec$, $\omega_3(0) = 0$, $\phi(0) = 2.5\ rad$, $\theta(0) = 0.5\ rad$ and $\psi(0) = 0.25\ rad$. The results with control law (15) and $\kappa = \alpha = 1$ are shown in Figs. 1-2.

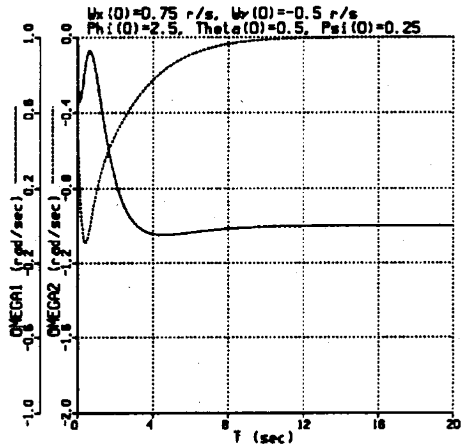

Figure 1: Angular velocities ω_1 and ω_2.

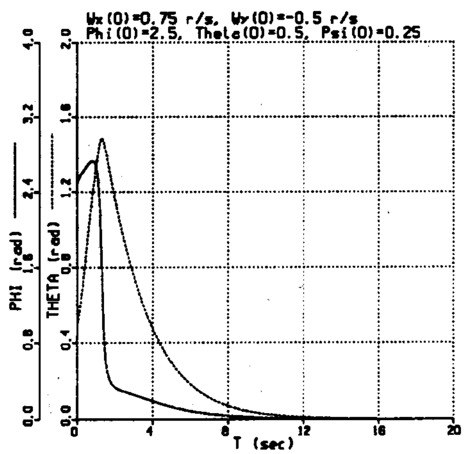

Figure 2: Eulerian angles ϕ and θ.

6 Conclusions

The problem of stabilization of a symmetric spacecraft with two gas jet actuators aligned about the principal axes of equal moments of inertia is investigated. Using a new formulation for the kinematic equations, asymptotically stabilizing controls have been derived for the restricted problem of spin axis stabilization. The asymptotic stability of the closed-loop system is proved by construction of appropriate Lyapunov functions. The stabilizing control laws derived are especially simple and elegant. Moreover, they do not depend on the particular choice of the Eulerian angle set, used to describe the attitude orientation in the inertial space. This provides a great deal of freedom in the analysis and design of attitude control laws.

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