Pursuit Evasion Game of Two Players under an External Flow Field

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***Abstract*— In this paper, we address the differential game of pursuit and evasion between two players in the presence of an external flow field. It is assumed that the two players move on the plane at fixed but different speeds, and they are both agile. That is, they steer by choosing at each instant their direction of travel and abrupt heading changes are allowed. The external flow field is approximated by a time-invariant affine function. By utilizing standard techniques from differential game theory, we characterize the regions of initial conditions that lead to capture, as well as the regions that result in evasion when the two players act optimally. We derive the optimal strategies of the pursuer and the evader within the capture regions. Finally, we present numerical simulations of the resulting pursuer and evader trajectories for several values of the parameters of the external flow field.**

I. Introduction

Two-player pursuit-evasion differential games have been studied extensively in the literature. Pursuit-evasion between an agile pursuer and an evader with a curvature constraint was studied in Isaacs' Homicidal Chauffeur game [1]. A reversed version of the Homicidal Chauffeur game, where the evader is agile and the player has a curvature constraint, was recently studied in [2], [3]. A stochastic version of the Homicidal Chauffeur game has been addressed in [4]. Another similar game, called the Game of Two Cars [5], focuses on two players, both having a finite maximum turning radius. A general result for the pursuit-evasion problem with curvature constraints was presented in [6]. The approach was extended to the three-dimensional space in [7]. Other pursuit-evasion games of two players under some special scenarios include the isotropic rocket problem addressed in Isaacs' book [1], where one of the players has a bounded magnitude of acceleration, and the Lion and Man problem [8], where the movements of both players are restricted to lie within a region of the whole plane.

One common assumption among all the previous results is that they do not take into consideration how environmental conditions may affect the outcome of the game. For instance, when either the pursuer or the evader (or both) is a small autonomous underwater vehicle (AUV) or an unmanned aerial vehicle (UAV), the presence of the sea current or the wind, will significantly affect their motion. As a result, their behavior and the solution of the differential game may be greatly affected by the existence of the external flow field.

The effect of an external flow field has received a considerable amount of attention in optimal control problems. The problem of optimally guiding a Dubins vehicle [9] to a specified position in a flow field was addressed in [10]. The problem of steering a Dubins vehicle in a stochastic wind field while minimizing the expected time of capture has been studied in [11], and the minimum-time guidance problem for the isotropic rocket in the presence of wind has been discussed in [12]. Pursuit-evasion games under the influence of external disturbances do not seem to have received the same level of attention in the literature, however.

In this paper, we consider the differential game of pursuit and evasion between two players on a plane under an external flow field. It is assumed that the pursuer and the evader move with constant but different speeds, and they are both agile, that is, they are allowed to change their headings instantaneously. To simplify the analysis, it will be assumed that the flow field is approximated by a time-invariant, spatially-affine function. Our goal is to find the region of initial conditions of both players that leads to capture when both players act optimally, and derive the corresponding optimal strategies of the two players when capture is guaranteed.

II. Problem Formulation

A. Problem Formulation

Consider a pursuer and an evader moving on a plane under the influence of an external flow field. The equations of motion for the pursuer and the evader in the inertial reference frame are given by

$$\dot{x}_P = v_P \cos\phi + w_1(x_P, y_P), \tag{1}$$
$$\dot{y}_P = v_P \sin\phi + w_2(x_P, y_P), \tag{2}$$
$$\dot{x}_E = v_E \cos\psi + w_1(x_E, y_E), \tag{3}$$
$$\dot{y}_E = v_E \sin\psi + w_2(x_E, y_E), \tag{4}$$

where $(x_P, y_P) \in \mathbb{R}^2$ and $(x_E, y_E) \in \mathbb{R}^2$ denote the positions of the pursuer and the evader, respectively, $\phi, \psi \in [-\pi, \pi]$ are the control of the pursuer and the evader, and v_P and v_E represent the speed of the pursuer and the evader, respectively. In this work, we will assume that $v_P > v_E$. Finally, $w_1(\cdot,\cdot)$ and $w_2(\cdot,\cdot)$ are the components of an external spatially varying flow field along x-axis and y-axis, respectively.

B. Differential Game Formulation in the Reduced Space

In order to simplify the analysis, it will be assumed that the external flow field is approximated, at least locally, by a time-invariant affine function. Specifically, let

$$w_1(x, y) = \alpha_1 x + \beta_1 y + \gamma_1, \tag{5}$$
$$w_2(x, y) = \alpha_2 x + \beta_2 y + \gamma_2, \tag{6}$$

where $\alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{R}$ $(i = 1, 2)$ are prescribed constants. By choosing a new reference frame whose origin is at

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the pursuer, the kinematic equations can be represented in a reduced two-dimensional space space. In particular, let $x = x_E - x_P$ and $y = y_E - y_P$ be the relative distance between the evader and the pursuer along the x-axis and y-axis, respectively. The kinematic equations in terms of x and y are then given by

$$\dot{x} = v_E \cos\psi - v_P \cos\phi + \alpha_1 x + \beta_1 y, \tag{7}$$

$$\dot{y} = v_E \sin\psi - v_P \sin\phi + \alpha_2 x + \beta_2 y. \tag{8}$$

By defining the reduced state as $\mathbf{x} = [x, y]^\intercal$, the equations can then be written compactly as

$$\dot{\mathbf{x}} = v_E \mathbf{v} - v_P \mathbf{u} + w(\mathbf{x}), \tag{9}$$

where $\mathbf{v} = [\cos\psi, \sin\psi]^\intercal$ and $\mathbf{u} = [\cos\phi, \sin\phi]^\intercal$ are the controls, and where the relative wind field is given by $w(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$. The game terminates when capture occurs, that is, when the evader is in the *interior* of a ball $\mathcal{B}$ of radius ℓ centered at the pursuer's current location, given by $\mathcal{B} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \le \ell\}$. The *terminal surface* is the manifold in the state space which, once penetrated, determines termination of the game. The terminal surface $\mathcal{C}$ is thus the circle centered at the origin of radius ℓ, i.e., $\mathcal{C} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = \ell\}$. Accordingly, the state space $\mathcal{E}$ is the portion of the x, y-plane exterior to $\mathcal{C}$, that is, $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| > \ell\}$.

Under this setup, we want to find the region in the state space such that the evader can be captured by the pursuer if their initial relative coordinates fall inside this region. This region is denoted as the *capture zone*. The region which leads to escape of the evader is the *escape zone*.

To this end, we formulate the problem as a *game of kind* [1], that is, the game has finitely many outcomes. The payoff is defined as

$$J(\mathbf{x}(t_0), \phi(\cdot)\psi(\cdot)) = \begin{cases} +1, & \text{for escape,} \\ 0, & \text{for neutral outcome,} \\ -1, & \text{for capture.} \end{cases} \tag{10}$$

The outcome is *neutral* [1] if the trajectory of the evader intersects the terminal surface, but it does not penetrate it.

Our goal is to find the region of the initial conditions that leads to capture or escape with conflicting actions of the pursuer's control ϕ and the evader's control ψ that minimize/maximize the payoff (10) under the dynamic equations (7) and (8).

III. Problem Analysis

A. Effect of External Field

Before we proceed with the formulation of the differential game of kind, some discussion that can help the reader intuitively understand the types of solutions one may expect to obtain is in order. From (9), it is clear that in the reduced state space the system is described by a linear differential equation, controlled by $\mathbf{v}$ and $\mathbf{u}$.

Broadly speaking, the objective of the pursuer is to make $|\mathbf{x}| \to 0$ (i.e., stabilization to the origin), whereas the objective of the evader is to ensure that this does not occur (and even make, perhaps, $|\mathbf{x}| \to \infty$ as time increases). From (9) it is clear that since the flow field is approximated by an affine function, the solution of the problem will depend on the relative contributions of the flow field term $A\mathbf{x}$ and the contribution by both players, namely, $v_E\mathbf{v} - v_P\mathbf{u}$. Since the latter term is uniformly bounded whereas the former term increases without bound, it follows that if the relative distance between the players is very large, the external flow field will be too strong to be overcome by the (constrained) control actions of either player. In that case, the trajectories of both players will tend to follow the vector field directions of the external flow. This observation necessarily restricts the results of the current analysis locally around the origin. In addition, and depending on the strength of the wind, controllability may become an issue given that the control authority of the pursuer and the evader are limited.

The problem of controllability/stabilizability of a linear system with bounded controls has been extensively studied in the literature [13]–[16]. The main result in this context states that global stabilizability with bounded controls is possible only if all the eigenvalues of the matrix A have non-positive real part. However, this is a global result that holds for all initial conditions. In our problem we restrict our analysis to a region locally around the origin where capture can occur even if the matrix has eigenvalues with positive real part.

B. The Game of Kind

In this subsection, we follow the standard approach of the game of kind introduced in [1]. First, we focus on identifying the *usable part* of the terminal surface. The UP is the subset of $\mathcal{C}$ in which the pursuer can cause termination immediately when both players act optimally. The remaining points on $\mathcal{C}$ form the nonuseable part, that is, termination will not occur even if the trajectory reaches this part of $\mathcal{C}$ under optimal play (i.e., when both players act optimally). The part of $\mathcal{C}$ that separates the usable part and the nonuseable part of $\mathcal{C}$ is called the *boundary of the usable part* (BUP).

In order to find the usable part, we parameterize $\mathcal{C}$ with the variable s according to

$$x = \ell \cos s, \qquad y = \ell \sin s. \tag{11}$$

Let $r^2 = x^2 + y^2 = |\mathbf{x}|^2$. Taking the time derivative on both sides of the last equation, and consider only points on $\mathcal{C}$, we have

$$\ell \dot{r} = \ell \cos s (v_E \cos\psi - v_P \cos\phi + \alpha_1 \ell \cos s + \beta_1 \ell \sin s) + \ell \sin s (v_E \sin\psi - v_P \sin\phi + \alpha_2 \ell \cos s + \beta_2 \ell \sin s).$$

The usable part of $\mathcal{C}$ is specified by the condition

$$\min_{\phi} \max_{\psi} \dot{r}(\mathbf{x}) < 0, \quad \mathbf{x} \in \mathcal{C}, \tag{12}$$

which implies that the relative trajectory is able to penetrate the terminal surface $\mathcal{C}$. From the last two inequalities, and using standard trigonometric identities, we have that, for $\mathbf{x} \in \mathcal{C}$,

$$\min_{\phi} \max_{\psi} \; \dot{r}(\mathbf{x}) = v_E - v_P + \frac{\ell}{2}(\alpha_1 + \beta_2) + \frac{\ell}{2}[(\alpha_1 - \beta_2)\cos 2s \quad + (\beta_1 + \alpha_2)\sin 2s]. \tag{13}$$

Let now $\sigma = \sqrt{(\alpha_1 - \beta_2)^2 + (\beta_1 + \alpha_2)^2}$. When $\sigma = 0$, we have $\alpha_1 - \beta_2 = 0$ and $\beta_1 + \alpha_2 = 0$. Hence, whether the game terminates depends on the sign of $v_E - v_P + \ell(\alpha_1 + \beta_2)/2$. Specifically, when $v_E - v_P + \ell(\alpha_1 + \beta_2)/2 < 0$, the usable part of the terminal surface is $\mathcal{C}$ itself, whereas when $v_E - v_P + \ell(\alpha_1 + \beta_2)/2 > 0$, the game will not terminate under any initial conditions of the pursuer and the evader, which means that the evader always escapes. In the latter case the whole state space $\mathcal{E}$ is the escape zone.

Henceforth, we assume that $\sigma \neq 0$. Then (12) and (13) imply that

$$\min_{\phi} \max_{\psi} \dot{r} = v_E - v_P + \frac{\ell}{2}(\alpha_1 + \beta_2) + \frac{\ell\sigma}{2}\sin(\theta + 2s) < 0, \tag{14}$$

where θ is given by $\sin\theta = (\alpha_1 - \beta_2)/\sigma$ and $\cos\theta = (\beta_1 + \alpha_2)/\sigma$. From (14) we reach the following conclusion:

Proposition 3.1: In the reduced space the game will *not* terminate if

$$\frac{2(v_P - v_E) - \ell(\alpha_1 + \beta_2)}{\ell\sigma} < -1, \tag{15}$$

where $\sigma = \sqrt{(\alpha_1 - \beta_2)^2 + (\beta_1 + \alpha_2)^2}$.

Proof: From (14), we have $\sin(\theta+2s) < [2(v_P - v_E) - \ell(\alpha_1 + \beta_2)]/\ell\sigma$. Let

$$\zeta = \frac{2(v_P - v_E) - \ell(\alpha_1 + \beta_2)}{\ell\sigma}. \tag{16}$$

Clearly, when $\zeta < -1$, the inequality above has no solution for s. That is, when (15) is satisfied, the game will not terminate since no usable part exists in this case. ■

Corollary 3.2: When $\zeta \geq 1$ the usable part is the whole terminal surface $\mathcal{C}$.

Remark 1: Note that (15) is a "controllability"-like condition that relates the elements of the matrix A and the bounds of the velocities of both the players so that capture is possible.

Henceforth, we assume that $-1 \leq \zeta < 1$. Under this assumption, the BUP is determined from $\sin(\theta + 2s) = \zeta$. This yields four solutions in $[0, 2\pi)$, denoted by s_1, s_2, $s_3 = s_1 + \pi$, $s_4 = s_2 + \pi$. Hence, the BUP contains four points on $\mathcal{C}$, represented by $P_i = (\cos s_i, \sin s_i)$, $i = 1, \ldots, 4$. A typical illustration of the terminal surface, which is divided into the usable and nonusable parts by the BUP, which consists of four points on the terminal surface, is shown in Figure 1.

Now we turn to the construction of the *barrier* [1]. The barrier is a surface in the state space that consists of initial conditions for which the outcome is neutral. One property of the barrier is that it is never crossed by either the pursuer or the evader during optimal play. In particular, the barrier emanates from the BUP and is tangent to $\mathcal{C}$ at the BUP. We denote the barrier by $\mathcal{S}$. At each point on $\mathcal{S}$ we define the normal vector $\nu = [\nu_1, \nu_2]^\intercal \in \mathbb{R}^2$ extending into the escape zone.

The *Isaacs equation* [1] for the game of kind for this

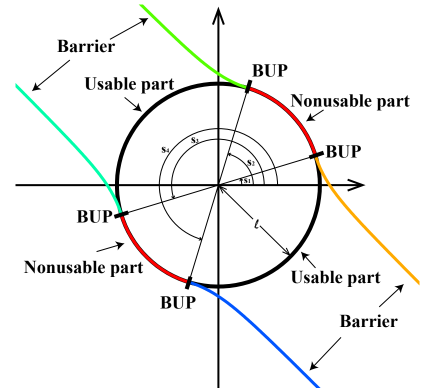

Fig. 1. The terminal surface $\mathcal{C}$ of the game is given by a circle of radius ℓ. The circle is separated by the BUP (4 points on the circle parameterized by s_1 through s_4) into the usable part (black lines) and the nonusable part (red lines). Every barrier meets the terminal surface at the BUP tangentially.

problem becomes

$$\begin{aligned} 0 &= \min_{\phi} \max_{\psi} \{\nu_1(v_E \cos\psi - v_P \cos\phi + \alpha_1 x + \beta_1 y) \\ &\quad + \nu_2(v_E \sin\psi - v_P \sin\phi + \alpha_2 x + \beta_2 y)\} \\ &= v_E\rho - v_P\rho + \nu_1(\alpha_1 x + \beta_1 y) + \nu_2(\alpha_2 x + \beta_2 y), \end{aligned} \tag{17}$$

where $\rho = \sqrt{\nu_1^2 + \nu_2^2}$ and the corresponding optimal control of the pursuer and the evader on the barrier are specified by

$$\cos\phi^* = \frac{\nu_1}{\rho}, \qquad \sin\phi^* = \frac{\nu_2}{\rho}, \tag{18}$$

$$\cos\psi^* = \frac{\nu_1}{\rho}, \qquad \sin\psi^* = \frac{\nu_2}{\rho}. \tag{19}$$

From [1], it follows that $\nu_i, i = 1, 2$, satisfy the differential equations

$$\dot{\nu}_i = -\sum_j \nu_j \frac{\partial f_j(\mathbf{x}, \phi^*, \psi^*)}{\partial \mathbf{x}_i}, \quad i = 1, 2, \tag{20}$$

where $f_j, j = 1, 2$, stands for the right-hand side of (7) and (8), respectively. We take these equations and the original equations under the optimal control ϕ^* and ψ^*, and reverse the time direction by replacing t with $\tau = t_c - t$ to obtain the Retrogressive Path Equations (RPE). These are the differential equations with respect to the retrograde time τ and indicate the fact that the game will be solved backwards in time starting from the terminal surface $\mathcal{C}$. Denoting with $(\mathring{\ })$ the derivative with respect to τ, the retrograde evolution of the states and the vector ν can be established as:

$$\begin{aligned} \mathring{x} &= (v_P - v_E)\frac{\nu_1}{\rho} - \alpha_1 x - \beta_1 y, \\ \mathring{y} &= (v_P - v_E)\frac{\nu_2}{\rho} - \alpha_2 x - \beta_2 y, \\ \mathring{\nu}_1 &= \alpha_1 \nu_1 + \alpha_2 \nu_2, \\ \mathring{\nu}_2 &= \beta_1 \nu_1 + \beta_2 \nu_2. \end{aligned} \tag{21}$$

By the definition of the BUP and the barrier, it is clear that the barrier starts at the BUP towards the state space $\mathcal{E}$ in a retrogressive sense. Moreover, the two surfaces meet tangentially, since no penetration occurs at the BUP and the vectorfields of both players are tangential to the barrier. Pick any $\bar{s} \in \{s_1, s_2, s_3, s_4\}$. The initial conditions for the RPEs are thus given by $x(\tau=0) = \ell\cos\bar{s},\ y(\tau=0) = \ell\sin\bar{s}$, $\nu_1(\tau=0) = \cos\bar{s},\ \nu_2(\tau=0) = \sin\bar{s}$. By integrating (21) subject to these initial conditions we obtain

$$\nu(\tau) = e^{A^\mathsf{T}\tau}\nu(\tau=0), \tag{22}$$

$$\mathbf{x}(\tau) = e^{-A\tau}\mathbf{x}(\tau=0) + \int_0^\tau e^{-A(\tau-\xi)}b(\xi)\,\mathrm{d}\xi, \tag{23}$$

where $b(\tau) = (v_P - v_E)\nu(\tau)/|\nu(\tau)|$. Therefore,

$$\mathbf{x}(\tau) = \ell e^{-A\tau}\begin{bmatrix}\cos\bar{s}\\ \sin\bar{s}\end{bmatrix} + (v_P - v_E)e^{-A\tau}\int_0^\tau \frac{e^{(A+A^\mathsf{T})\xi}}{|\nu(\xi)|}\begin{bmatrix}\cos\bar{s}\\ \sin\bar{s}\end{bmatrix}\mathrm{d}\xi. \tag{24}$$

By plotting the trajectories of (24) given the four initial conditions of the BUP, we can determine whether these are valid barriers and whether the state space $\mathcal{E}$ is separated by the barriers.

If $\mathcal{E}$ is indeed separated by the barriers, then the regions of $\mathcal{E}$ that contains the usable part of the terminal surface will form the capture zone. Otherwise, the whole state space is either the capture zone or the escape zone.

C. Game of Degree in the Capture Region

Now that we have identified the capture region, we aim at determining the optimal trajectory of $\mathbf{x}$ inside this region by solving a game of degree. Within the capture region, the performance index is $\mathcal{J} = \int_0^{t_c}\mathrm{d}t$. To this end, we define the value function $\mathcal{V}(\mathbf{x})$, which satisfies the Hamilton-Jacobi-Isaacs (HJI) equation [1]

$$0 = \min_\phi \max_\psi \mathcal{H}(\mathbf{x}, \mathcal{V}_\mathbf{x}), \tag{25}$$

where the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = 1 + \frac{\partial\mathcal{V}}{\partial x}\Big(v_E\cos\psi - v_P\cos\phi + \alpha_1 x + \beta_1 y\Big) + \frac{\partial\mathcal{V}}{\partial y}\Big(v_E\sin\psi - v_P\sin\phi + \alpha_2 x + \beta_2 y\Big). \tag{26}$$

Let $\mathcal{V}_x = \frac{\partial\mathcal{V}}{\partial x}$, $\mathcal{V}_y = \frac{\partial\mathcal{V}}{\partial y}$. Then (25) can be rewritten as

$$\begin{aligned}0 =& 1 + \mathcal{V}_x(\alpha_1 x + \beta_1 y) + \mathcal{V}_y(\alpha_2 x + \beta_2 y)\\ &+ \min_\phi\{-v_P(\mathcal{V}_x\cos\phi + \mathcal{V}_y\sin\phi)\}\\ &+ \max_\psi\{v_E(\mathcal{V}_x\cos\psi + \mathcal{V}_y\sin\psi)\}.\end{aligned} \tag{27}$$

Hence, the optimal controls ϕ^* and ψ^* are given by

$$\begin{aligned}\cos\phi^* &= \frac{\mathcal{V}_x}{\mu}, & \sin\phi^* &= \frac{\mathcal{V}_y}{\mu},\\ \cos\psi^* &= \frac{\mathcal{V}_x}{\mu}, & \sin\psi^* &= \frac{\mathcal{V}_y}{\mu},\end{aligned} \tag{28}$$

where $\mu = \sqrt{\mathcal{V}_x^2 + \mathcal{V}_y^2}$. Plugging (28) back into the Hamiltonian, we get the optimal Hamiltonian

$$\mathcal{H}^* = 1 + (v_E - v_P)\mu + \mathcal{V}_x(\alpha_1 x + \beta_1 y) + \mathcal{V}_y(\alpha_2 x + \beta_2 y). \tag{29}$$

The RPEs can then be expressed as

$$\mathring{x} = (v_P - v_E)\frac{\mathcal{V}_x}{\mu} - \alpha_1 x - \beta_1 y, \tag{30}$$

$$\mathring{y} = (v_P - v_E)\frac{\mathcal{V}_y}{\mu} - \alpha_2 x - \beta_2 y, \tag{31}$$

$$\mathring{\mathcal{V}}_x = \alpha_1\mathcal{V}_x + \alpha_2\mathcal{V}_y, \tag{32}$$

$$\mathring{\mathcal{V}}_y = \beta_1\mathcal{V}_x + \beta_2\mathcal{V}_y. \tag{33}$$

On the terminal surface $\mathcal{C}$, we have $\mathcal{V} = 0$. Along with the parameterization of $\mathcal{C}$ by $x = \ell\cos s$, $y = \ell\sin s$, we get

$$0 = \frac{\partial\mathcal{V}}{\partial s} = \ell(-\mathcal{V}_x\sin s + \mathcal{V}_y\cos s).$$

Upon solving these equations, we further get, for some $\delta > 0$,

$$\mathcal{V}_x(\tau=0) = \delta\cos s, \qquad \mathcal{V}_y(\tau=0) = \delta\sin s. \tag{34}$$

By substituting (34) into the expression for $\mathcal{H}^*$, one can solve for δ to obtain $\delta = 1/\eta$ where $\eta = v_P - v_E - \ell(\alpha_1\cos^2 s + (\beta_1 + \alpha_2)\sin s\cos s + \beta_2\sin^2 s)$. Integrating the RPE's (30) through (33) subject to the initial conditions (34) yields

$$\begin{bmatrix}\mathcal{V}_x(\tau)\\ \mathcal{V}_y(\tau)\end{bmatrix} = e^{A^\mathsf{T}\tau}\begin{bmatrix}\delta\cos s\\ \delta\sin s\end{bmatrix}, \tag{35}$$

and hence

$$\begin{bmatrix}x(\tau)\\ y(\tau)\end{bmatrix} = e^{-A\tau}\begin{bmatrix}\ell\cos s\\ \ell\sin s\end{bmatrix} + (v_P - v_E)e^{-A\tau}\int_0^\tau \frac{e^{(A+A^\mathsf{T})\xi}}{\mu(\xi)}\begin{bmatrix}\delta\cos s\\ \delta\sin s\end{bmatrix}\mathrm{d}\xi, \tag{36}$$

which yields the optimal trajectory for the game of degree.

IV. Simulation Results

In this section, we present numerical simulations to illustrate the previous analysis. In the following cases, we vary the matrix A while we keep ℓ, v_P and v_E fixed to compute different types of barriers under different flow fields. Henceforth, we let $\ell = 0.1$, $v_P = 1.0$, $v_E = 0.9$. For the parameters of the flow field in the inertial frame, we set $\gamma_1 = \gamma_2 = 0$.

Case 1: $A = \begin{bmatrix}0 & 10\\ -5 & 0\end{bmatrix}$. This matrix has two pure imaginary eigenvalues (center). In this case, $\sigma = 5$, $\zeta = 0.4$ and the corresponding values for the BUP are $s_1 = 0.2058, s_2 = 1.3650, s_3 = 3.3474$ and $s_4 = 4.5066$. As shown in Figure 2, the trajectories of the RPEs emanating from P_1 and P_3 are inside $\mathcal{B}$; these two trajectories are outside the state space $\mathcal{E}$. Hence, they are not valid barriers and are discarded. On the other hand, the trajectories emanating from P_2 and P_4 are valid barriers. They have spiral-like shapes but they fail to separate the state space into two parts. The whole state space is a capture zone; regardless of the initial conditions of the two players, capture is guaranteed. The dashed magenta lines in Figure 2 show the optimal trajectories in relative coordinates with respect to different initial positions on the usable part of the terminal surface. Although the barrier does not separate the state space into capture and escape zones, it is still not crossed during optimal play, which gives us some information as to how the optimal trajectories look like. The barrier also marks a discontinuity in the value function.

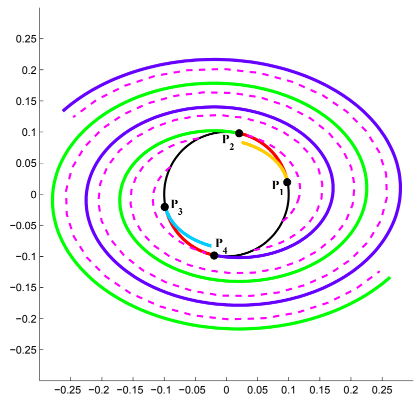

Fig. 2. Barriers in frame $\mathcal{M}$ when $A = [0, 10; -5, 0]$. The dashed magenta lines are the optimal trajectories of the relative coordinates emanating from the usable part of the terminal surface.

Given the initial positions for the evader and the pursuer as $\mathbf{x}_E(0) = [-0.764, 0.337]$ and $\mathbf{x}_P(0) = [-0.524, 0.336]$, respectively, the optimal trajectories of the evader and the pursuer in the inertial frame are depicted in Figure 3. These trajectories are consistent with the external flow field represented by the black arrows. The results suggest that both players are trying to take advantage of the flow field. Intuitively, this makes sense. Since the matrix A has purely imaginary eigenvalues, the uncontrolled system trajectories are circles around the origin. The flow field does not give an advantage to either the pursuer or the evader. It is then reasonable that under optimal controls of both players, the trajectories in the reduced state move in spiral-like patterns, as confirmed in Figure 2.

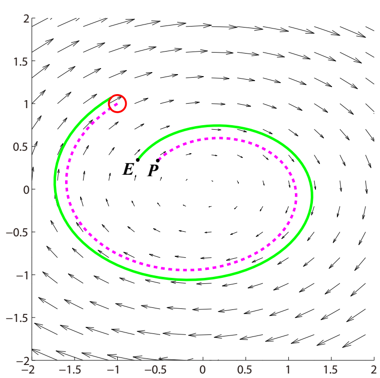

Fig. 3. Optimal trajectories of the evader in green and the pursuer in dashed magenta, respectively, where $A = [0, 10; -5, 0]$. Red circle around the final position of the pursuer represents the terminal surface. The flow field is depicted by the black arrows in the background.

Case 2: $A = \begin{bmatrix} 1.4020 & -1.0772 \\ 1.4770 & 0.7756 \end{bmatrix}$. In this case, the eigenvalues are a complex conjugate pair with positive real part (unstable spiral). These values correspond to $\sigma = 0.7431$, $\zeta = -0.2390$ and the corresponding parameters for the BUP are $s_1 = 5.6612, s_2 = 1.1901, s_3 = 2.5196, s_4 = 4.3317$. As depicted in Figure 4, and similarly to the first case, the trajectories of the RPEs emanating from P_2 and P_4 are inside $\mathcal{B}$ and are thus discarded. The trajectories starting from P_1 and P_3 intersect $\mathcal{C}$ after some time, and thus the trajectories after the intersection are discarded. In this case, the barrier separates the capture zone from the escape zone. The capture zone is represented by the shaded region in Figure 4. All the remaining space outside the circle is the escape zone.

The optimal trajectories of the evader and the pursuer in the inertial frame are depicted in Figure 5 with initial positions $\mathbf{x}_E(0) = [-0.20, 0.683]$ and $\mathbf{x}_P(0) = [-0.230, 0.579]$, respectively. Notice that in this case, there is a small region of relative initial positions for the pursuer and the evader such that capture occurs.

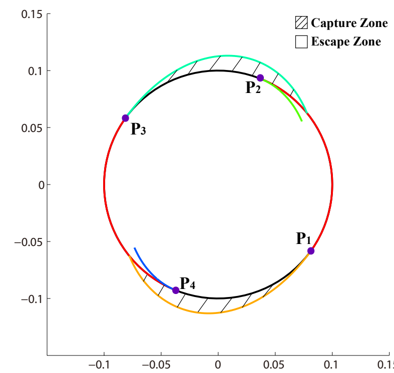

Fig. 4. Barriers in frame $\mathcal{M}$ for $A = [1.4020, -1.0772; 1.4770, 0.7756]$, the shaded region is the escape zone and the white region outside the circle is the escape zone.

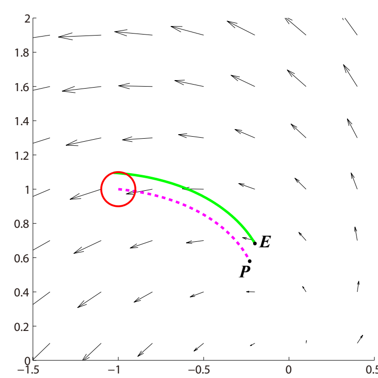

Fig. 5. Optimal trajectories of the evader in green and the pursuer in dashed magenta, for $A = [1.4020, -1.0772; 1.4770, 0.7756]$.

In this case the matrix A has two complex eigenvalues with positive real parts, which implies that the origin is an unstable spiral. The trajectories of the uncontrolled system would result in $|x| \to \infty$ as time goes on. In this case, the flow field gives an advantage to the evader. Indeed, as shown in Figure 4, the capture zone is very small compared to the escape zone.

Case 3: $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. In this case, $\sigma = 4$, $\zeta = 0$ and the corresponding parameters for the BUP are $s_1 = 0, s_2 = \pi/2, s_3 = \pi$ and $s_4 = 3\pi/2$. As illustrated in Figure 6, all four trajectories emanating from P_1, P_2, P_3 and P_4 are valid barriers. They separate the state space into two capture zones and two escape zones, depicted in the figure by the two shaded regions and the two white regions, respectively. Typical optimal trajectories of the evader and the pursuer in the inertial frame with initial positions $\mathbf{x}_E(0) = [0.951, -0.852]$ and $\mathbf{x}_P(0) = [1.265, -1.165]$ are shown in Figure 7.

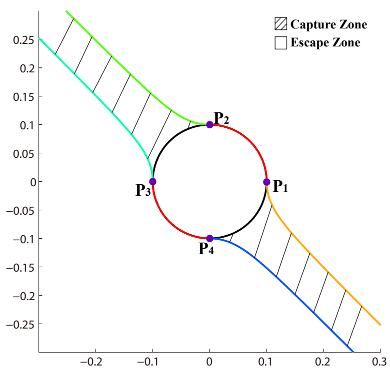

Fig. 6. Four valid barriers in frame $\mathcal{M}$ emanating from P_1 through P_4, for $A = [1, 2; 2, 1]$. The state space is divided by two shaded capture zones and two white escape zones.

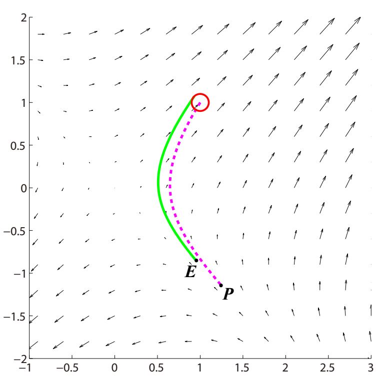

Fig. 7. Optimal trajectories of the evader in green and the pursuer in dashed magenta, for $A = [1, 2; 2, 1]$.

In Cases 3, the matrix A has one positive eigenvalue and one negative eigenvalue. Hence, the origin is a saddle point, and in some part of the plane the flow field points towards the origin (helping the pursuer), whereas in other parts it points away from the origin (thus giving an advantage to the evader), as indicated by the black vector fields in Figure 7. This suggests that the pursuer tries to steer the game in the part of the space that the flow field is beneficial to him and the evader does the same, i.e., tries to steer the state to the parts of the state space that are more helpful to him. In this case, the game will terminate (or not) depending on whether the pursuer can capture the evader before the latter moves in the part of the space that the former has an advantage.

V. Conclusions

This paper deals with a differential game between two players in a plane subject to an external flow field. Under the assumption that the flow field is approximated by a time-invariant affine function, we reformulate the problem as a game of kind and characterize the initial conditions that secure capture of the evader, as well as the initial conditions that lead to escape of the evader, when both players act optimally. The optimal controls of both players inside the capture zone are derived, and numerical simulations with different parameters of the flow field are presented to illustrate the corresponding capture and escape zones. Follow-up work includes the study of pursuit evasion problem under more realistic wind fields, as well as the study of the optimal control of the pursuer and the evader when stochastic environment disturbances are taken into consideration.

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