On the Existence and Synthesis of Curvature-Bounded Paths Inside Nonuniform Rectangular Channels

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***Abstract*— Motion planners for autonomous mobile vehicles that are based on rectangular cell decompositions are often required to construct kinematically feasible path – typically curvature-bounded paths – traversing rectangular channels. In this paper, we present a numerical algorithm for determining the existence of a curvature-bounded path contained within a rectangular channel. The rectangular cells comprising the channel are assumed to be of arbitrary, non-uniform dimensions and the bounds on curvature are allowed to be different for different cells. The proposed algorithm is based on the explicit construction of the cone of feasible directions for a bounded-curvature path at the cell exit edge, given the entry point for each cell in the channel. Based on this analysis, we devise a path construction scheme that retains the convenience of cell-by-cell path synthesis but eliminates the guesswork involved in choosing terminal conditions within each cell.**

I. Introduction

We propose an algorithm to construct paths of given bounded curvature inside a channel of (perhaps non-uniform) cells. In what follows, we refer to "feasible paths" to mean those paths satisfying a specified upper bound on the curvature. We assume the upper bound on the curvature to be specified *a priori*[1]. The work presented in this paper is intended to bridge the gap between the (high-level) geometric path planning and the (low-level) trajectory generation layers in the motion planning and control hierarchy [1].

The result of a path planning algorithm based on a cell decomposition of the environment is a finite sequence of obstacle-free rectangular cells – which we refer to as a rectangular channel – leading from the initial point to the destination. Such a channel is a convenient intermediate result for hierarchical motion planning, since trajectory planning can then be performed on a cell-by-cell basis.

However, there are two major caveats associated with this approach to motion planning: (a) there may not exist a feasible path within the given channel, and (b) within each cell, there is no systematic method of choosing the terminal conditions. Stated differently, there are two major issues that must be addressed before a rectangular cell decomposition can be meaningfully used in path planning algorithms for vehicles with kinematic and dynamic constraints: first, ensuring that a feasible path exists in any channel resulting from the higher level of planning, and second, constructing a feasible path within an arbitrary channel, if it exists.

In [2], we address the first issue by proposing a geometric path planning algorithm based on rectangular cell decomposition which can incorporate any information on the kinematic and dynamic limitations of the vehicle in the high-level planning itself. We proposed the use of histories of cells – finite length channels – and of costs associated with such histories to find an overall sequence of cells from the initial point to the destination. In the context of planning curvature-bounded paths, a natural candidate for such a cost function is the length of the shortest curvature-bounded path within the given history of cells. In this paper, we address the second issue, namely, that of planning curvature-bounded paths within a given rectangular channel. Since our ultimate objective is to use this trajectory generation scheme in conjunction with *multi-resolution* cell decompositions [3], [4] we assume channels comprised of rectangular cells of arbitrary dimensions.

Curvature-bounded path planning has been studied extensively, starting with the seminal work of Dubins [5]. Reference [6] reproduces Dubins' result using optimal control theory, while Ref. [7] presents a synthesis algorithm for the Dubins problem. The problem of planning curvature-bounded paths in environments *with obstacles* is far more involved. References [8] and [9] address, respectively, the questions of decidability of existence and of complexity of computation of shortest curvature- bounded paths in a polygonal environment with polygonal obstacles. References [10]–[12] present algorithms for efficiently finding approximations to shortest curvature-bounded paths in polygonal environments. The common theme in Refs. [10]–[12] is to appropriately choose a finite set of configurations (position and orientation) in the environment, and connecting these configurations by Dubins paths, and then perform a graph search using the length of the Dubins paths as edge costs. Reference [13] proposes a method to generate curvature-bounded paths contained within uniform width channels, but do not require the path to have any specific initial or final orientations. The reader is referred to Ref. [14] for a survey of motion planning and control for nonholonomic vehicles (of which the Dubins car is a specific example).

Our work explores the following novel ideas:

1) All of the above references (except [13]) assume that the initial and final configurations (position and orientation) are specified. In view of our application, we assume that the initial position and orientation is specified, but the terminal configuration is not specified, rather it is

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[1]In practice, such a may be determined by the vehicle dynamics and the environment; e.g., the upper bound may be determined by the friction coefficient at the tire/ground interface in the case of a ground vehicle.

constrained to lie inside a specified set.

2) We assume only a local upper bound on the curvature of allowable paths, that is, we allow different maximum curvature inside different cells.
3) We assume that the channel of rectangular cells through which we wish to find a curvature-bounded path is of arbitrary and non-uniform width.
4) We propose a cell-by-cell construction of feasible paths that eliminates the guesswork involved in choosing terminal conditions for planning within each cell.

The choice of terminal conditions is a crucial issue for cell-by-cell path construction, because an inappropriate or naïve choice may result in the planner erroneously reporting absence of a feasible path, and by consequence, may render the motion planning algorithm undesirably conservative.

II. Existence of Curvature-bounded Paths

In this section, we state precisely the problem of existence of a continuously differentiable curvature-bounded path in a rectangular channel[2]. In what follows, we denote a configuration in $\mathbb{R}^2 \times \mathbb{S}^1$ defined by an orientation $\theta \in [-\pi, \pi]$ at a point $P \in \mathbb{R}^2$ by (P, θ).

Problem 1: Let $\bar{\mathcal{R}}^C$ be a rectangular channel, and let W be a point on any of the three edges of R_1 which does not intersect R_2. Let $\alpha \in [-\pi, \pi]$ be a specified angle. For any set of positive real numbers $r_n > 0$, $n = 1, \ldots, C$, determine if there exists a path Π such that:

1) The path Π starts with the initial configuration (W, α),
2) The path Π terminates with a final configuration that lies in a set defined by the Cartesian product of a specified edge of the rectangle R_C (different from the edge coinciding with rectangle R_{C-1}) with a specified set of allowable terminal tangent angles,
3) The path Π does not leave $\bar{\mathcal{R}}^C$, i.e. $(x(s), y(s)) \in \cup_{n=1}^{C} R_n$ for every $s \in [0, 1]$,
4) The curvature of Π at any point in rectangle R_n is at most r_n^{-1}, for every $n = 1, \ldots, C$.

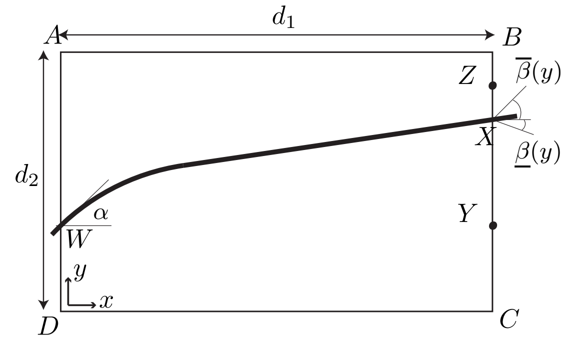
We propose a numerical algorithm to solve Problem 1 by recursively solving two basic problems defined on a single rectangle. We discuss the two basic problems below.

Let $ABCD$ be a rectangle. We attach a Cartesian axes system with origin at D, with positive x-axis along DC, and positive y-axis along DA, as shown in Fig. 1. Let the dimensions of the rectangle be d_1 and d_2.

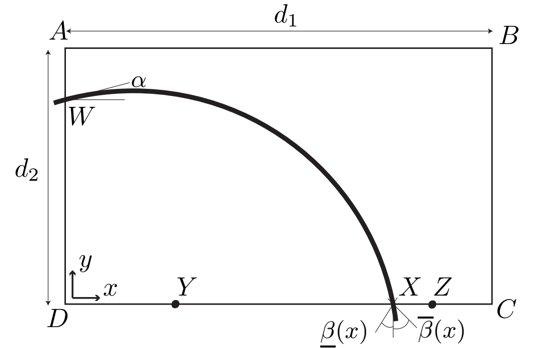
Definition 1: Let $\underline{\beta} : [0, d_2] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\overline{\beta} : [0, d_2] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be functions such that $\underline{\beta}(x) \leqslant \overline{\beta}(x)$, for every $x \in [0, d_2]$. Let $Y = (d_1, y), Z = (d_1, z)$ be points on the segment BC with $y \leqslant z$, and let $r > 0$ be fixed. A path Π is a *Type 1 admissible path* if it satisfies the following:

1) The curvature at any point on Π is at most r^{-1},
2) Π intersects the segment BC in exactly one point $X = (d_1, x)$ such that $x \in [y, z]$, and it may intersect segment AB and/or segment CD in at most one point each, and

[2] We define the term *rectangular channel* to mean a channel $\bar{\mathcal{R}}^C$ composed of a sequence $\{R_n\}$ of disjoint rectangles of arbitrary dimensions such that exactly one edge of R_n has a non-empty intersection with exactly one edge of R_{n+1}, for $n = 1, \ldots, C-1$ of arbitrary dimensions.



(a) Type 1 admissible path.



(b) Type 2 admissible path.

Fig. 1. Setup for the basic problems.

3) $\Pi'(X) \in \left[\underline{\beta}(x), \overline{\beta}(x)\right]$.

We may define a *Type 2 admissible path* analogously, for the case of traversal across adjacent edges (see Fig. 1(b)).

We refer to the closed interval $\left[\underline{\beta}(y), \overline{\beta}(y)\right]$ as a *terminal orientation cone* since the set of directions at point X corresponding the angles in this interval forms a closed pointed cone. We may now state the two basic problems as follows. Let $\underline{\beta}$ and $\overline{\beta}$ be functions and let Y and Z be points as in Definitions 1. Let $W = (0, w)$, $r > 0$, and $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be fixed.

Problem 2 (Traversal across parallel edges): Determine bounds $\underline{\alpha}, \overline{\alpha}$ such that for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, there exists a Type 1 admissible path with initial configuration (W, α).

Problem 3 (Traversal across adjacent edges): Determine bounds $\underline{\alpha}, \overline{\alpha}$ such that for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, there exists a Type 2 admissible path with initial configuration (W, α).

A. Recursive Analysis for Terminal Cones

Before discussing the solutions to Problems 2 and 3, which are based on detailed plane geometrical analysis, we present the overall algorithm that uses the solutions to these problems recursively to solve the main problem, i.e., Problem 1.

We attach a coordinate axes system to each rectangle of $\bar{\mathcal{R}}^C$ in a manner consistent with the axes system shown in Fig. 1. The dimensions of each rectangle along the x and y axes are denoted, respectively, as $d_{n,1}$ and $d_{n,2}$. For each rectangle R_n, $n = 2, 3, \ldots, C-1$, we refer to the segments formed by the intersections $R_{n-1} \cap R_n$ and $R_n \cap R_{n+1}$, respectively, as the *entry and exit segments*. For the rectangle

R_1 (resp. rectangle R_C), the entry segment (resp. exit segment) is specified arbitrarily. We denote the endpoints of the entry segment by U_n and V_n, and the endpoints of the exit segment by Y_n and Z_n. We specify the coordinates of the points U_n, V_n, Y_n, Z_n, by the corresponding lower case letters, i.e., $V_n = (0, v_n)$, etc.

For every point $Q = (q, 0)$ (or $Q = (d_{n,1}, q)$, as applicable), $q \in [y_n, z_n]$, on the segment Y_nZ_n, $n = 1, \ldots, C$, we denote the lower and upper boundaries of the terminal orientation cone by $\underline{\beta}_n(q)$ and $\overline{\beta}_n(q)$ respectively. Similarly, for every point $P = (0, p)$, $p \in [u_n, v_n]$, on the segment U_nV_n, $n = 1, \ldots, C$, we denote the upper and lower bounds resulting from the solution of Problem 2 (or Problem 3, as applicable), by $\underline{\alpha}_n(p)$ and $\overline{\alpha}_n(p)$ respectively. Note that the angles $\underline{\alpha}_n(\cdot), \overline{\alpha}_n(\cdot), \underline{\beta}_n(\cdot)$, and $\overline{\beta}_n(\cdot)$ are all measured w.r.t. the local coordinate axes system attached to R_n.

Finally, we denote by ϱ_n denote the number of reflection operations involved in the geometric transformation (consisting of rotation and reflection operations) required to align the entry and exit segments of R_n to the segments AD and BC, respectively, for traversal across parallel edges, or to segments AD and CD, respectively, for traversal across adjacent edges. The recursive analysis for determining the existence of curvature-bounded paths in rectangular channels is then described using the above notation as follows.

$\overline{\alpha}_{C+1}(q) \leftarrow \frac{\pi}{2}, \underline{\alpha}_{C+1}(q) = -\frac{\pi}{2}$, $q \in [0, d_{C,2}]$ for traversal across parallel edges of R_C, or $q \in [0, d_{C,1}]$ for traversal across adjacent edges of R_C
$\varrho_{C+1} \leftarrow 0$
for $n = C$ to 1 **do**
  if $\varrho_n + \varrho_{n+1}$ even **then**
    $\underline{\beta}_n \leftarrow \underline{\alpha}_{n+1}$, $\overline{\beta}_n \leftarrow \overline{\alpha}_{n+1}$
  else
    $\underline{\beta}_n(q) \leftarrow -\overline{\alpha}_{n+1}(y_n - (q - v_{n+1}))$, $q \in [y_n, z_n]$
    $\overline{\beta}_n(q) \leftarrow -\underline{\alpha}_{n+1}(y_n - (q - v_{n+1}))$, $q \in [y_n, z_n]$
  end if
  $\underline{\alpha}_n(q), \overline{\alpha}_n(q) \leftarrow$ solution of Problem 2 or Problem 3, as applicable to R_n for $q \in [u_n, v_n]$
end for

In light of the preceding algorithm, we focus on the solutions of Problem 2 and 3. Owing to the lack of space, we discuss the solution of Problem 2, and we refer the reader to Ref. [15] for the (analogous) solution of Problem 3.

III. Traversal across Parallel Edges

Recall that Problem 2 is of finding lower and upper bounds $\underline{\alpha}$ and $\overline{\alpha}$ such that, if $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, then there exists a Type 1 admissible path Π with initial configuration (W, α). To solve this problem, we construct, for every point $X = (d_1, x)$ on the exit segment YZ, a family of Type 1 admissible paths Υ_x and Λ_x between the points W and X which satisfy the following properties:

(P1) $\Lambda'_x(W) \leqslant \Upsilon'_x(W)$,
(P2) For all $\alpha \in [\Lambda'_x(W), \Upsilon'_x(W)]$, there exists a Type 1 admissible path from (W, α) to X,
(P3) $\Upsilon'_x(W)$ and $\Lambda'_x(W)$ vary continuously with x,
(P4) There exists no other Type 1 admissible path Π from W to X with $\Pi'(W) > \Upsilon'_x(W)$ that satisfies (P1)-(P3),
(P5) There exists no other Type 1 admissible path Π from W to X with $\Pi'(W) < \Lambda'_x(W)$ that satisfies (P1)-(P3).

Proposition 2: Let $\mathcal{I} \subset [y, z]$ be any closed interval such that Type 1 admissible paths Υ_x and Λ_x satisfying (P1)-(P5) exist for each $x \in \mathcal{I}$. Then $\overline{\alpha}$ and $\underline{\alpha}$ defined as

$$\underline{\alpha} \stackrel{\text{def}}{=} \min_{x \in \mathcal{I}} \{\Lambda'_x(W)\}, \qquad \overline{\alpha} \stackrel{\text{def}}{=} \max_{x \in \mathcal{I}} \{\Upsilon'_x(W)\},$$

solve Problem 2.

Proof: (*sketch*) Since $\mathcal{I}$ is closed and bounded, and $\Lambda'_x(W)$, $\Upsilon'_x(W)$ are continuous in x by (P3), $\underline{\alpha}$ and $\overline{\alpha}$ exist. It can be shown using (P3) that the collection of intervals $\{[\Lambda'_x(W), \Upsilon'_x(W)]\}_{x \in \mathcal{I}}$ covers the interval $[\underline{\alpha}, \overline{\alpha}]$. Hence, for all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, there exists $x_\alpha \in \mathcal{I}$ such that $\alpha \in \left[\Lambda'_{x_\alpha}(W), \Upsilon'_x(W)\right]$. It follows by (P2) that there exists a Type 1 admissible path Π between W and $X = (d_1, x_\alpha)$ such that $\Pi'(W) = \alpha$. Thus $\underline{\alpha}$ and $\overline{\alpha}$ solve Problem 2. ■

Proposition 2 allows us to transform Problem 2 to the problem of constructing paths Λ_x and Υ_x that satisfy properties (P1)-(P5). More specifically, we are interested in $\Lambda'_x(W)$ and $\Upsilon'_x(W)$. In what follows, we present a “cookbook” approach to computing $\Upsilon'_x(W)$ for all $x \in [y, z]$. Owing to the lack of space, we omit the details of the constructions of Υ_x and Λ_x (see Ref. [15] for these details). As we show at the end this section, $\Lambda'_x(W)$ can be computed with the same procedure used for computing $\Upsilon'_x(W)$ after an elementary geometric transformation. For the ease of analysis, we identify the following mutually exclusive cases.
Case 1: $0 < r < (d_1/4)$,
Case 2: $(d_1/4) \leqslant r < (d_1/2)$,
Case 3: $(d_1/2) \leqslant r$.

Depending on which of above conditions holds, we perform either of the following series of steps. For the sake of brevity, we denote the sine and cosine of an angle α by, respectively, $\mathsf{s}(\alpha)$ and $\mathsf{c}(\alpha)$; and the corresponding inverse trigonometric functions by, respectively, $\mathsf{as}(\alpha)$ and $\mathsf{ac}(\alpha)$. Also, we denote by $\mathfrak{S}(a, b)$ the function $\sqrt{a^2 - b^2}$, by $\mathfrak{D}$ the expression $((x - w)^2 + d_1^2)/2r^2$, by $\mathfrak{R}_1$ the expression $(x - w)/r$, by $\mathfrak{R}_2$ the expression d_1/r, and by $\mathfrak{R}_3$ the expression $(d_2 - w)/r$.

Steps for Case 1
Define the function $\alpha^* : [0, d_2] \to \left[0, \frac{\pi}{2}\right]$ as

$$\alpha^*(w) \quad \stackrel{\text{def}}{=} \quad \begin{cases} \frac{\pi}{2}, & \mathfrak{R}_3 > 1, \\ \mathsf{ac}(1 - \mathfrak{R}_3), & \mathfrak{R}_3 \leqslant 1. \end{cases} \tag{1}$$

The family of paths Υ_x satisfies $\Upsilon'_x(W) = \alpha^*(w)$ for all $x \in [y, z]$, *irrespective* of the values taken by $\underline{\beta}, \overline{\beta}$ on $[y, z]$. Identify $\mathcal{I} = [y, z]$, and $\overline{\alpha} = \max_{x \in \mathcal{I}} \{\Upsilon'_x(W)\} = \alpha^*(w)$.

For the sake of brevity, we denote $\alpha^*(w)$ by $\mathfrak{a}^*$.

Steps for Case 2
1) If $3r + r\mathsf{s}(\mathfrak{a}^*) < d_1$, then the family of paths Υ_x satisfies $\Upsilon'_x(W) = \mathfrak{a}^*$ for all $x \in [y, z]$, irrespective of the values taken by $\underline{\beta}, \overline{\beta}$ on $[y, z]$.

2) If $3r + r\mathsf{s}(\mathfrak{a}^*) \geqslant d_1$, then define points $M_1 = (d_1, m_1)$ and $M_2 = (d_2, m_2)$, where

$$m_1 \stackrel{\text{def}}{=} w - \mathfrak{S}(2r, (r\mathsf{s}(\mathfrak{a}^*) - d_1)) - r\mathsf{c}(\mathfrak{a}^*), \tag{2}$$

$$m_2 \stackrel{\text{def}}{=} w + \mathfrak{S}(2r, (r(1 + \mathsf{s}(\mathfrak{a}^*)) - d_1)) - r\mathsf{c}(\mathfrak{a}^*). \tag{3}$$

a) If $m_1 \geqslant y$ and/or if $m_2 \leqslant z$, then $\Upsilon'_x(W) = \mathfrak{a}^*$ for all $x \in [y, m_1] \cup [m_2, z]$.

b) For all $x \in (m_1, m_2) \cap [y, z]$, define the function $\beta^* : [0, d_2] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which takes value $\beta^*(x)$ for all $x \in [0, d_2]$ that solves the equation

$$(\mathsf{c}(\mathfrak{a}^*) + \mathfrak{R}_1)\,\mathsf{c}(\beta^*(x)) + (\mathsf{s}(\mathfrak{a}^*) - \mathfrak{R}_2)\,\mathsf{s}(\beta^*(x)) + \mathfrak{R}_1\mathsf{c}(\mathfrak{a}^*) - \mathfrak{R}_2\mathsf{s}(\mathfrak{a}^*) + \mathfrak{D} = 1. \tag{4}$$

(i) If $\beta^*(x) < \underline{\beta}(x)$, then $\Upsilon'_x(W)$ satisfies

$$\left(\mathsf{c}\left(\underline{\beta}(x)\right) + \mathfrak{R}_1\right)\mathsf{c}(\Upsilon'_x(W)) + \left(\mathsf{s}\left(\underline{\beta}(x)\right) - \mathfrak{R}_2\right)\mathsf{s}(\Upsilon'_x(W)) + \mathfrak{R}_1\mathsf{c}\left(\underline{\beta}(x)\right) - \mathfrak{R}_2\mathsf{s}\left(\underline{\beta}(x)\right) + \mathfrak{D} = 1. \tag{5}$$

(ii) If $\underline{\beta}(x) \leqslant \beta^*(x)$, then $\Upsilon'_x(W) = \mathfrak{a}^*$.

Since Υ_x exists for all $x \in [y, z]$, we identify $\mathcal{I} = [y, z]$.

Steps for Case 3

1) If $2\mathfrak{R}_3 \leqslant \mathfrak{R}_2^2 + \mathfrak{R}_3^2$ and $\mathfrak{S}(1, (1 - \mathfrak{R}_3)) < \mathfrak{R}_2 - 1$, then follow the same steps described for Case 2 above.

2) If $2\mathfrak{R}_3 \leqslant \mathfrak{R}_2^2 + \mathfrak{R}_3^2$ and $\mathfrak{S}(1, (1 - \mathfrak{R}_3)) \geqslant \mathfrak{R}_2 - 1$ or if $2\mathfrak{R}_3 > \mathfrak{R}_2^2 + \mathfrak{R}_3^2$ then define points $M_1 = (d_1, m_1)$, $M_2 = (d_1, m_2)$, $N_1 = (d_1, n_1)$, and $N_2 = (d_1, n_2)$, where m_1 and m_2 are as in (2) and (3), and

$$n_1 \stackrel{\text{def}}{=} w - \mathfrak{S}(r, (r - d_1)), \tag{6}$$

$$n_2 \stackrel{\text{def}}{=} w + \mathfrak{S}(r, (r\mathsf{s}(\mathfrak{a}^*) - d_1)) - r\mathsf{c}(\mathfrak{a}^*). \tag{7}$$

a) If $m_1 \geqslant y$ and/or if $m_2 \leqslant z$, then $\Upsilon'_x(W) = \mathfrak{a}^*$ for all $x \in [y, m_1] \cup [m_2, z]$.

b) If $n_1 \geqslant y$ and/or $n_2 \leqslant z$, then for all $x \in \{(m_1, n_2) \cup (n_2, m_2)\} \cap [y, z]$, follow Step 2b) of Case 2.

c) For all $x \in [n_1, n_2] \cap [y, z]$, define the function $\gamma^* : [0, d_2] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ which takes value $\gamma^*(x)$ for all $x \in [0, d_2]$ that solves the equation

$$\mathfrak{R}_1\mathsf{c}(\gamma^*(x)) - \mathfrak{R}_2\mathsf{s}(\gamma^*(x)) = \mathfrak{D}. \tag{8}$$

(i) If $\gamma^*(x) < \underline{\beta}(x)$, then $\Upsilon'_x(W)$ satisfies (5).

(ii) If $\underline{\beta}(x) \leqslant \gamma^*(x) < \overline{\beta}(x)$, then $\Upsilon'_x(W)$ satisfies

$$-\mathfrak{R}_1\mathsf{c}(\Upsilon'_x(W)) + \mathfrak{R}_2\mathsf{s}(\Upsilon'_x(W)) = \mathfrak{D}. \tag{9}$$

(iii) If $\overline{\beta}(x) < \gamma^*(x)$, then there does not exist a Type 1 admissible path between W and X. In particular, Υ_x does not exist.

Finally, identify $\mathcal{I}$ as the largest interval in the set $[y, z] \setminus \{x \in [n_1, n_2] \cap [y, z] : \overline{\beta}(x) < \gamma^*(x)\}$.

By virtue of the rigid transformation of reflection about the x-axis, the problem of construction of the path Λ_x between the points $W = (0, w)$ and $X = (d_1, x)$ is equivalent to the problem of construction of the path $\Upsilon_{\bar{x}}$ between the points $\bar{W} \stackrel{\text{def}}{=} (0, d_2 - w)$ and $\bar{X} = (d_1, \bar{x}) \stackrel{\text{def}}{=} (d_1, d_2 - x)$, and $\Lambda'_x(W) = \Upsilon'_{\bar{x}}(\bar{W})$. Thus, the steps described above are sufficient for analyzing the family of paths Λ_x.

IV. Synthesis of Composite Paths

In this section, we discuss the problem of constructing a path that satisfies the conditions stated in Problem 1 for a given rectangular channel $\bar{\mathcal{R}}^C$. Specifically, we discuss the construction of the overall path by concatenating paths constructed individually within each rectangle. Consequently, we consider a rectangle $ABCD$ as before; we assume that the initial configuration (W, α) is specified, and that the terminal configuration is constrained to lie in the set $[y, z] \times \left\{\left[\underline{\beta}(x), \overline{\beta}(x)\right]\right\}_{x \in [y,z]}$.

The crucial issue with this approach is of selection of the terminal configuration for construction within each rectangle. This issue is much simplified in light of the analyses discussed in Sections II-A and III; to see this, let $X_n = (x_n, 0)$ or $X = (d_{n,1}, x_n)$, as applicable, be any point on the exit segment Y_nZ_n of the rectangle R_n. If we construct a path in rectangle R_n with terminal configuration (X_n, β_n), where $\beta_n \in \left[\underline{\beta}_n(x_n), \overline{\beta}_n(x_n)\right]$, then there exists a feasible path from (X_n, β_n) that traverses the remainder of the channel, i.e., through rectangles $R_{n+1}, \ldots, R_C$. Then the synthesis of paths within each rectangle may be performed as follows:

1) There exists a shortest path Φ from the initial configuration (W, α) to the exit segment YZ. If the path Φ is Type 1 admissible (or Type 2 admissible, as applicable), the synthesis problem is solved.
2) If Φ is not admissible, we search for a terminal configuration (X, β) such that X lies on the segment YZ, $\beta \in \left[\underline{\beta}(x), \overline{\beta}(x)\right]$, and such that there exists a Type 1 admissible (or Type 2 admissible, as applicable) path from (W, α) to (X, β).

Owing to the lack of space, we omit a formal, detailed description of the above steps; instead, we refer the reader interested to [15] for further details.

V. Simulation Results

We present here numerical examples demonstrating the results of the implementing our analysis and synthesis techniques. First, we consider the problem of traversal across parallel edges of a rectangle with the (randomly generated) dimensions $d_1 = 16.294$ units, $d_2 = 18.116$ units, $r = 9.1818$ units. The rest of the problem data are as follows: $w = 12.940$ units, $y = 0$, $z = d_2$, i.e., $Y = C$, $Z = B$. The functions $\underline{\beta}$ and $\overline{\beta}$ are assumed to be constant functions with values $\underline{\beta}(x) = -29.557°$ and $\overline{\beta}(x) = 74.408°$, $x \in [0, d_2]$.

First, note that $r > d_1/2$, and hence Case 3 of Section III holds. Also, by $d_2 - w = 5.1760 < r$, and hence, by (1), $\mathfrak{a}^* = \mathsf{ac}(1 - \mathfrak{R}_3) = 64.133°$.

Following the steps for Case 3, note that $\mathfrak{S}(1, (1 - \mathfrak{R}_3)) = 1.0115$ units, while $\mathfrak{R}_2 - 1 = 0.6596$, i.e., $\mathfrak{S}(1, (1 - \mathfrak{R}_3)) > \mathfrak{R}_2 - 1$, and hence we execute Step 2. We compute the locations of points $M_1 = (d_1, m_1)$,

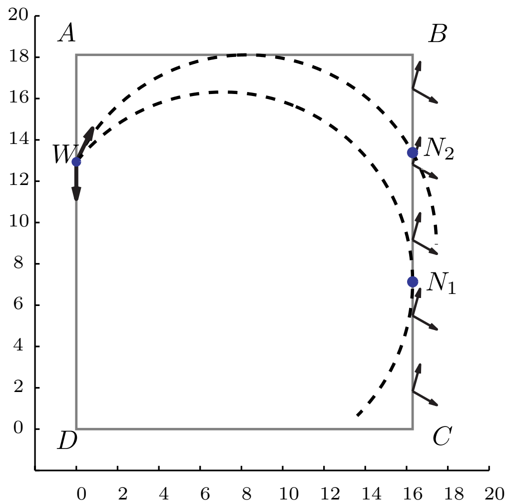
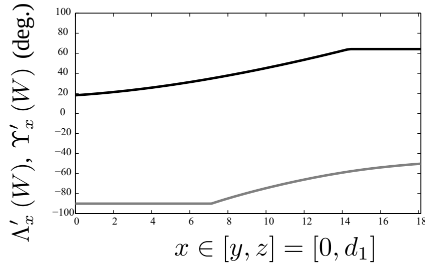
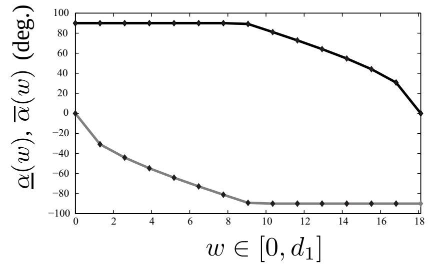


Fig. 2. The arrows on segment BC indicate $\underline{\beta}$ and $\overline{\beta}$, while the arrows at point W indicate $\underline{\alpha}$ and $\overline{\alpha}$.



(a) Lower (gray) curve shows $\Lambda'_x(W)$, upper (black) curve shows $\Upsilon'_x(W)$.



(b) Lower curve shows $\underline{\alpha}(w)$, upper curve shows $\overline{\alpha}(w)$.

Fig. 3. Results of cone analysis for (a) a specific point on the entry segment and (b) for all points on the entry segment.

$M_2 = (d_1, m_2)$, $N_1 = (d_1, n_1)$, and $N_2 = (d_1, n_2)$, using, respectively, (2), (3), (6), and (7), as $m_1 = -7.5800$ units, $m_2 = 27.261$ units, $n_1 = 7.1329$ units, and $n_2 = 13.382$ units. Note that $m_1 < y$ and $m_2 > z$. Figure 2 shows the locations of the points N_1 and N_2. We execute Steps 2b) and 2c) for Case 3, and calculate $\Upsilon'_x(W)$ for $x \in [0, d_1]$. Figure 3(a) shows the values of $\Lambda'_x(W)$ (gray curve) and $\Upsilon'_x(W)$ (black curve). The interval $[0, d_1]$ was uniformly discretized into 100 sample points for the calculation of $\Lambda'_x(W)$ and $\Upsilon'_x(W)$.

The numbers $\underline{\alpha}$ and $\overline{\alpha}$ can be calculated from $\Lambda'_x(W)$ and $\Upsilon'_x(W)$ according to Theorem 2. Figure 3(b) shows the values of $\underline{\alpha}$ (gray curve) and $\overline{\alpha}$ (black curve) as a function of w. The diamonds in Fig. 3(b) indicate the values of w for which computation was performed, and the complete curve was obtained by interpolation.

Figure 4 shows a rectangular channel of non-uniform width. The size of the entire environment is 200 units. First consider the problem of planning a path with a (constant) minimum radius of curvature $r_{\min} = 13$ units. Bereg and Kirkpatrick [13] guarantee the existence of a path of minimum radius of curvature $r_{\min}$ if the (uniform) width of the channel is at least $\tau r_{\min}$, where $\tau \approx 1.55$. The channel shown in Fig. 4 is of non-uniform width, but suppose we consider a uniform width channel lying completely within the given channel. The width d of such a channel is less than or equal to the width of the first rectangle, which is chosen as 18 units. Then, the specification of $r_{\min} = 13$ units violates the condition $d \geqslant \tau r_{\min}$. Thus, there is *no guarantee* that a path with a minimum radius of curvature of 13 units exists within the given channel.

It may be argued that the given rectangular channel may be treated as a special case of a polygonal channel, and the geometric techniques presented in [11] or [12] could be applied. This is true if the specified lower bound on the radius of curvature is *constant*. However, if the specified bounds are different inside different rectangle, then these methods cannot be applied directly because there is no method to select the terminal conditions within each rectangle.

Our algorithm addresses both these issues. The gray path shown in Fig. 4 has a minimum radius of curvature of 13 units, while the black path satisfies the local curvature conditions. Figure 5 shows the result of executing the cone analysis algorithm of Section II-A for the channel shown in Fig. 4. Each of the gray curves indicates the functions $\underline{\alpha}_n$ in degrees, and each of the black curves indicates the functions $\overline{\alpha}_n$ in degrees, measured in the local coordinate system attached to each rectangle. The dots on each plot indicate the initial condition (position and orientation) for each rectangle, measure from the local coordinate system attached to each rectangle. The cone analysis shows that, for any initial condition (position and orientation) at the entry segment of *any* of the intermediate rectangles, there exists a path which satisfies the specified curvature conditions and traverses through the remainder of the channel if and only if the initial condition lies in region enclosed by the upper and lower curves of Fig. 5 for that rectangle.

VI. Conclusions

The main contribution of this paper is a constructive algorithm for establishing the existence of curvature-bounded paths for the traversal of rectangular channels of arbitrary width. The algorithm is numerically efficient, since it relies on geometric arguments, and it is based on the explicit computation of the cone of the feasible entry and exit directions of all possible curvature-bounded paths inside a given rectangular cell. The cone analysis simplifies the feasible path synthesis, since paths may now be constructed within individual rectangles by selecting terminal conditions

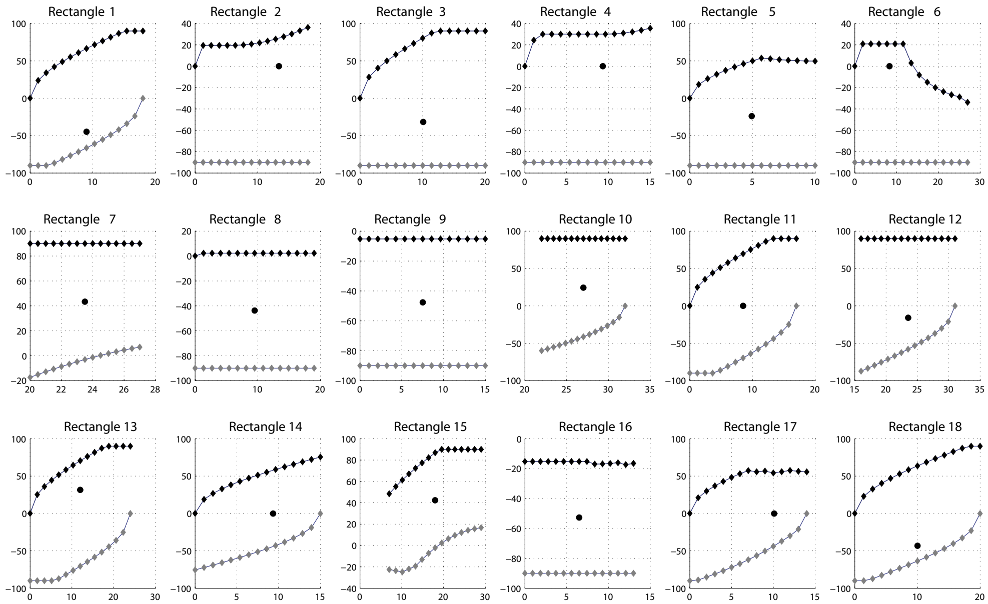


Fig. 5. Cone analysis for each rectangle.

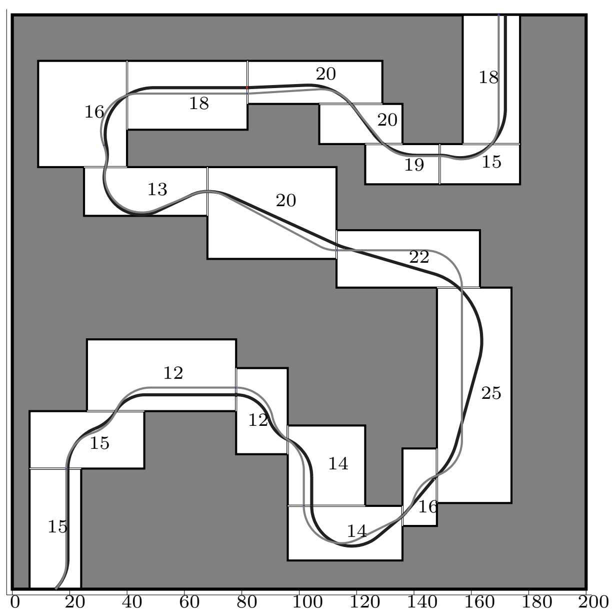


Fig. 4. The gray path has a constant curvature bound of 13 units, while the black path has variable curvature bound. The numbers within each rectangle indicate local curvature bounds for the latter.

based on the cone analysis data.

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