

Speed Profile Optimization for Optimal Path Tracking

Yiming Zhao and Panagiotis Tsiotras

Abstract—In this paper, we study the problem of minimum-time, and minimum-energy speed profile optimization along a given path, which is a key step for solving the optimal path tracking problems for a particular class of dynamical systems. We focus on characterizing the optimal switching structure between extremal controls using optimal control theory, and present semi-analytical solutions to both problems. It is shown that the optimal solutions of these two problems are closely related.

Index Terms—Optimal control, path tracking, minimum energy, minimum time

I. INTRODUCTION

The minimum-time path-tracking problem for robotic manipulators, ground vehicles, and aircraft has been studied in [8], [6], [7], [9], [11]. The optimal solution to these problems can help improve plant productivity [8], [6], [7], racing car performance [9], or achieve faster aircraft landing in case of an emergency [11], [14]. These solutions maximize pointwise the speed along the path, and do not contain any singular arcs. When tracking time is not of primary concern, it is often desirable to minimize the energy or the fuel consumption of the system. Along this direction, the minimum-work problem has been studied in [1], [5], [3]. Unlike the solution to the minimum-time problem, minimum-work or minimum-energy solutions usually contain singular control arcs, in addition to the bang-bang control arcs. It is observed that for all the dynamical systems mentioned above, the optimal path tracking problem can be simplified to a speed profile optimization problem subject to speed and control constraints and boundary conditions. After the optimal speed is obtained, the other state and control variables can be obtained using inverse dynamics. In this paper, we analyze the basic speed optimization problem showing up in these different applications.

The main contributions of this paper include: a) The identification of optimal switching structures in the minimum-time and minimum-energy solutions, and b) The characterization of the relation between optimal solutions of minimum-time, maximum-time, and minimum-energy path-tracking problems.

II. PROBLEM FORMULATION

The system dynamics considered in this paper have the following form

$$\begin{aligned} \dot{s} &= v, \\ \dot{v} &= -d(v, s) + u, \end{aligned} \quad (1)$$

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where s is the path coordinate with $s \in [s_0, s_f] \subset \mathbb{R}$, and v is the speed at which the system moves along the path, whereas $d : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function representing the accelerations (e.g., drag) affecting the speed along the path. The system is subject to position-dependent speed constraints $0 \geq v_{\min}(s) \leq v \leq v_{\max}(s)$, and control constraints $u_{\min}(s) \leq u(s) \leq u_{\max}(s)$. We would like to find u that minimizes a certain cost function while keeping all constraints satisfied. Despite its simple form, system (1) is suitable for many industrial and transportation systems, including those mentioned in the Introduction.

By letting $E = v^2/2$ and by using the path length as the new independent variable, then (1) reduces to a single differential equation along the path

$$E'(s) = -D(E, s) + u, \quad (2)$$

where $(\cdot)'$ denotes the derivative with respect to s . We will assume that $D(E, s) = d(\sqrt{2E}, s)$ satisfies the following assumptions.

Assumption 1: For all $E \in [v_{\min}^2(s)/2, v_{\max}^2(s)/2]$, and $s \in [s_0, s_f]$, the following conditions hold

- i) $D(E, s)$ is at least twice differentiable with respect to E , and $D(E, s)$, $\partial D/\partial E$ and $\partial^2 D/\partial E^2$ are continuous with respect to s .
- ii) $\frac{\partial^2 D(E, s)}{\partial E^2} + \frac{3}{2E} \frac{\partial D(E, s)}{\partial E} > 0$.

Assumption 1 is a necessary condition for the main theoretical results later in this paper. In particular, condition ii) implies that $v^2 \partial d(v, s)/\partial v$ is monotonically increasing with respect to v . This assumption holds in many cases, for example, when d is the summation of aerodynamic drag and the component of the gravity forces.

We consider optimal path tracking problems of the form
Problem 1 (Optimal Control for Path Tracking): Solve

$$\min_u J(t_f, E, u), \quad (3)$$

$$\text{subject to } E'(s) = -D(E, s) + u, \quad (4)$$

$$t'(s) = \frac{1}{\sqrt{2E(s)}}, \quad (5)$$

$$\underline{g}(s) \leq E(s) \leq \bar{g}(s), \quad (6)$$

$$E(s_0) = v_0^2/2, \quad E(s_f) = v_f^2/2, \quad (7)$$

$$u_{\min}(s) \leq u(s) \leq u_{\max}(s), \quad (8)$$

$$t(s_0) = 0, \quad t(s_f) = t_f, \quad (9)$$

where $\bar{g}(s) = v_{\max}^2(s)/2$ and $\underline{g}(s) = v_{\min}^2(s)/2$ are bounds on the specific kinetic energy \bar{E} . It is assumed that \bar{g} and \underline{g} are piecewise differentiable on $[s_0, s_f]$.

III. MINIMUM-TIME PATH TRACKING

The cost function for the minimum-time problem is

$$J(t_f, E, u) = t_f = \int_{s_0}^{s_f} \frac{ds}{\sqrt{2E(s)}}.$$

When the state constraints (6) are not active, the Hamiltonian of the optimal control Problem 1 is

$$H = \frac{1}{\sqrt{2E}}(\lambda_t + 1) + \lambda_E(-D(E, s) + u),$$

where λ_t and λ_E are the costate variables for the t and E dynamics, respectively. The costate dynamics are given by

$$\lambda'_t = -\frac{\partial H}{\partial t} = 0, \quad (10)$$

$$\lambda'_E = -\frac{\partial H}{\partial E} = \frac{1}{2\sqrt{2}}E^{-3/2}(\lambda_t + 1) + \lambda_E \frac{\partial D(E, s)}{\partial E}. \quad (11)$$

It follows that λ_t is constant. Since t_f is free, $\lambda_t \equiv 0$ according to the transversality condition. The switching function is

$$\frac{\partial H}{\partial u} = \lambda_E.$$

According to the Pontryagin's Maximum Principle (PMP), in general, the optimal control u^* may contain bang-bang control, singular control, and control arcs associated with active state constraints, as described by the following expression

$$u^*(s) = \begin{cases} u_{\min}, & \text{for } \lambda_E > 0, s \in [s_0, s_f] \setminus \mathcal{K}, \\ \tilde{u}(s), & \text{for } \lambda_E = 0, s \in [s_0, s_f] \setminus \mathcal{K}, \\ u_{\max}, & \text{for } \lambda_E < 0, s \in [s_0, s_f] \setminus \mathcal{K}, \\ u_u(s), & \text{for } s \in \mathcal{K}_U, \\ u_l(s), & \text{for } s \in \mathcal{K}_L. \end{cases} \quad (12)$$

where \tilde{u} is the singular control, $\mathcal{K}_U = \{s \in [s_0, s_f] | E^*(s) = \bar{g}(s)\}$, $\mathcal{K}_L = \{s \in [s_0, s_f] | E^*(s) = \underline{g}(s)\}$, and $\mathcal{K} = \mathcal{K}_U \cup \mathcal{K}_L$. At those points where \bar{g} (respectively, \underline{g}) is differentiable,

$$u_u(s) = \bar{g}'(s) + D(\bar{g}(s), s) \quad (13)$$

and

$$u_l(s) = \underline{g}'(s) + D(\underline{g}(s), s). \quad (14)$$

At the points where \bar{g} (respectively, \underline{g}) is discontinuous and/or non-differentiable, the left and right limits of u_u and u_l can be defined similarly.

Proposition 1: The time-optimal control solution does not contain any singular control.

Proof: It is sufficient to show that there does not exist any sub-interval $[s_a, s_b] \subseteq [s_0, s_f]$ on which $\lambda_E(s) \equiv 0$ and $\underline{g}_l(s) < E(s) < \bar{g}_u(s)$ (strict inequalities) for all $s \in [s_a, s_b]$. Suppose, ad absurdum, that $\lambda_E(s) \equiv \lambda'_E(s) \equiv 0$ for all $s \in [s_a, s_b]$, and the state constraints are not active on $[s_a, s_b]$. It follows that on $[s_a, s_b]$, equation (11) yields $0 = E^{-3/2}/2\sqrt{2} > 0$, which is impossible. Hence λ_E cannot remain constantly zero on any interval, and the proof is complete. ■

Proposition 2: The optimal control $u^*(s)$ is bang-bang, and does not contain any switch from u_{\min} to u_{\max} on $[s_0, s_f] \setminus \mathcal{K}$.

Proposition 3: Assume $\bar{g}(s) \neq \underline{g}(s)$ and $u_l(s) < u_{\max}(s)$ for all $s \in [s_0, s_f]$. Let $E^*(s)$ be the optimal kinetic energy solution to the min-time problem. Then the set \mathcal{K}_L does not contain any nontrivial interval.

The proofs of Proposition 2 and Proposition 3 can be found in [10].

Corollary 1: The time optimal control u^* can be constructed as a combination of u_{\max} , u_{\min} and u_u .

Based on the theoretical results in this section, an efficient algorithm has been proposed in [14] to solve Problem 1. This algorithm can be modified to also provide the maximum t_f along a given geometric path. The details are omitted due to the page limitations. As will be shown later in this paper, the maximum time solution, although not very useful in practice, is important for constructing minimum energy solutions.

IV. MINIMUM-ENERGY PATH TRACKING

Next, we consider the minimization of the energy consumed for tracking the path:

Problem 2 (Minimum-energy path tracking):

$$\min J(t_f, v, u) = \int_0^{t_f} v(t)u(t) dt = \int_{s_0}^{s_f} u(s) ds \quad (15)$$

subject to the same constraints as in Problem 1.

Without loss of generality, we assume that t_f is fixed. Note that $v(t)u(t)$ represents the power input to the system.

A. Optimality Conditions

Consider first the case when the state constraints (6) are not active. Then the Hamiltonian is

$$H = (\lambda_E + 1)u + \frac{\lambda_t}{\sqrt{2E}} - \lambda_E D(E, s).$$

The costate dynamics are

$$\lambda'_t = 0, \quad \lambda'_E = \frac{1}{2\sqrt{2}}E^{-3/2}\lambda_t + \lambda_E \frac{\partial D(E, s)}{\partial E}. \quad (16)$$

Therefore, the costate λ_t is constant. The switching function is $\lambda_E + 1$. By PMP, the extremal control is given by

$$u = \begin{cases} u_{\max}, & 1 + \lambda_E < 0, \\ \tilde{u}, & 1 + \lambda_E = 0, \\ u_{\min}, & 1 + \lambda_E > 0, \end{cases} \quad (17)$$

where \tilde{u} is the singular control. Suppose that the optimal specific kinetic energy E^* contains a singular arc represented by \tilde{E} , i.e., $E^*(s) = \tilde{E}(s)$ on some subinterval of $[s_0, s_f]$. For notational convenience, let us denote

$$\frac{\partial^k D(\tilde{E}, s)}{\partial E^k} = \frac{\partial^k \tilde{D}}{\partial E^k}, \quad k = 1, 2,$$

and let λ_t^* be the optimal costate value. Since the switching function is identically zero along the singular arc, its derivative must also vanish, which yields $(\lambda_E = -1)$

$$\frac{d}{ds} \left(\frac{\partial H}{\partial u} \right) = -\frac{\partial \tilde{D}}{\partial E} + \frac{1}{2\sqrt{2}}\tilde{E}^{-3/2}\lambda_t \equiv 0, \quad (18)$$

from which the singular specific kinetic energy profile can be computed. For notational convenience, equation (18) is rewritten as

$$P(\tilde{E}(s), s) = \lambda_t^*, \quad (19)$$

where, for any $E > 0$,

$$P(E, s) = 2\sqrt{2}E^{3/2} \frac{\partial D}{\partial E} \Big|_{(E(s), s)}. \quad (20)$$

Proposition 4: Let $E^*(s)$ be the optimal specific kinetic energy profile for the energy-optimal problem with corresponding optimal costate value λ_t^* . Let the function \tilde{E} be defined as in (19). Then, for all s , $P(E^*(s), s) > \lambda_t^*$ if and only if $E^*(s) > \tilde{E}(s)$, and $P(E^*(s), s) < \lambda_t^*$ if and only if $E^*(s) < \tilde{E}(s)$.

Proof: Note that

$$\frac{\partial}{\partial E} \left(E^{3/2} \frac{\partial D}{\partial E} \right) = E^{3/2} \left(\frac{\partial^2 D}{\partial E^2} + \frac{3}{2E} \frac{\partial D}{\partial E} \right) > 0,$$

which is positive according to Assumption 1. Therefore, $E^{3/2}(\partial D/\partial E)$ increases monotonically with respect to E for any fixed $s \in [s_0, s_f]$. The following expression holds from the definition of P and λ_t^* :

$$P(E^*(s), s) - \lambda_t^* = 2\sqrt{2} \left(E^{3/2} \frac{\partial D(E^*, s)}{\partial E} - \tilde{E}^{3/2} \frac{\partial \tilde{D}}{\partial E} \right),$$

and the claim of this proposition follows from the monotonicity of $E^{3/2}(\partial D/\partial E)$ with respect to E . ■

With $E^*(s)$, λ_t^* and $\tilde{E}(s)$ as in Proposition 4, the singular control \tilde{u} can be obtained by

$$\tilde{u}(s) = \tilde{E}'(s) + D(\tilde{E}(s), s). \quad (21)$$

According to the PMP, when t_f is free, we have $\lambda_t^* = 0$ following the transversality condition at t_f . When t_f is fixed, we need to first calculate the optimal value of λ_t^* .

B. Optimality of the Singular Arcs

An admissible singular control $\tilde{u}(s)$, in addition to the constraint $u_{\min} \leq \tilde{u}(s) \leq u_{\max}$, must satisfy the generalized Legendre-Clebsch condition [2]

$$\frac{\partial}{\partial u} \left[\frac{d^2}{ds^2} \left(\frac{\partial H}{\partial u} \right) \right] \leq 0, \quad (22)$$

if it is to be part of the optimal trajectory. Along the singular arc, differentiating the Hamiltonian with respect to s , one obtains

$$\frac{d^2}{ds^2} \left(\frac{\partial H}{\partial u} \right) = \lambda_E' \frac{\partial \tilde{D}}{\partial E} + \left(\lambda_E \frac{\partial^2 \tilde{D}}{\partial E^2} - \frac{3}{4\sqrt{2}} \tilde{E}^{-5/2} \lambda_t^* \right) \tilde{E}'.$$

Using (4) and the fact that $\lambda_E = -1$ along the singular arc, it follows that

$$\frac{\partial}{\partial u} \left[\frac{d^2}{ds^2} \left(\frac{\partial H}{\partial u} \right) \right] = -\frac{\partial^2 \tilde{D}}{\partial E^2} - \frac{3}{4\sqrt{2}} \tilde{E}^{-5/2} \lambda_t^*. \quad (23)$$

Since $\tilde{E}(s)$ satisfies (19), it follows that

$$\lambda_t^* = 2\sqrt{2} \tilde{E}^{3/2} \frac{\partial \tilde{D}}{\partial E}, \quad (24)$$

By eliminating λ_t^* from (23), and by using equation (19), equation (23) can be written as:

$$\frac{\partial}{\partial u} \left[\frac{d^2}{ds^2} \left(\frac{\partial H}{\partial u} \right) \right] = -\frac{\partial^2 \tilde{D}}{\partial E^2} - \frac{3}{2\tilde{E}} \frac{\partial \tilde{D}}{\partial E}, \quad (25)$$

which is negative by Assumption 1. Hence, along the singular arcs, the generalized Legendre-Clebsch condition is satisfied when Assumption 1 is valid, in which case these arcs can be part of the optimal trajectory.

C. Optimal Switching Structure Involving Singular Arcs

When solving an optimal control problem, it is a common practice to assume a certain fixed switching structure. This approach, although convenient, may lead to a suboptimal solution. According to the following theorem, we can actually identify the optimal switching structure for the energy-optimal path tracking problem.

Theorem 1: Let $E^*(s)$ be the energy-optimal specific kinetic energy profile, let λ_t^* be the optimal costate value, and let $\tilde{E} : [s_0, s_f] \rightarrow \mathbb{R}_+$ be the function defined by $P(\tilde{E}(s), s) = \lambda_t^*$. Consider a subinterval $(s_a, s_b) \subset [s_0, s_f]$ such that $\underline{g}(s) < E^*(s) < \bar{g}(s)$ for all $s \in (s_a, s_b)$. If $E^*(s) < \tilde{E}(s)$ (respectively, $E^*(s) > \tilde{E}(s)$) for all $s \in (s_a, s_b) \subset [s_0, s_f]$, then the corresponding optimal control $u^*(s)$ does not contain any switching from u_{\min} to u_{\max} (respectively, u_{\max} to u_{\min}) on (s_a, s_b) .

Proof: Assume that $E^*(s) < \tilde{E}(s)$ for all $s \in (s_a, s_b)$, and assume $u^*(s) = u_{\min}$ on (s_a, τ) and $u^*(s) = u_{\max}$ on (τ, s_b) , where $\tau \in (s_a, s_b)$ is the switching point from u_{\min} to u_{\max} . Because the state constraints are not saturated on (s_a, s_b) , the optimal costate λ_E^* is continuous on (s_a, s_b) . Since $u^*(s) = u_{\min}$ on (s_a, τ) , and $u^*(s) = u_{\max}$ on (τ, s_b) , it follows that $1 + \lambda_E^*(s) > 0$ on (s_a, τ) and $1 + \lambda_E^*(s) < 0$ on (τ, s_b) according to (17), and $\lambda_E^*(\tau) = -1$ by the continuity of λ_E^* .

According to equation (16), the derivative of the costate at τ is given by

$$\begin{aligned} \lambda_E'^*(\tau) &= \lambda_E^*(\tau) \frac{\partial D(E^*, \tau)}{\partial E} + \frac{1}{2\sqrt{2}} E^{*-3/2}(\tau) \lambda_t^* \\ &= -\frac{1}{2\sqrt{2}} (E^*)^{-3/2}(\tau) (P(E^*(\tau), \tau) - \lambda_t^*), \end{aligned}$$

where (19) and (20) are used for the derivation. Following Proposition 4, $\lambda_E'^*(\tau) > 0$ since the above expression is positive when $E^*(\tau) < \tilde{E}(\tau)$. Since $\partial D/\partial E$ is continuous with respect to s , $\lambda_E'^*(s)$ is also continuous with respect to s . Hence, $\lambda_E'^*(s) > 0$ in a neighborhood of τ . However, this implies that given $1 + \lambda_E^*(s) > 0$ on (s_a, τ) , there exists $\epsilon > 0$ such that $1 + \lambda_E^*(s) > 0$ for all $s \in (\tau, \epsilon) \subseteq (\tau, s_b)$, which is a contradiction to the fact that $1 + \lambda_E^*(s) < 0$ on (τ, s_b) . Therefore, if $E^*(s) < \tilde{E}(s)$ the optimal thrust contains no switch from u_{\min} to u_{\max} on (s_a, s_b) . The proof for the case $E^*(s) > \tilde{E}(s)$ is similar, and hence it is omitted. ■

Theorem 1 narrows down the possible switching combinations of the optimal control u^* for the energy-optimal problem. The valid switching structures are illustrated in Fig. 1(a). In contrast, the switching structures in Fig. 1(b) are not optimal.

D. State Constraints and the Relaxed Problem

When either the upper or lower bound of the state constraint (6) is active along a certain part of the optimal specific kinetic energy solution E^* , this part of E^* we have a *state constrained arc*. For the corresponding *state*

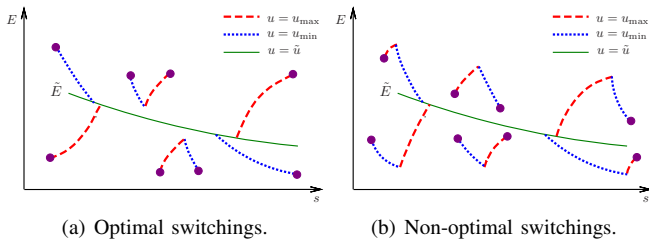


Fig. 1. Possible switching structures.

constrained control it is necessary to identify the intervals on which state constraints (6) are active, which is usually not straightforward.

In this section, we formulate a relaxed version of Problem 2 by partially relaxing the state constraints (6) on certain intervals. The optimal solution to this relaxed problem can be determined in a semi-analytic way, and will be used in the proof regarding the optimal solution to Problem 2.

Before introducing the relaxed problem, some additional notation needs to be presented first. For $\Gamma_U \subseteq [s_0, s_f]$, define

$$\bar{g}_{\Gamma_U}(s) = \begin{cases} \bar{g}(s), & s \in \Gamma_U, \\ M, & s \in [s_0, s_f] \setminus \Gamma_U, \end{cases}$$

where $M > 0$ is a number large enough such that $E(s) < M$ is always satisfied on $[s_0, s_f]$ by any feasible specific kinetic energy profile $E(s)$. By choosing a subset Γ_U of interest and enforcing the state constraint $E(s) \leq \bar{g}_{\Gamma_U}(s)$ for all $s \in [s_0, s_f]$, it can be ensured that the optimal solution E^* satisfies $E^*(s) \leq \bar{g}(s)$ on Γ_U , while remaining unconstrained on $[s_0, s_f] \setminus \Gamma_U$. Similarly, also define

$$\underline{g}_{\Gamma_L}(s) = \begin{cases} \underline{g}(s), & s \in \Gamma_L, \\ 0, & s \in [s_0, s_f] \setminus \Gamma_L. \end{cases}$$

By enforcing the constraint $E(s) \geq \underline{g}_{\Gamma_L}(s)$ instead of the constraint $E(s) \geq \underline{g}(s)$, the later constraint is relaxed on $[s_0, s_f] \setminus \Gamma_L$. Next, a modified version for Problem 2 is introduced by relaxing the original state constraints (6) on certain subintervals.

Problem 3 (Relaxed Min-Energy Path Tracking Problem): Let $\Gamma_U, \Gamma_L \subseteq [s_0, s_f]$. Minimize the energy cost (15) subject to constraints (4), (5), (7), (8), (9), and the state bounds

$$E(s) - \bar{g}_{\Gamma_U}(s) \leq 0, \quad \underline{g}_{\Gamma_L}(s) - E(s) \leq 0. \quad (26)$$

for all $s \in [s_0, s_f]$.

Similarly, one can form the relaxed minimum-time and the relaxed maximum-time path tracking problem with state constraints (26) instead of (6). For the sake of brevity, the formal definitions for these problems are not presented here since they are self-evident from the definition of Problem 3.

Since the unconstrained solution to an optimal control problem has the same, or better, optimality characteristics than a constrained one, a constraint is, in general, not active unless it is violated by the optimal solution of the unconstrained problem¹. This property is stated formally by the next lemma.

¹The only exception is the trivial case when along the unconstrained optimal solution certain constraints are active but not violated.

Lemma 1: If the optimal solution of Problem 3 does not violate constraints (6), then it is also an optimal solution for Problem 2.

E. The Optimal Switching Structure Involving State-Constrained Arcs

For an arbitrary geometric path, the energy-optimal control u^* for the minimum energy path tracking problem is composed of bang-bang control u_{\min} and u_{\max} , singular control \tilde{u} , and state constrained control u_u and u_l arcs.

Lemma 2: Let $E_U^*(s)$ be the minimum-time path-following specific kinetic energy profile with flight time t_{\min} , and let $E_L^*(s)$ be the maximum-time path-following specific kinetic energy profile with time t_{\max} , subject to the same boundary conditions and state constraints as in (4)-(8). Let $E^*(s)$ be the optimal specific kinetic energy profile for the minimum-energy path-following problem with fixed time t_f . Then the following inequalities hold

$$t_{\min} \leq t_f \leq t_{\max}, \quad E_L^*(s) \leq E^*(s) \leq E_U^*(s), \quad (27)$$

for all $s \in [s_0, s_f]$.

Proof: See [10]. ■

According to Lemma 2, the fixed-time, energy-optimal specific kinetic energy E^* is bounded by the minimum-time solution E_U^* and the maximum-time solution E_L^* . Furthermore, based on Theorem 1, it can be shown that $E^*(s) = E_U^*(s)$ or $E^*(s) = E_L^*(s)$ on certain subintervals. This property of E^* is characterized by the following Lemma.

Lemma 3: Let $E^*(s)$ be the optimal specific kinetic energy solution to Problem 2 and let \tilde{E} be defined on $[s_0, s_f]$ by $P(\tilde{E}(s), s) = \lambda_t^*$, where λ_t^* is the corresponding optimal costate value. Let $E_U^*(s)$ and $E_L^*(s)$ be the optimal specific kinetic energy solutions to the minimum-time and maximum-time path-tracking problems, respectively. Furthermore, let

$$\Gamma_U = \{s | E_U^*(s) < \tilde{E}(s), s \in [s_0, s_f]\} \quad (28)$$

$$\Gamma_L = \{s | E_L^*(s) > \tilde{E}(s), s \in [s_0, s_f]\} \quad (29)$$

and suppose that $E^*(s) > \underline{g}(s)$ for all $s \in [s_0, s_f] \setminus \Gamma_L$, and $E^*(s) < \bar{g}(s)$ for all $s \in [s_0, s_f] \setminus \Gamma_U$. Then $E^*(s) = E_U^*(s)$ for all $s \in \Gamma_U$, and $E^*(s) = E_L^*(s)$ for all $s \in \Gamma_L$.

Proof: First, it will be shown that $E^*(s) = E_U^*(s)$ for all $s \in \Gamma_U$. Let u_U^* and u^* be the thrust control associated with E_U^* and E^* , respectively. From Lemma 2, it follows that $E^*(s) \leq E_U^*(s)$ for all $s \in [s_0, s_f]$. Assume, on the contrary, that there exists $\tau \in \Gamma_U$ such that $E^*(\tau) < E_U^*(\tau)$. Then by the definition of Γ_U , $E^*(\tau) < \tilde{E}(\tau)$. Let $q = \inf\{s | E^*(s) = E_U^*(s), s \in [\tau, s_f]\}$. Since $E^*(s_f) = E_U^*(s_f)$, q is well-defined. Similarly, let $p = \sup\{s | E^*(s) = E_U^*(s), s \in [s_0, \tau]\}$ and since $E^*(s_0) = E_U^*(s_0)$, p is also well-defined. Note that $E^*(s) < E_U^*(s)$ for all $s \in (p, q)$ by the fact $E^*(\tau) < E_U^*(\tau)$, the definitions of p, q , and the continuity of E^* and E_U^* (see Fig. 2). Since $E^*(s) < E_U^*(s) \leq \bar{g}_w(s)$ for all $s \in (p, q)$, the upper bound of the state constraint (6) is inactive along E^* for $s \in (p, q)$. Hence, $u^*(s)$ can only take the values of u_{\max} , u_{\min} , $\tilde{u}(s)$, or $\underline{u}_w(s)$ on (p, q) . Since $E^*(\tau) < \tilde{E}(\tau)$, it is true that $u^*(\tau) \neq \tilde{u}(\tau)$. Also, since $E^*(\tau) < \tilde{E}(\tau)$, it follows that $\tau \notin \Gamma_L$, and therefore $E^*(\tau) > \underline{g}_w(\tau)$, and it follows that either $u^*(\tau) = u_{\max}$ or

$u^*(\tau) = u_{\min}$. Next, it will be shown that neither of these two options is possible.

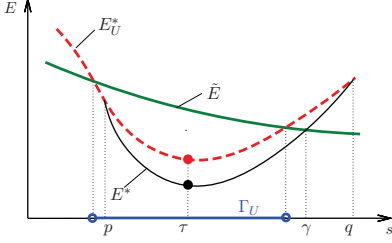


Fig. 2. Illustration for the proof of Lemma 3

First, consider the case $u^*(\tau) = u_{\min}$. It is claimed that $E^*(s) < \tilde{E}(s)$ for all $s \in (\tau, q)$. To see this, assume that $E^*(s) \geq \tilde{E}(s)$ for some $s \in (\tau, q)$. It then follows from the fact $E^*(\tau) < \tilde{E}(\tau)$ and the continuity of E^* and \tilde{E} that the equation $E^*(\gamma) = \tilde{E}(\gamma)$ has at least one solution on (τ, q) (see Fig. 3). Let $\gamma = \inf\{s | E^*(s) = \tilde{E}(s), s \in (\tau, q)\}$. It follows that $E^*(\gamma) = \tilde{E}(\gamma)$, and $E^*(s) < \tilde{E}(s)$ for all $s \in (\tau, \gamma)$. Therefore, $(\tau, \gamma) \subseteq [s_0, s_f] \setminus \Gamma_L$, and it is true that $E^*(s) > g_w(s)$ for all $s \in (\tau, \gamma)$. It follows that on (τ, γ) , $u^*(s)$ can only take the values of u_{\min} and u_{\max} . Since $E^*(s) < \tilde{E}(s)$ for all $s \in (\tau, \gamma)$, $u^*(s)$ cannot switch from u_{\min} to u_{\max} according to Theorem 1, and $u^*(s) = u_{\min}$ for all $s \in [\tau, \gamma)$. The trajectories $E^*(s)$ and $\tilde{E}(s)$ on (τ, γ) can be computed starting from $E^*(\gamma) = \tilde{E}(\gamma)$ at $s = \gamma$ by integrating backwards (4) with $u^*(s) = u_{\min}$ and \tilde{u} , respectively. Since $u_{\min} \leq \tilde{u}(s)$, a straightforward application of the Comparison Lemma [4] yields that $E^*(\tau) \geq \tilde{E}(\tau)$, leading to a contradiction. Hence $E^*(s) < \tilde{E}(s)$ for all $s \in (\tau, q)$, and thus $u^*(s) = u_{\min}$ for all $s \in (\tau, q)$ according to Theorem 1. The last statement implies however that one can compute $E^*(\tau)$ and $E_U^*(\tau)$ on the interval (τ, q) starting at $s = q$ with initial conditions $E^*(q) = E_U^*(q)$ and integrating backwards (4) using $u^*(s) = u_{\min}$ and $u_U^*(s)$, respectively, for all $s \in (\tau, q)$. Since $u_U^*(s) \geq u_{\min} = u^*(s)$, an application of the Comparison Lemma as before yields that $E^*(\tau) \geq E_U^*(\tau)$, which contradicts the assumption $E^*(\tau) < E_U^*(\tau)$.

Similarly, if $u^*(\tau) = u_{\max}$, one can prove in a similar manner that $E^*(\tau) < E_U^*(\tau)$ is also impossible. Hence, there does not exist $\tau \in \Gamma_U$ such that $E^*(\tau) < E_U^*(\tau)$, and thus it must be true that $E^*(s) = E_U^*(s)$ on Γ_U .

The proof of the other statement, namely, $E^*(s) = E_L^*(s)$ for all $s \in \Gamma_L$, is similar, hence, is omitted. ■

Lemma 3, along with Lemma 1, is used to characterize the state constrained arcs in the optimal specific kinetic energy profile $E^*(s)$. Specifically, given the state constraints, one needs first to compute the optimal solution of a certain relaxed problem in order to identify the state constrained arcs. Subsequently, the solution of the relaxed (non-constrained) problem can be used to construct the solution of the original problem with state constraints.

Typically, the relaxation of constraints will affect the optimal solution. However, as shown by the following proposition, by choosing carefully where the constraints are relaxed, the minimum-time and maximum-time solutions do

not change on certain subintervals after the relaxation of constraints.

Proposition 5: Let \tilde{E} be defined by $P(\tilde{E}(s), s) = \lambda_t$ for a certain costate value λ_t such that $\tilde{u} \in [u_{\min}, u_{\max}]$, where \tilde{u} is given by (21). Let Γ_U and Γ_L as in (28) and (29), where $E_U^*(s)$ and $E_L^*(s)$ are the specific kinetic energy solutions to the minimum-time and maximum-time path-tracking problems, respectively, with constraints (6). Let $E_{U_r}^*(s)$ and $E_{L_r}^*(s)$ be the specific kinetic energy solutions to the relaxed minimum-time and maximum-time path-tracking problems, respectively, with constraints $E(s) \leq \bar{g}_{\Gamma_U}(s)$ and $E(s) \geq \underline{g}_{\Gamma_L}(s)$ instead of (6). Then $E_U^*(s) = E_{U_r}^*(s)$ for all $s \in \Gamma_U$, and $E_L^*(s) = E_{L_r}^*(s)$ for all $s \in \Gamma_L$.

Proof: See [10]. ■

F. The Optimal Specific Kinetic Energy Solution

In this section, the optimal solution to Problem 2 is given by Theorem 2 below. The proof of the theorem takes advantage of the optimal solution of the relaxed Problem 3 given in the previous section. First, the optimal solution to the relaxed Problem 3 is characterized with the state constraints relaxed on some carefully selected subintervals. Then it is shown that this solution satisfies the state constraints in Problem 2, hence is also the optimal solution to Problem 2. The optimal solution to Problem 2 is a combination of the minimum-time solution, the maximum-time solution, and energy-saving singular arcs. The detailed proof of this fact is rather involved, hence, is omitted. The interested reader is referred to [10] for the proof.

Theorem 2: Suppose there exists a real number λ_t and a function \tilde{E} given by $P(\tilde{E}(s), s) = \lambda_t$ for all $s \in [s_0, s_f]$, such that the specific kinetic energy E^* given by

$$E^*(s) = \begin{cases} E_L^*(s), & s \in \Gamma_L, \\ \tilde{E}(s), & s \in [s_0, s_f] \setminus (\Gamma_U \cup \Gamma_L), \\ E_U^*(s), & s \in \Gamma_U \end{cases} \quad (30)$$

satisfies the desired total tracking time, where $\Gamma_U = \{s | E_U^*(s) < \tilde{E}(s), s \in [s_0, s_f]\}$, and $\Gamma_L = \{s | E_L^*(s) > \tilde{E}(s), s \in [s_0, s_f]\}$. Then E^* is the optimal solution to Problem 2.

Proof: See [10]. ■

Despite the simple form of the energy-optimal solution in (30), one is not readily able to choose the correct value of $\tilde{E}(s)$ for each $s \in [s_0, s_f]$ in order to construct the optimal specific kinetic energy according to (30) because the optimal costate value λ_t^* is unknown.

To identify the correct λ_t^* value and the associated singular arcs for a specific total tracking time, a numerical algorithm has been introduced in Ref. [10]. This algorithm searches among a family of extremals for the correct value of λ_t^* . This allows the computation of the associated function $\tilde{E}(s)$ from (19) and, subsequently, the optimal solution $E^*(s)$ from (30). It has been shown in Ref. [10] that such an algorithm is guaranteed to converge to the optimal solution.

V. NUMERICAL EXAMPLE

We computed energy-optimal speed profiles for a fixed-wing aircraft tracking a landing path shown in Fig. 3.

A standard point mass aircraft model is used for all calculations [11]. The optimal speed profiles are shown in Fig. 4, which also illustrates the relation between minimum-time, maximum-time, and minimum-energy solutions with different t_f . The same problem was solved using a Nonlinear Programming solver [12]. The comparison of the optimal speed profiles are shown in Fig. 5. It is clear from these figures that the results are extremely close to the optimal ones. Furthermore, the Matlab implementation of the energy-optimal path-tracking control algorithm found the optimal solution in 3-6 seconds, while the Nonlinear Programming solver took at least 5 minutes (and for some cases much more) to find a converged optimal solution. See Ref. [13] for more details about this numerical example.

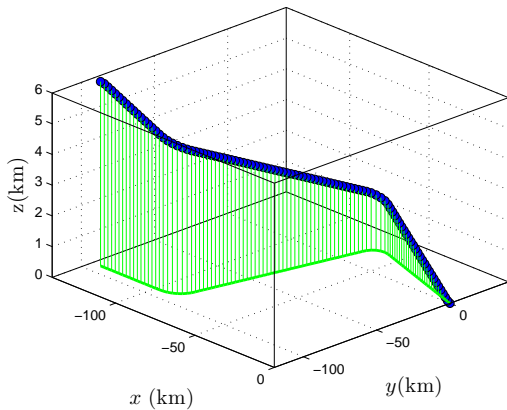


Fig. 3. 3D Geometric Path.

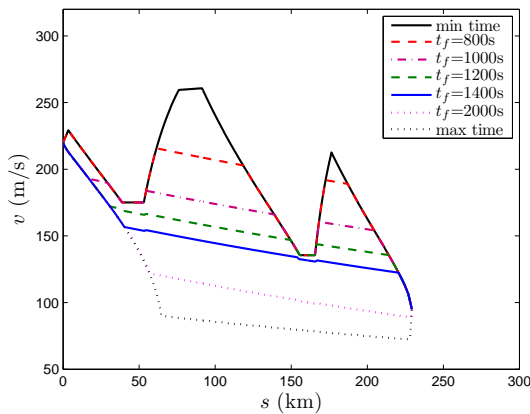


Fig. 4. Energy-optimal speed profiles with different t_f .

VI. CONCLUSION

In this paper, we have studied a speed optimization problem subject to path-dependent speed and control constraints, which is key to the path tracking problems for many industrial and transportation systems. The optimal switching structure in the minimum-time and minimum-energy solutions are analyzed based on optimal control theory. It is shown that the minimum-energy solution is a

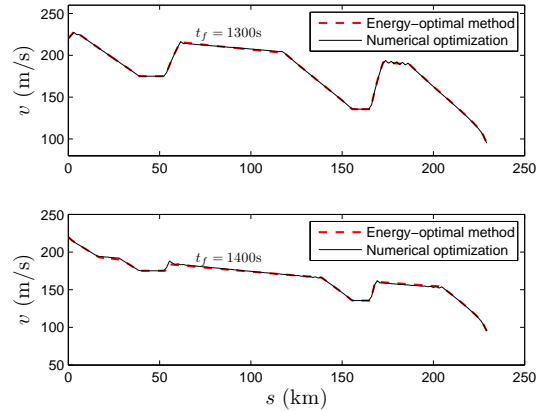


Fig. 5. Comparison with numerical optimization solver for $t_f = 1300s$ and $t_f = 1400s$.

concatenation of the minimum-time solution, the maximum-time solution, and singular arcs. Based on our analysis, an efficient algorithm has been proposed for computing the energy-optimal solution.

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