Extended Kalman Filter for Spacecraft Pose Estimation Using Dual Quaternions*

Nuno Filipe[1] Michail Kontitsis[2] Panagiotis Tsiotras[3]

Abstract— **Based on the highly successful Quaternion Multiplicative Extended Kalman Filter (Q-MEKF) for spacecraft attitude estimation using unit quaternions, this paper proposes a Dual Quaternion Multiplicative Extended Kalman Filter (DQ-MEKF) for spacecraft pose (i.e., attitude and position) and linear and angular velocity estimation using unit dual quaternions. By using the concept of error unit dual quaternion, defined analogously to the concept of error unit quaternion in the Q-MEKF, this paper proposes, as far as the authors know, the first multiplicative EKF for pose estimation. Compared to existing literature, the state of the DQ-MEKF only includes six elements of a unit dual quaternion, instead of eight, resulting in obvious computational savings. A version of the DQ-MEKF is presented that takes only discrete-time pose measurements with noise and, hence, is suitable for uncooperative satellite proximity operation scenarios where the chaser satellite has only access to measurements of the relative pose, but requires the relative velocities for control. Finally, the DQ-MEKF is experimentally validated and compared with two alternative EKF formulations on a 5-DOF air-bearing platform.**

I. INTRODUCTION

The highly successful Quaternion Multiplicative Extended Kalman Filter (Q-MEKF) based on unit quaternions for spacecraft attitude estimation, described in detail in Section XI of [1], has been used extensively in several NASA spacecraft [2]. Although newer approaches, such as nonlinear observers, have been shown to have some advantages over the classical EKF, a comprehensive survey of nonlinear attitude estimation methods [2] concluded that the classical EKF is still the most useful solution.

A major advantage of the Q-MEKF is that the 4-by-4 covariance matrix of the four elements of the unit quaternion does not need to be computed. As stated in [1], propagating this covariance matrix, i.e., the state covariance matrix, is the largest computational burden in any Kalman filter implementation. By rewriting the state of the EKF in terms of the three elements of the vector part of the unit *error* quaternion between the true unit quaternion and its estimate, only a 3-by-3 covariance matrix needs to be computed. The unavoidable drawback of this approach is that all three-dimensional attitude representations are singular or discontinuous for certain attitudes [3]. Indeed, by construction, the Q-MEKF described in Section XI of [1] will fail if the attitude error between the true attitude and its estimate is larger than 180 deg. However, unlike the true attitude of the body which can vary arbitrarily, the attitude error between the true attitude of the body and its estimate is expected to be close to zero, especially after the Q-MEFK has converged. Hence, in the Q-MEKF described in [1], whereas the attitude covariance matrix is only 3-by-3, the body can still have any arbitrary attitude.

This paper derives a Dual Quaternion Multiplicative EKF (DQ-MEKF) for spacecraft pose estimation based on the classical Q-MEKF for attitude estimation. As far as the authors know, this is the first multiplicative EKF for pose estimation. Unit dual quaternions offer a compact representation of the pose of a frame with respect to another frame. Their properties, including examples of previous applications, are discussed in length in [4].

The traditional approach to estimate the pose of a body consists of developing separate estimators for attitude and position. For example, [5] suggests two discrete-time linear Kalman filters to estimate the relative attitude and position separately. Since the translation Kalman filter requires the attitude estimated by the rotation Kalman filter, the former is only switched on after the latter has converged. Owing to this inherent coupling between rotation and translation, several authors have proposed estimating the attitude and position simultaneously. For example, in [6], a lander's terrain-relative pose is estimated using an EKF. The state of the EKF contains the vector part of the unit error quaternion and the position vector of the lander with respect to the inertial frame expressed in the inertial frame. Also in [7], the relative pose of two satellites is estimated using an EKF. In this case, the state of the EKF contains the vector part of the unit error quaternion and the position vector of the chaser satellite with respect to the target satellite expressed in a reference frame attached to the target satellite. Finally, [8] also estimates the pose between two frames using a discrete-time additive EKF. In [8], the state contains the position vector of a body with respect to some reference frame expressed in that reference frame along with the four elements of the true quaternion describing the orientation of the body.

As far as the authors know, the only previous EKF formulations where the state includes a unit dual quaternion are given in [9], [10]. However, these EKF formulations include the true unit dual quaternion describing the pose of the body and not the error unit dual quaternion between the true unit dual quaternion and its best estimate. Therefore, the state of the EKFs in [9], [10] contains all eight elements of the unit dual quaternion. Moreover, in [9], [10] the optimal Kalman state update is added to and not multiplied with the current best unit dual quaternion estimate. As a consequence, the predicted value of the unit dual quaternion

*This work was supported by AFRL research award FA9453-13-C-0201.

[1]N. Filipe is a Ph.D. Candidate at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA. Email: nuno.filipe@gatech.edu

[2]M. Kontitsis is a Postdoctoral Research Associate at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA.

[3]P. Tsiotras is Dean's Professor at the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150, USA.

immediately after a measurement update does not fulfill the two algebraic constraints of a unit dual quaternion. Hence, in [9], the predicted value after a measurement update is further modified to satisfy these constraints through a process that includes parameters that must be tuned by the user. On the other hand, in [10], these two algebraic constraints are simply not enforced after a measurement update, which can lead to numerical problems.

The main contributions of this paper are: 1) By using the concept of error unit dual quaternion defined analogously to the concept of error unit quaternion of the Q-MEKF, this paper proposes, as far as the authors know, the first multiplicative EKF for pose estimation. As a consequence, the predicted value of the unit dual quaternion immediately after a measurement update automatically satisfies the two algebraic constraints of a unit dual quaternion. 2) By using the error unit dual quaternion instead of the true unit dual quaternion, the state of the DQ-MEKF is reduced from eight elements to just six. As a consequence, the DQ-MEKF requires less computational resources. Moreover, the state estimate of the DQ-MEKF can be directly used by the pose controllers given in [4] without additional conversions. 3) Similarly to the Q-MEKF, the DQ-MEKF is a *continuous-discrete Kalman filter*, i.e., the state and its covariance matrix are propagated continuously between discrete-time measurements. As a consequence, discrete-time measurements do not need to be equally spaced in time and integrating different sensors with different update rates is relatively straightforward. In particular, the DQ-MEKF is designed to take continuous-time linear and angular velocity measurements with noise and bias and discrete-time pose measurements with noise. An extension of the DQ-MEKF is also proposed that takes only discrete-time pose measurements with noise and estimates the linear and angular velocities. This version is suitable for uncooperative satellite proximity operation scenarios where the chaser satellite has only access to measurements of the relative pose (e.g., from a camera), but requires the relative velocities for control. 4) Finally, the DQ-MEKF without velocity measurements is validated experimentally on a 5-DOF air-bearing platform and compared with two alternative EKF formulations. It is shown that the DQ-MEKF compares favorably with these alternative formulations.

II. THE EXTENDED KALMAN FILTER (EKF)

First, the main equations of the EKF are reviewed based on [1]. The state equation of the EKF is $\dot{x}_n(t) = f_n(x_n(t), t) + g_{n\times p}(x_n(t), t)w_p(t)$, where $x_n(t) \in \mathbb{R}^n$ is the state and $w_p(t) \in \mathbb{R}^p$ is the process noise. The process noise is assumed to be a Gaussian white-noise process, whose mean and covariance function are given by $E\{w_p(t)\} = 0_{p\times 1}$ and $E\left\{w_p(t)w_p^{\mathsf{T}}(\tau)\right\} = Q_{p\times p}(t)\delta(t-\tau)$, where $Q_{p\times p}(t) \in \mathbb{R}^{p\times p}$ is a symmetric positive semidefinite matrix. The initial mean and covariance of the state are given by $E\{x_n(t_0)\} \triangleq \hat{x}_n(t_0) = x_{n,0} \in \mathbb{R}^n$ and $E\{(x_n(t_0) - x_{n,0})(x_n(t_0) - x_{n,0})^{\mathsf{T}}\} \triangleq P_{n\times n}(t_0) = P_{n\times n,0} \in \mathbb{R}^{n\times n}$ and are assumed to be known.

A. Time Update

Given $x_{n,0}$, the minimum covariance estimate of the state at a future time t in the absence of measurements is given by $\hat{x}_n(t) = E\{x_n(t)|\hat{x}_n(t_0) = x_{n,0}\}$. This estimate satisfies the differential equation $\dot{\hat{x}}_n(t) = E\{f_n(x_n(t), t)\}$, which is typically approximated as

$$\dot{\hat{x}}_n(t) \approx f_n(\hat{x}_n(t), t). \tag{1}$$

Hence, in the absence of measurements, the state estimate is propagated using (1). The covariance matrix of the state is given by $P_{n\times n}(t) = E\{\Delta x_n(t)\Delta x_n^{\mathsf{T}}(t)\} \in \mathbb{R}^{n\times n}$, where $\Delta x_n(t) = x_n(t) - \hat{x}_n(t) \in \mathbb{R}^n$ is the state error. As a first-order approximation, the derivative of the state error is given by $\frac{\mathrm{d}}{\mathrm{d}t}\Delta x_n(t) = F_{n\times n}(t)\Delta x_n(t) + G_{n\times p}(t)w_p(t)$ and the covariance matrix of the state satisfies the Riccati equation

$$\dot{P}_{n\times n} = F_{n\times n}P_{n\times n} + P_{n\times n}F_{n\times n}^{\mathsf{T}} + G_{n\times p}Q_{p\times p}G_{n\times p}^{\mathsf{T}}, \tag{2}$$

where

$$F_{n\times n}(t) \triangleq \left.\frac{\partial f_n(x_n, t)}{\partial x_n}\right|_{\hat{x}_n(t)}, \quad G_{n\times p}(t) \triangleq g_{n\times p}(\hat{x}_n(t), t). \tag{3}$$

In the absence of measurements, the covariance matrix of the state is propagated using (2).

B. Measurement Update

Assume that a measurement is taken at time t_k that is related with the state of the EKF through the nonlinear output equation $z_m(t_k) = h_m(x_n(t_k)) + v_m(t_k) \in \mathbb{R}^m$, where $v_m(t_k) \in \mathbb{R}^m$ is the measurement noise assumed to be a discrete Gaussian white-noise process, whose mean and covariance are given by $E\{v_m(t_k)\} = 0_{m\times 1}$ and $E\{v_m(t_k)v_m^{\mathsf{T}}(t_\ell)\} = R_{m\times m}(t_k)\delta_{t_k t_\ell}$, where $R_{m\times m}(t_k) \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix. Immediately following the measurement at time t_k, the minimum variance estimate of $x_n(t_k)$ is given by

$$\hat{x}_n^+(t_k) = \hat{x}_n^-(t_k) + \Delta^\star\hat{x}_n(t_k), \tag{4}$$

where $\Delta^\star\hat{x}_n(t_k) = K_{n\times m}(t_k)[z_m(t_k) - \hat{z}_m(t_k)]$ is the optimal Kalman state update, $\hat{z}_m(t_k) = E\{z_m(t_k)\} \approx h_m(\hat{x}_n^-(t_k))$, $\hat{x}_n^-(t_k)$ and $\hat{x}_n^+(t_k)$ are the predicted values of the state immediately before and after the measurement, and $K_{n\times m}(t_k)$ is the Kalman gain. The Kalman gain is given by

$$K_{n\times m} = P_{n\times n}^- H_{m\times n}^{\mathsf{T}}[H_{m\times n}P_{n\times n}^- H_{m\times n}^{\mathsf{T}} + R_{m\times m}]^{-1}, \tag{5}$$

where $P_{n\times n}^-(t_k)$ is the predicted state covariance matrix immediately before the measurement and

$$H_{m\times n}(t_k) = \left.\frac{\partial h_m(x_n)}{\partial x_n}\right|_{\hat{x}_n^-(t_k)} \in \mathbb{R}^{m\times n} \tag{6}$$

is the measurement sensitivity matrix. Immediately after the measurement, the state covariance matrix is given by $P_{n\times n}^+ = (I_{n\times n} - K_{n\times m}H_{m\times n})P_{n\times n}^-(I_{n\times n} - K_{n\times m}H_{m\times n})^{\mathsf{T}} + K_{n\times m}R_{m\times m}K_{n\times m}^{\mathsf{T}}$.

III. DQ-MEKF

A. Mathematical Preliminaries

1) Quaternions: A quaternion can be represented as the ordered pair $q = (q_0, \overline{q})$, where $\overline{q} = [q_1\ q_2\ q_3]^{\mathsf{T}} \in \mathbb{R}^3$ and $q_0 \in \mathbb{R}$. The basic operations between quaternions are:

Addition: $a + b = (a_0 + b_0, \overline{a} + \overline{b})$,

Multiplication by a scalar: $\lambda a = (\lambda a_0, \lambda\overline{a})$,

$$\text{Multiplication: } ab = (a_0 b_0 - \bar{a}\cdot\bar{b}, a_0\bar{b} + b_0\bar{a} + \bar{a}\times\bar{b}),$$
$$\text{Conjugation: } a^* = (a_0, -\bar{a}),$$
$$\text{Cross product: } a\times b = (0, b_0\bar{a} + a_0\bar{b} + \bar{a}\times\bar{b}),$$

where $\lambda \in \mathbb{R}$. Note that the quaternion multiplication is not commutative. The quaternions $(1, 0_{3\times 1})$ and $(0, 0_{3\times 1})$ will be denoted by $\mathtt{1}$ and $\mathtt{0}$, respectively.

The bijective mapping between the set of quaternions and $\mathbb{R}^4$ will be denoted by $[\,\cdot\,] : \mathbb{H} \to \mathbb{R}^4$, where $[q] = [q_0\ q_1\ q_2\ q_3]^\mathsf{T}$. Using this mapping, the cross product of $a \in \mathbb{H}^v = \{q \in \mathbb{H} :\ q_0 = 0\}$ with $b \in \mathbb{H}^v$ can be computed as $[a \times b] = [a]^\times [b]$, where $[\,\cdot\,]^\times : \mathbb{H}^v \to \mathbb{R}^{4\times 4}$ is defined as

$$[a]^\times = \begin{bmatrix} 0 & 0_{1\times 3} \\ 0_{3\times 1} & \bar{a}^\times \end{bmatrix}, \text{ where } \bar{a}^\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Likewise, the left quaternion multiplication of $a \in \mathbb{H}$ with $b \in \mathbb{H}$ can be computed as $[ab] = [\![a]\!][b]$, where $[\![\,\cdot\,]\!] : \mathbb{H} \to \mathbb{R}^{4\times 4}$ is defined as

$$[\![a]\!] = \begin{bmatrix} a_0 & -\bar{a}^\mathsf{T} \\ \bar{a} & [\tilde{a}] \end{bmatrix}, \text{ where } [\tilde{a}] = a_0 I_{3\times 3} + \bar{a}^\times. \tag{7}$$

The relative attitude of a frame fixed to a body with respect to another frame, say, the I-frame, can be represented by the *unit* quaternion $q_{\text{B/I}} = (\cos(\frac{\phi}{2}), \sin(\frac{\phi}{2})\bar{n})$, where the B-frame is said to be rotated with respect to the I-frame about the unit vector $\bar{n}$ by an angle ϕ. A unit quaternion is defined as a quaternion that belongs to the set $\mathbb{H}^u = \{q \in \mathbb{H} :\ qq^* = q^*q = \mathtt{1}\}$. From this constraint, assuming that $-180 < \phi < 180$ deg, the scalar part of a unit quaternion can be computed from $q_0 = \sqrt{1 - \|\bar{q}\|^2}$. The coordinates of a vector in the B-frame, $\bar{v}^{\text{B}}$, can be calculated from the coordinates of that same vector in the I-frame, $\bar{v}^{\text{I}}$, and vice-versa, via $v^{\text{B}} = q^*_{\text{B/I}} v^{\text{I}} q_{\text{B/I}}$ and $v^{\text{I}} = q_{\text{B/I}} v^{\text{B}} q^*_{\text{B/I}}$, where $v^{\text{X}} = (0, \bar{v}^{\text{X}})$. This is equivalent to $\bar{v}^{\text{B}} = R(q_{\text{B/I}})\bar{v}^{\text{I}}$ and $\bar{v}^{\text{I}} = R(q^*_{\text{B/I}})\bar{v}^{\text{B}}$, where $R(q_{\text{X/Y}})$ is the rotation matrix formed from $q_{\text{X/Y}}$.

2) Dual Quaternions: A dual quaternion is defined as $\boldsymbol{q} = q_r + \epsilon q_d$, where ϵ is the *dual unit* defined by $\epsilon^2 = 0$ and $\epsilon \neq 0$. The quaternions $q_r, q_d \in \mathbb{H}$ are the *real part* and the *dual part* of the dual quaternion, respectively. The basic operations between dual quaternions are [4]:

$$\text{Addition: } \boldsymbol{a} + \boldsymbol{b} = (a_r + b_r) + \epsilon(a_d + b_d),$$
$$\text{Multiplication by a scalar: } \lambda\boldsymbol{a} = (\lambda a_r) + \epsilon(\lambda a_d),$$
$$\text{Multiplication: } \boldsymbol{ab} = (a_r b_r) + \epsilon(a_r b_d + a_d b_r),$$
$$\text{Conjugation: } \boldsymbol{a}^* = a_r^* + \epsilon a_d^*,$$
$$\text{Cross product: } \boldsymbol{a}\times\boldsymbol{b} = a_r\times b_r + \epsilon(a_d\times b_r + a_r\times b_d).$$

Note that the dual quaternion multiplication is not commutative. The dual quaternions $\mathtt{1} + \epsilon\mathtt{0}$ and $\mathtt{0} + \epsilon\mathtt{0}$ will be denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively.

The bijective mapping between the set of dual quaternions and $\mathbb{R}^8$ will be denoted by $[\,\cdot\,] : \mathbb{H}_d \to \mathbb{R}^8$, where $[\boldsymbol{q}] = [[q_r]^\mathsf{T} [q_d]^\mathsf{T}]^\mathsf{T}$. Using this mapping, the left dual quaternion multiplication of $\boldsymbol{a} \in \mathbb{H}_d$ with $\boldsymbol{b} \in \mathbb{H}_d$ can be computed as $[\boldsymbol{ab}] = [\![\boldsymbol{a}]\!][\boldsymbol{b}]$, where $[\![\,\cdot\,]\!] : \mathbb{H}_d \to \mathbb{R}^{8\times 8}$ is defined as

$$[\![\boldsymbol{a}]\!] = \begin{bmatrix} [\![a_r]\!] & 0_{4\times 4} \\ [\![a_d]\!] & [\![a_r]\!] \end{bmatrix}. \tag{8}$$

Finally, it is convenient to define $[\tilde{\cdot}] : \mathbb{H}_d \to \mathbb{R}^{6\times 6}$ and $\bar{\cdot}^\times : \mathbb{H}_d \to \mathbb{R}^{6\times 6}$ as, respectively,

$$[\tilde{\boldsymbol{a}}] = \begin{bmatrix} [\tilde{a}_r] & 0_{3\times 3} \\ [\tilde{a}_d] & [\tilde{a}_r] \end{bmatrix} \text{ and } \bar{\boldsymbol{a}}^\times = \begin{bmatrix} \bar{a}_r^\times & 0_{3\times 3} \\ \bar{a}_d^\times & \bar{a}_r^\times \end{bmatrix}.$$

Similarly, $(\,\cdot\,)_r : \mathbb{H}_d \to \mathbb{H}$ is defined as $(\boldsymbol{a})_r = a_r$, $(\,\cdot\,)_d : \mathbb{H}_d \to \mathbb{H}$ is defined as $(\boldsymbol{a})_d = a_d$, and $\bar{\cdot} : \mathbb{H}_d \to \mathbb{R}^6$ is defined as $\bar{\boldsymbol{a}} = [\bar{a}_r^\mathsf{T}\ \bar{a}_d^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^6$.

The pose of a body frame with respect to another frame, say, the I-frame, can be represented by a *unit* quaternion $q_{\text{B/I}} \in \mathbb{H}^u$ and by a translation vector $\bar{r}^{\text{I}}_{\text{B/I}} \in \mathbb{R}^3$ or $\bar{r}^{\text{B}}_{\text{B/I}} \in \mathbb{R}^3$, where $\bar{r}^{\text{X}}_{\text{Y/Z}}$ is the translation vector from the origin of the Z-frame to the origin of the Y-frame expressed in the X-frame. Alternatively, this pose can be represented more compactly by the *unit* dual quaternion [4]

$$\boldsymbol{q}_{\text{B/I}} = q_{\text{B/I},r} + \epsilon q_{\text{B/I},d} = q_{\text{B/I}} + \epsilon\tfrac{1}{2} r^{\text{I}}_{\text{B/I}} q_{\text{B/I}} = q_{\text{B/I}} + \epsilon\tfrac{1}{2} q_{\text{B/I}} r^{\text{B}}_{\text{B/I}}, \tag{9}$$

where $r^{\text{X}}_{\text{Y/Z}} = (0, \bar{r}^{\text{X}}_{\text{Y/Z}})$. Given $\boldsymbol{q}_{\text{B/I}}$, the position of the body frame with respect to the I-frame can be obtained in I-frame coordinates from $r^{\text{I}}_{\text{B/I}} = 2q_{\text{B/I},d} q^*_{\text{B/I}}$ and in B-frame coordinates from $r^{\text{B}}_{\text{B/I}} = 2q^*_{\text{B/I}} q_{\text{B/I},d}$. Note that whereas the relation between $r^{\text{B}}_{\text{B/I}}$ and $r^{\text{I}}_{\text{B/I}}$ is quadratic in $q_{\text{B/I}}$, $q_{\text{B/I},d}$ is related linearly in $q_{\text{B/I}}$ with $r^{\text{B}}_{\text{B/I}}$ and $r^{\text{I}}_{\text{B/I}}$. A *unit* dual quaternion is defined as a dual quaternion that belongs to the set $\mathbb{H}^u_d = \{\boldsymbol{q} \in \mathbb{H}_d :\ \boldsymbol{qq}^* = \boldsymbol{q}^*\boldsymbol{q} = \mathbf{1}\}$. Assuming that $-180 < \phi < 180$ deg, the scalar parts of the real and dual parts of a unit dual quaternion can be computed from their respective vector parts from

$$q_{r,0} = \sqrt{1 - \|\bar{q}_r\|^2} \quad \text{and} \quad q_{d,0} = \frac{-\bar{q}_r^\mathsf{T}\bar{q}_d}{q_{r,0}}. \tag{10}$$

The rotational and translational kinematic equations of the body frame with respect to another frame can be written compactly in terms of dual quaternions as [4]

$$\dot{\boldsymbol{q}}_{\text{B/I}} = \tfrac{1}{2}\boldsymbol{\omega}^{\text{I}}_{\text{B/I}}\boldsymbol{q}_{\text{B/I}} = \tfrac{1}{2}\boldsymbol{q}_{\text{B/I}}\boldsymbol{\omega}^{\text{B}}_{\text{B/I}}, \tag{11}$$

where $\boldsymbol{\omega}^{\text{X}}_{\text{Y/Z}}$ is the *dual velocity* of the Y-frame with respect to the Z-frame expressed in the X-frame, $\boldsymbol{\omega}^{\text{B}}_{\text{B/I}} \triangleq \omega^{\text{B}}_{\text{B/I}} + \epsilon v^{\text{B}}_{\text{B/I}}$, $\boldsymbol{\omega}^{\text{I}}_{\text{B/I}} \triangleq \omega^{\text{I}}_{\text{B/I}} + \epsilon(v^{\text{I}}_{\text{B/I}} - \omega^{\text{I}}_{\text{B/I}} \times r^{\text{I}}_{\text{B/I}})$, $\omega^{\text{X}}_{\text{Y/Z}} = (0, \bar{\omega}^{\text{X}}_{\text{Y/Z}})$, $\bar{\omega}^{\text{X}}_{\text{Y/Z}}$ is the angular velocity of the Y-frame with respect to the Z-frame expressed in the X-frame, $v^{\text{X}}_{\text{Y/Z}} = (0, \bar{v}^{\text{X}}_{\text{Y/Z}})$, and $\bar{v}^{\text{X}}_{\text{Y/Z}}$ is the linear velocity of the origin of the Y-frame with respect to the Z-frame expressed in the X-frame.

B. Angular and Linear Velocity Measurement Model

The dual velocity measurement model is defined analogously to the angular velocity measurement model typically used in literature [1], [2], i.e.,

$$\boldsymbol{\omega}^{\text{B}}_{\text{B/I},m} = \boldsymbol{\omega}^{\text{B}}_{\text{B/I}} + \boldsymbol{b}_{\boldsymbol{\omega}} + \boldsymbol{\eta}_{\boldsymbol{\omega}}, \tag{12}$$

where $\boldsymbol{\omega}^{\text{B}}_{\text{B/I},m} = \omega^{\text{B}}_{\text{B/I},m} + \epsilon v^{\text{B}}_{\text{B/I},m} \in \mathbb{H}^v_d$, $\omega^{\text{B}}_{\text{B/I},m} = (0, \bar{\omega}^{\text{B}}_{\text{B/I},m})$, $\bar{\omega}^{\text{B}}_{\text{B/I},m}$ is a measurement of $\bar{\omega}^{\text{B}}_{\text{B/I}}$, $v^{\text{B}}_{\text{B/I},m} = (0, \bar{v}^{\text{B}}_{\text{B/I},m})$, $\bar{v}^{\text{B}}_{\text{B/I},m}$ is a measurement of $\bar{v}^{\text{B}}_{\text{B/I}}$, $\boldsymbol{b}_{\boldsymbol{\omega}} = b_\omega + \epsilon b_v$ is the dual bias, $b_\omega = (0, \bar{b}_\omega)$, $\bar{b}_\omega \in \mathbb{R}^3$ is the bias of the angular velocity measurement, $b_v = (0, \bar{b}_v)$, $\bar{b}_v \in \mathbb{R}^3$ is the bias of the linear velocity measurement, $\boldsymbol{\eta}_{\boldsymbol{\omega}} = \eta_\omega + \epsilon\eta_v$, $\eta_\omega = (0, \bar{\eta}_\omega)$, $\bar{\eta}_\omega \in \mathbb{R}^3$ is the noise of the angular velocity measurement assumed to be a Gaussian white-noise process, $\eta_v = (0, \bar{\eta}_v)$, and $\bar{\eta}_v \in \mathbb{R}^3$ is the noise of the linear velocity measurement assumed to be a Gaussian white-noise process with $E\left\{\overline{\boldsymbol{\eta}}_{\boldsymbol{\omega}}\right\}=0_{6\times 1}$, $E\left\{\overline{\boldsymbol{\eta}}_{\boldsymbol{\omega}}(t)\overline{\boldsymbol{\eta}}_{\boldsymbol{\omega}}^{\mathsf{T}}(\tau)\right\}=\overline{Q}_{\boldsymbol{\omega}}(t)\delta(t-\tau)$, where

$$\overline{Q}_{\boldsymbol{\omega}}(t)=\begin{bmatrix}\overline{Q}_{\omega}(t) & \overline{Q}_{\omega v}(t)\\ \overline{Q}_{\omega v}(t) & \overline{Q}_{v}(t)\end{bmatrix}\in\mathbb{R}^{6\times 6}$$

is a symmetric positive semidefinite matrix. The dual bias is not constant, but assumed to be driven by another Gaussian white-noise process through

$$\dot{\boldsymbol{b}}_{\boldsymbol{\omega}}=\boldsymbol{\eta}_{\boldsymbol{b}_{\boldsymbol{\omega}}},\tag{13}$$

where $\boldsymbol{\eta}_{\boldsymbol{b}_{\boldsymbol{\omega}}}=(0,\overline{\eta}_{b_{\omega}})+\epsilon(0,\overline{\eta}_{b_v})$, $E\left\{\overline{\boldsymbol{\eta}}_{\boldsymbol{b}_{\boldsymbol{\omega}}}\right\}=0_{6\times 1}$, $E\left\{\overline{\boldsymbol{\eta}}_{\boldsymbol{b}_{\boldsymbol{\omega}}}(t)\overline{\boldsymbol{\eta}}_{\boldsymbol{b}_{\boldsymbol{\omega}}}^{\mathsf{T}}(\tau)\right\}=\overline{Q}_{\boldsymbol{b}_{\boldsymbol{\omega}}}(t)\delta(t-\tau)$, and

$$\overline{Q}_{\boldsymbol{b}_{\boldsymbol{\omega}}}(t)=\begin{bmatrix}\overline{Q}_{b_{\omega}}(t) & \overline{Q}_{b_{\omega}b_v}(t)\\ \overline{Q}_{b_{\omega}b_v}(t) & \overline{Q}_{b_v}(t)\end{bmatrix}\in\mathbb{R}^{6\times 6}$$

is a symmetric positive semidefinite matrix.

If the I-frame is inertial, $\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}$ should be interpreted as the inertial angular and linear velocities of the satellite. In that case, $\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}$ can be measured from a combination of, e.g., rate-gyros, Doppler radar, and GPS. On the other hand, if the I-frame is not inertial, $\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}$ should be interpreted as the relative angular and linear velocities of the satellite with respect to a moving frame, for example, a frame attached to another satellite. In that case, $\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}$ can be measured from a combination of, e.g., rate-gyros on both satellites [7], Doppler radar, differential GPS, and LIDAR.

C. Derivation of the DQ-MEKF

The state and process noise of the DQ-MEKF are initially selected as $x_{16}=[[\delta\boldsymbol{q}_{\mathrm{B/I}}]^{\mathsf{T}},[\boldsymbol{b}_{\boldsymbol{\omega}}]^{\mathsf{T}}]^{\mathsf{T}}\in\mathbb{R}^{16}$ and $w_{16}=[[\boldsymbol{\eta}_{\boldsymbol{\omega}}]^{\mathsf{T}},[\boldsymbol{\eta}_{\boldsymbol{b}_{\boldsymbol{\omega}}}]^{\mathsf{T}}]^{\mathsf{T}}\in\mathbb{R}^{16}$, where the dual error quaternion $\delta\boldsymbol{q}_{\mathrm{B/I}}\in\mathbb{H}_d^u$ is defined analogously to the error quaternion [1] $\delta q_{\mathrm{B/I}}=\hat{q}_{\mathrm{B/I}}^{*}q_{\mathrm{B/I}}\in\mathbb{H}^u$ as $\delta\boldsymbol{q}_{\mathrm{B/I}}=\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{*}\boldsymbol{q}_{\mathrm{B/I}}\in\mathbb{H}_d^u$, i.e., $\delta\boldsymbol{q}_{\mathrm{B/I}}$ is the dual quaternion between the actual dual quaternion $\boldsymbol{q}_{\mathrm{B/I}}\in\mathbb{H}_d^u$ and its current best estimate $\hat{\boldsymbol{q}}_{\mathrm{B/I}}\in\mathbb{H}_d^u$. Analogously to the propagation of $\hat{q}_{\mathrm{B/I}}\in\mathbb{H}^u$ in [3], $\hat{\boldsymbol{q}}_{\mathrm{B/I}}$ is propagated using

$$\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{q}}_{\mathrm{B/I}})\approx\tfrac{1}{2}\hat{\boldsymbol{q}}_{\mathrm{B/I}}\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\mathrm{B}},\tag{14}$$

where, from (12),

$$\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\mathrm{B}}\triangleq E\left\{\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}\right\}=\boldsymbol{\omega}_{\mathrm{B/I},m}^{\mathrm{B}}-\hat{\boldsymbol{b}}_{\boldsymbol{\omega}},\tag{15}$$

with $\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}\triangleq E\left\{\boldsymbol{b}_{\boldsymbol{\omega}}\right\}$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}\right)=E\left\{\boldsymbol{\eta}_{\boldsymbol{b}_{\boldsymbol{\omega}}}\right\}=\mathbf{0}.\tag{16}$$

The approximation in (14) is a result of using the typical EKF approximation given by (1) [3]. Note that the current best guess of $\boldsymbol{q}_{\mathrm{B/I}}$, given by $\hat{\boldsymbol{q}}_{\mathrm{B/I}}$, is not defined as the standard expected value of the random variable $\boldsymbol{q}_{\mathrm{B/I}}$ as this would require the expectation to be defined with respect to a non-trivial probability density function in $\mathbb{H}_d^u$. As shown in [11], even the definition of probability density function on $\mathbb{H}^u$ is not trivial. The reader is referred to [11] for a discussion of possible probability density functions in $\mathbb{H}^u$.

Analogously to [3], for $-180<\phi<180$ deg, $\delta\boldsymbol{q}_{\mathrm{B/I}}$ is parameterized by $\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}}$ and the expected value of $\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}}$ is required to be zero, i.e., $E\left\{\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}}\right\}=0_{6\times 1}$. Hence, $E\left\{\delta\boldsymbol{q}_{\mathrm{B/I}}\left(\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}}\right)\right\}=\mathbf{1}$.

To determine the state equation of the DQ-MEKF, the time derivative of $\delta\boldsymbol{q}_{\mathrm{B/I}}$ needs to be calculated. Taking the time derivative of $\delta\boldsymbol{q}_{\mathrm{B/I}}$ yields $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}_{\mathrm{B/I}})=\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{*})\boldsymbol{q}_{\mathrm{B/I}}+\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{*}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{q}_{\mathrm{B/I}})$. Substituting in (11) and (14) yields $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}_{\mathrm{B/I}})\approx\frac{1}{2}(\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}})^{*}\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{*}\boldsymbol{q}_{\mathrm{B/I}}+\frac{1}{2}\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{*}\boldsymbol{q}_{\mathrm{B/I}}\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}=-\frac{1}{2}\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}\delta\boldsymbol{q}_{\mathrm{B/I}}+\frac{1}{2}\delta\boldsymbol{q}_{\mathrm{B/I}}\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}$. Combining (15) and (12) yields $\boldsymbol{\omega}_{\mathrm{B/I}}^{\mathrm{B}}\approx\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}+\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}-\boldsymbol{b}_{\boldsymbol{\omega}}-\boldsymbol{\eta}_{\boldsymbol{\omega}}$. Finally, inserting the previous equation in the expression for $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}_{\mathrm{B/I}})$ results in $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}_{\mathrm{B/I}})\approx\frac{1}{2}(\delta\boldsymbol{q}_{\mathrm{B/I}}(\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}+\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}-\boldsymbol{b}_{\boldsymbol{\omega}}-\boldsymbol{\eta}_{\boldsymbol{\omega}})-\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}\delta\boldsymbol{q}_{\mathrm{B/I}})$.

At this point, as in the derivation of the Q-MEKF, reduced state and process noise vectors are selected, namely, $x_{12}=[\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}}^{\mathsf{T}},\overline{\boldsymbol{b}}_{\boldsymbol{\omega}}^{\mathsf{T}}]^{\mathsf{T}}\in\mathbb{R}^{12}$ and $w_{12}=[\overline{\boldsymbol{\eta}}_{\boldsymbol{\omega}}^{\mathsf{T}},\overline{\boldsymbol{\eta}}_{\boldsymbol{b}_{\boldsymbol{\omega}}}^{\mathsf{T}}]^{\mathsf{T}}\in\mathbb{R}^{12}$. The state equations of the DQ-MEKF are then given by the vector parts of $\frac{\mathrm{d}}{\mathrm{d}t}(\delta\boldsymbol{q}_{\mathrm{B/I}})$ and (13), yielding $f_{12}(x_{12}(t),t)$ and $g_{12\times 12}(x_{12}(t),t)$ equal to, respectively,

$$\begin{bmatrix}\overline{\frac{1}{2}(\delta\boldsymbol{q}_{\mathrm{B/I}}(\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}+\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}-\boldsymbol{b}_{\boldsymbol{\omega}}-\boldsymbol{\eta}_{\boldsymbol{\omega}})-\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\hat{\mathrm{B}}}\delta\boldsymbol{q}_{\mathrm{B/I}})}\\ 0_{6\times 1}\end{bmatrix},\ \begin{bmatrix}-\frac{1}{2}[\widetilde{\delta\boldsymbol{q}_{\mathrm{B/I}}}] & 0_{6\times 6}\\ 0_{6\times 6} & I_{6\times 6}\end{bmatrix}.$$

By replacing $\delta q_{\mathrm{B/I},r,0}$ and $\delta q_{\mathrm{B/I},d,0}$ through (10) in $f_{12}(x_{12}(t),t)$ and $g_{12\times 12}(x_{12}(t),t)$ and using (3), $F_{12\times 12}(t)$ and $G_{12\times 12}(t)$ can be determined to be, respectively,

$$\begin{bmatrix}-\overline{\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\mathrm{B}}}^{\times} & -\frac{1}{2}I_{6\times 6}\\ 0_{6\times 6} & 0_{6\times 6}\end{bmatrix}\ \text{and}\ \begin{bmatrix}-\frac{1}{2}I_{6\times 6} & 0_{6\times 6}\\ 0_{6\times 6} & I_{6\times 6}\end{bmatrix}.$$

1) Time Update: For the time update of the DQ-MEKF, $\hat{\boldsymbol{q}}_{\mathrm{B/I}}$, $\hat{\boldsymbol{\omega}}_{\mathrm{B/I}}^{\mathrm{B}}$, and $\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}$ are propagated using (14), (15), and (16), respectively, given $\hat{\boldsymbol{q}}_{\mathrm{B/I}}(t_0)$ and $\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}(t_0)$. Numerical errors in the propagation of $\hat{\boldsymbol{q}}_{\mathrm{B/I}}$ through (14) can result in the violation of the algebraic constraints associated with $\mathbb{H}_d^u$. Hence, after each integration step, these algebraic constraints are enforced by calculating $[q_{\mathrm{B/I},r}]=[q_{\mathrm{B/I},r}]/\|[q_{\mathrm{B/I},r}]\|$ and $[q_{\mathrm{B/I},d}]=(I_{4\times 4}-[q_{\mathrm{B/I},r}][q_{\mathrm{B/I},r}]^{\mathsf{T}}/\|[q_{\mathrm{B/I},r}]\|^2)[q_{\mathrm{B/I},d}]$. The covariance matrix of x_{12} is propagated according to (2) given $P_{12\times 12}(t_0)$.

2) Measurement Update: Here, it is assumed that a measurement of $\boldsymbol{q}_{\mathrm{B/I}}$ is available. If the I-frame is a moving frame, this measurement can come, for example, from a vision-based system. If the I-frame is an inertial frame, this measurement can come, for example, from a combination of a star sensor and a GPS. If the pose measurement is available in terms of a quaternion and a translation vector, then the corresponding dual quaternion can be computed from (9). Then, the output equation is defined analogously to the output equation used in [7], [3] when a quaternion measurement is available, i.e., $\overline{(\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{-}(t_k))^{*}\boldsymbol{q}_{\mathrm{B/I},m}(t_k)}=\overline{\delta\boldsymbol{q}_{\mathrm{B/I}}(t_k)}+v_6(t_k)$. Using (6) to calculate the measurement sensitivity matrix yields $H_{6\times 12}(t_k)=[I_{6\times 6}\ 0_{6\times 6}]$. In summary, for the measurement update of the DQ-MEKF, the Kalman gain is calculated from (5) and the optimal Kalman state update is calculated from

$$\Delta^{\star}\hat{x}_{12}(t_k)\triangleq\begin{bmatrix}\Delta^{\star}\overline{\delta\hat{\boldsymbol{q}}_{\mathrm{B/I}}}(t_k)\\ \Delta^{\star}\overline{\hat{\boldsymbol{b}}}_{\boldsymbol{\omega}}(t_k)\end{bmatrix}=K_{12\times 6}\overline{(\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{-}(t_k))^{*}\boldsymbol{q}_{\mathrm{B/I},m}(t_k)}.$$

The estimate of the state at time t_k after the measurement is then calculated from

$$\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{+}(t_k)=\hat{\boldsymbol{q}}_{\mathrm{B/I}}^{-}(t_k)\Delta^{\star}\delta\hat{\boldsymbol{q}}_{\mathrm{B/I}}(t_k),\tag{17}$$

$$\overline{\hat{\boldsymbol{b}}}_{\boldsymbol{\omega}}^{+}(t_k)=\overline{\hat{\boldsymbol{b}}}_{\boldsymbol{\omega}}^{-}(t_k)+\Delta^{\star}\overline{\hat{\boldsymbol{b}}}_{\boldsymbol{\omega}}(t_k),\tag{18}$$

where $\Delta^{\star}\delta\hat{\boldsymbol{q}}_{\mathrm{B/I}}$ is the unit dual quaternion with vector part given by $\Delta^{\star}\overline{\delta\hat{\boldsymbol{q}}_{\mathrm{B/I}}}(t_k)$ and scalar part given by (10). Finally,

$P^{+}_{12\times 12}$ is computed. Note that whereas (18) is a direct application of (4), (17) is not. Since $\Delta^{\star}\delta\hat{\boldsymbol{q}}_{\mathrm{B/I}}(t_k)$ is a unit dual quaternion, $\hat{\boldsymbol{q}}^{+}_{\mathrm{B/I}}(t_k)$ is calculated using the dual quaternion multiplication, making the proposed EKF multiplicative.

Any measurement that is a nonlinear function of the state of the DQ-MEKF can be used in the measurement update. If another measurement is used, only the measurement sensitivity matrix needs to be recalculated. For example, if the measurements are $q_{\mathrm{B/I},m}$ and $r^{\mathrm{I}}_{\mathrm{B/I},m}$, then the output equation is defined as

$$\begin{bmatrix} \overline{(\hat{q}^{-}_{\mathrm{B/I}})^{*} q_{\mathrm{B/I},m}} \\ \bar{r}^{\mathrm{I}}_{\mathrm{B/I},m} \end{bmatrix} = \begin{bmatrix} \overline{\delta q_{\mathrm{B/I},r}} \\ 2\left(\hat{\boldsymbol{q}}_{\mathrm{B/I}}\delta\boldsymbol{q}_{\mathrm{B/I}}\right)_d \delta q^{*}_{\mathrm{B/I}}\hat{q}_{\mathrm{B/I}} \end{bmatrix} + v_6. \tag{19}$$

The new sensitivity matrix can be determined to be

$$H_{6\times 12}(t_k) = \begin{bmatrix} I_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 2R\left((\hat{q}^{-}_{\mathrm{B/I}})^{*}\right) & 0_{3\times 3} & 0_{3\times 3}. \end{bmatrix}. \tag{20}$$

D. DQ-MEKF Without Velocity Measurements

A special case of particular interest is when pose measurements are available, but angular and linear velocity measurements are not. Although velocity measurements are not available, velocity estimates might be required for pose control [4]. In this section, it is shown how this special case can be handled by modifying the inputs and parameters of the DQ-MEKF algorithm, without any modifications to the structure and basic equations of the DQ-MEKF algorithm.

As before, the I-frame may or may not be inertial. However, this version of the DQ-MEKF is specially suited for satellite proximity operations where the relative pose is measured using vision-based systems, which typically do not provide relative velocity measurements. In this scenario, the I-frame is the moving frame of the target satellite.

If angular and linear velocity measurements are not available, but estimates are required, $\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I},m}$ and $\boldsymbol{\eta}_{\boldsymbol{\omega}}$ are set to zero in (12). This results in $\boldsymbol{b}_{\boldsymbol{\omega}} = -\boldsymbol{\omega}^{\mathrm{B}}_{\mathrm{B/I}}$ and $\overline{Q}_{\boldsymbol{\omega}} = 0_{6\times 6}$. The dual velocity estimate is still given by (15), which now has the form $\hat{\boldsymbol{\omega}}^{\mathrm{B}}_{\mathrm{B/I}} = -\hat{\boldsymbol{b}}_{\boldsymbol{\omega}}$. The time derivative of $\boldsymbol{b}_{\boldsymbol{\omega}}$ is still calculated as in (13). However, since $\boldsymbol{b}_{\boldsymbol{\omega}}$ is now expected to be time-varying and not constant, the effect of the noise $\boldsymbol{\eta}_{\boldsymbol{b}_{\boldsymbol{\omega}}}$ might have to be increased by increasing $\overline{Q}_{\boldsymbol{b}_{\boldsymbol{\omega}}}$.

IV. EXPERIMENTAL RESULTS

In this section, the DQ-MEKF without velocity measurements is validated experimentally on the Autonomous Spacecraft Testing of Robotic Operations in Space (ASTROS) facility at the School of Aerospace Engineering of Georgia Tech [12]. This experimental facility includes a 5-DOF platform supported on hemispherical and linear air-bearings moving over a flat epoxy floor in order to simulate as best as possible the frictionless environment of space. It also includes a VICON motion capture system mounted on an aluminum grid above the experimental area. The VICON system measures the pose of the platform with respect to a reference frame fixed to the room. The standard deviation of these measurements is smaller than 7×10^{-5} and 1 mm for $q_{\mathrm{B/I},m}$ and $\bar{r}^{\mathrm{I}}_{\mathrm{B/I},m}$, respectively.

The ground truth for the pose was obtained from VICON measurements at 100 Hz. The ground truth for the linear velocity was obtained by passing the position measurements through a Linear Time-Invariant (LTI) system with transfer matrix $H(s) = \frac{3s}{s+3}I_{3\times 3}$. Finally, the ground truth for the angular velocity was obtained by passing the quaternion measurements through a LTI system with transfer matrix $H(s) = \frac{3s}{s+3}I_{4\times 4}$ and by using the relation $\omega^{\mathrm{B}}_{\mathrm{B/I}} = 2q^{*}_{\mathrm{B/I}}\dot{q}_{\mathrm{B/I}}$.

The DQ-MEKF was fed pose measurements from the VICON system at 0.5 Hz modeled through output equation (19). The initial estimate of the state is given in Table I. The same table also shows an a posteriori guess of the initial state based on the measurements. The DQ-MEKF was initialized with the covariance matrices: $P_{12\times 12}(0) = 1\times 10^{-9} I_{12\times 12}$, $Q_{12\times 12} = \mathrm{diag}([0,0,0,0,0,0,1\times 10^{-3},1\times 10^{-3},1\times 10^{-3},1\times 10^{-1},1\times 10^{-1},1\times 10^{-1}])$, and $R_{6\times 6} = \mathrm{diag}([1.4\times 10^{-6},1.4\times 10^{-6},1.4\times 10^{-6},2.25\times 10^{-6},2.25\times 10^{-6},2.25\times 10^{-6}])$.

TABLE I

INITIAL ESTIMATE AND A POSTERIORI GUESS OF THE STATE.

Variable	Initial Estimate	Best A Posteriori Guess
$q_{\mathrm{B/I}}(0)$	$[0.71,0,0,0.71]^{\mathsf{T}}$	$[0.80,-0.02,-0.02,0.60]^{\mathsf{T}}$
$\bar{r}^{\mathrm{I}}_{\mathrm{B/I}}(0)$	$[-0.5,2,-1]^{\mathsf{T}}$ (m)	$[-0.53,2.04,-0.99]^{\mathsf{T}}$ (m)
$b_{\omega}(0)$	$[0,0,0]^{\mathsf{T}}$ (deg/s)	$[0,0,0]^{\mathsf{T}}$ (deg/s)
$b_{v}(0)$	$[0,0,0]^{\mathsf{T}}$ (m/s)	$[0,0,0]^{\mathsf{T}}$ (m/s)

The pose estimated by the DQ-MEKF is compared with the ground truth in Fig. 1. Note that the motion only starts around 20 sec after the beginning of the experiment.

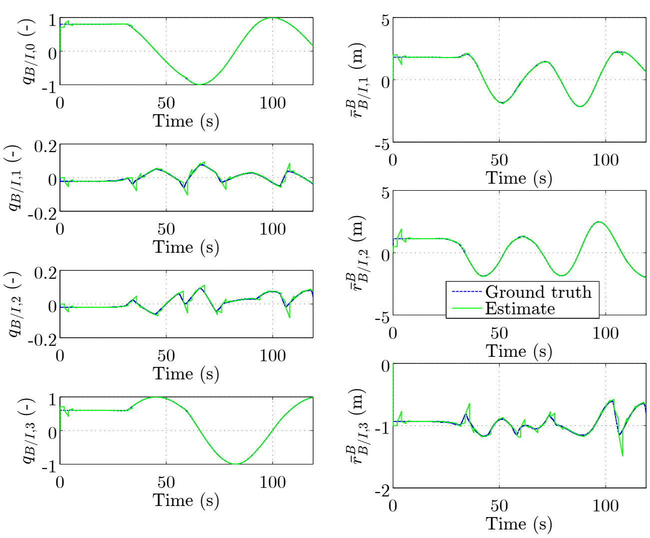

Fig. 1. Estimated and true pose.

The pose estimation error obtained with the DQ-MEKF is plotted in Fig. 2. Note that the pose error increases at around 20 sec, when the motion starts. The same figure also shows the pose estimation error obtained with two alternative EKF formulations. The first alternative EKF formulation, hereby referred to as the QV-AEKF, is an additive EKF, where the state contains the vector part of the unit error quaternion and the position vector of the body with respect to the inertial frame expressed in the body frame. The second alternative EKF formulation, hereby referred to as the SQV-AEKF, is essentially the QV-AEKF split into two additive EKFs, one for the attitude and another one for the position. The QV-AEKF and SQV-AEKF are derived in detail in [13]. In terms of computational resources, the DQ-MEKF, the QV-AEKF, and the SQV-AEKF require the propagation of 92, 91, and 55 states, respectively. For the comparison between the filters to be fair, the three filters were fed the same measurements, were initialized with the same initial estimate of the state, and were tuned with the same covariance matrices. The linear and angular velocity estimation errors obtained with the three filters are also shown in Fig. 2.

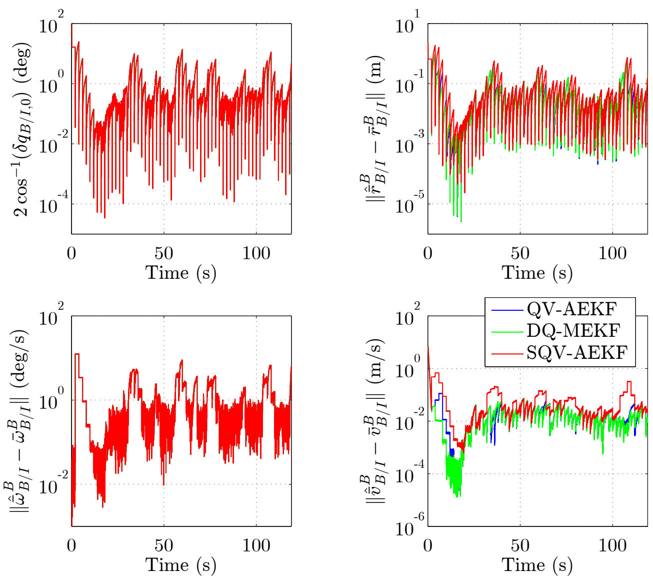

Fig. 2. Pose and velocity estimation error.

The Root-Mean-Square (RMS) attitude, position, angular velocity, and linear velocity estimation errors after 20 sec obtained with the three filters are given in Table II. Note that the RMS attitude and angular velocity estimation errors obtained with the three filters are the same. This is not surprising since the DQ-MEKF, the QV-AEKF, and the SQV-AEKF represent and update the attitude in the same way and the attitude is independent from the position. The SQV-AEKF exhibits the highest RMS position and linear velocity estimation errors. This is understandable since the DQ-MEKF and the QV-AEKF take into consideration the fact that the position vector of the body with respect to the inertial frame expressed in the body frame depends on the attitude of the body, whereas the SQV-AEKF does not. Moreover, the RMS position and linear velocity estimation errors obtained with the DQ-MEKF are smaller than the ones obtained with the QV-AEKF. This can be justified in part by the fact that the relation between $r^{\mathrm{B}}_{\mathrm{B/I}}$ and $r^{\mathrm{I}}_{\mathrm{B/I}}$ is quadratic in $q_{\mathrm{B/I}}$, whereas the relation between $q_{\mathrm{B/I},d}$ and $r^{\mathrm{I}}_{\mathrm{B/I}}$ is linear in $q_{\mathrm{B/I}}$. Hence, the linearization error committed when linearizing the output equations of the QV-AEKF and of the DQ-MEKF with respect to $\delta q_{\mathrm{B/I}}$ is smaller in the DQ-MEKF case.

V. CONCLUSION

This paper proposes a Dual Quaternion Multiplicative Extended Kalman Filter for pose estimation that is an extension of the well-known and widely used Quaternion Multiplicative Extended Kalman Filter for spacecraft attitude estimation. By using the dual quaternion multiplication and the concept of error unit dual quaternion, the two algebraic constraints of unit dual quaternions are automatically satisfied during the measurement update of the DQ-MEKF and the number of states is reduced from eight to six. The experimental results show that the DQ-MEKF does not encounter singularities and is accurate, precise, and fast enough for operational use. Moreover, when compared with two other EKF formulations, the experimental results suggest that the DQ-MEKF might be the best formulation if the measurements are expressed in a different reference frame than the variable to be estimated. This is the case, for example, when one needs the inertial position of a satellite expressed in the body frame, e.g., to implement a control law, but the measurements are expressed in the inertial frame, like the inertial position measurements produced by a GPS.

TABLE II

RMS ESTIMATION ERRORS AFTER 20 SEC.

RMS Estimation Error	DQ-MEKF	QV-AEKF	SQV-AEKF
Attitude (deg)	2.22	2.22	2.22
Position (mm)	70.8	69.5	122.8
Angular Velocity (deg/s)	1.91	1.91	1.91
Linear Velocity (mm/s)	22.7	22.2	80.7

REFERENCES

[1] E. Lefferts, F. Markley, and M. Shuster, "Kalman filtering for spacecraft attitude estimation," *Journal of Guidance, Control, and Dynamics*, vol. 5, no. 5, pp. 417–429, September-October 1982.

[2] J. L. Crassidis, F. L. Markley, and Y. Cheng, "Survey of nonlinear attitude estimation methods," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 12–28, January-February 2007.

[3] F. L. Markley, "Attitude error representations for Kalman filtering," *Journal of Guidance, Control, and Dynamics*, vol. 26, no. 2, pp. 311–317, March-April 2003.

[4] N. Filipe and P. Tsiotras, "Adaptive position and attitude-tracking controller for satellite proximity operations using dual quaternions," *Journal of Guidance, Control, and Dynamics*, 2014, DOI: 10.2514/1.G000054.

[5] M. Romano, D. A. Friedman, and T. J. Shay, "Laboratory experimentation of autonomous spacecraft approach and docking to a collaborative target," *Journal of Spacecraft and Rockets*, vol. 44, no. 1, pp. 164–173, January-February 2007.

[6] A. I. Mourikis, N. Trawny, S. I. Roumeliotis, A. E. Johnson, A. Ansar, and L. Matthies, "Vision-aided inertial navigation for spacecraft entry, descent, and landing," *IEEE Transactions on Robotics*, vol. 25, no. 2, pp. 264–280, April 2009.

[7] S.-G. Kim, J. L. Crassidis, Y. Cheng, A. Fosbury, and J. L. Junkins, "Kalman filtering for relative spacecraft attitude and position estimation," *Journal of Guidance, Control, and Dynamics*, vol. 30, no. 1, pp. 133–143, January-February 2007.

[8] J. S. Goddard, "Pose and motion estimation from vision using dual quaternion-based extended Kalman filtering," Ph.D. dissertation, The University of Tennessee, Knoxville, 1997.

[9] E. Bayro-Corrochano and Y. Zhang, "The motor extended Kalman filter: A geometric approach for rigid motion estimation," *Journal of Mathematical Imaging and Vision*, vol. 13, no. 3, pp. 205–228, December 2000.

[10] Y. Zu, U. Lee, and R. Dai, "Distributed motion estimation of space objects using dual quaternions," in *AIAA/AAS Astrodynamics Specialist Conference*, no. 2014-4296, San Diego, CA, August 4-7 2014.

[11] F. L. Markley, "Attitude estimation or quaternion estimation?" *Advances in the Astronautical Sciences*, vol. 115, pp. 113–127, 2003.

[12] P. Tsiotras, "ASTROS: A 5DOF experimental facility for research in space proximity operations," in *AAS Guidance and Control Conference*, no. 2014-114, Breckenridge, CO, January 31 - February 5 2014.

[13] N. Filipe, M. Kontitsis, and P. Tsiotras, "Extended Kalman filter for spacecraft pose estimation using dual quaternions," *Journal of Guidance, Control, and Dynamics*, 2015, to appear.