Time-Optimal Synthesis for the Zermelo-Markov-Dubins Problem: the Constant Wind Case

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***Abstract*— We consider a combination of the classical Markov-Dubins problem and Zermelo's navigation problem. In particular, we consider the problem of characterizing minimum-time paths with prescribed positions and tangents for a vehicle with Dubins-type kinematics in the presence of strong winds/currents. By utilizing optimal control theory, we characterize the structure of the optimal paths and subsequently solve the time-optimal synthesis problem.**

I. INTRODUCTION

In this work, we consider the time-optimal synthesis for an aerial/marine vehicle with Dubins-type kinematics operating in the presence of strong winds/currents. Our problem is essentially a combination of two well-known problems, one posted by A. A. Markov in 1887 and the other by E. Zermelo in 1931. Zermelo's navigation problem [1] deals with the characterization of optimal paths for a small ship traversing a river in the presence of currents. Zermelo solved this problem for the general case of a both temporally and spatially varying velocity current field using "an extraordinary ingenious method" according to Caratheodory [2]. Markov's problem was solved by Dubins by means of a number of constructive, geometric arguments [3]. We shall refer to it as the Markov-Dubins problem (MD) as suggested by Sussmann [4]. Reeds and Shepp solved a generalization of the MD problem when the path may contain cusps [5] (RS problem). Both the MD and the RS problems can be interpreted as minimum-time control problems. In particular, the MD (RS) problem is equivalent to the minimum-time problem for a vehicle that travels only forward (both forward and backward) with constant speed and, furthermore, the minimum allowable turning radius of the vehicle is bounded a priori. Sussmann and Tang [6] and Boissonnat et al [7] solved the MD and RS problems using the Maximum Principle of Pontryagin along with geometric control ideas, and provided more general and rigorous proofs, refining the original results of [3] and [5].

The optimal control formulation of the MD problem allowed McGee and Hedrick [8], [9] to characterize the solution of the Zermelo-Markov-Dubins problem (ZMD) for the case of constant winds. After interpreting the problem as a moving target problem (an idea that goes back to Kelley's interpretation of the Zermelo's problem [10]), they determined that the family of paths that solves the MD problem is not sufficient to solve the ZMD problem. Another solution to the ZMD based on numerical optimization techniques when the wind velocity field varies uniformly with time is presented in McNeely et al [11].

While the characterization of the structure of the optimal paths for the ZMD problem was developed in [8] and [9], a fundamental problem that still remains unresolved is that of the *time-optimal synthesis*. The solution of the synthesis problem consists of the following tasks: 1) characterize a family of extremals that is sufficiently large to solve the ZMD for arbitrary boundary conditions, 2) determine the reachable set that corresponds to each extremal of this family, and 3) provide a state-feedback minimum-time control scheme, that is, partition the state space such that each subset in the partition corresponds to a set of boundary conditions that can be interconnected in minimum time by a means of a specific control strategy. For the complete characterization of the control strategy that solves an arbitrary steering problem, besides the construction of the state-space partition, one needs to specify the switching points (in time and/or space) where the path concatenations (control switches) take place. In this work, we call the characterization of the switching times as the *inverse problem*. The synthesis problem for the MD has been solved by Bui et al [12], [13], while the same problem for the RS was addressed by Souères and Laumond in [14]. The inverse problem for the RS and the MD has been addressed in [15] and [16]. Both the synthesis and the inverse problems for the ZMD problem have never been addressed in the literature, as far as the authors know, and their detailed analysis and presentation is the main contribution of this paper. We demonstrate that the solution of both the inverse and the synthesis problems exhibit characteristics that are not present neither in the standard MD nor the Zermelo's problems.

The rest of the paper is organized as follows. In Section II we present the kinematic model, we examine its controllability, and we formulate the ZMD as an optimal control problem. In Section III we establish the existence of solutions for the ZMD problem. In Section IV we present the family of extremals of the problem that is sufficient for optimality based on PMP analysis. In Section V we carry out a detailed reachability analysis and subsequently we solve the time-optimal synthesis problem in Section VI. Finally, Section VII provides concluding remarks.

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II. Kinematic Model and Problem Formulation

Adopting to the approach of [6] and [7] we cast the ZMD as a minimum-time problem for a vehicle whose motion is described by the following set of equations

$$\dot{x} = \cos\theta + w_x, \quad \dot{y} = \sin\theta + w_y, \quad \dot{\theta} = u/\rho, \qquad (1)$$

where x, y are the cartesian coordinates of a reference point of the vehicle, θ is the vehicle's heading, u is the control input, $w \triangleq (w_x, w_y)$ is the velocity field induced by the winds/currents, and ρ is the minimum turning radius. In this work, we assume that the field w is constant.

We assume that $(x, y, \theta) \in \mathcal{M} \triangleq \mathbb{R}^2 \times \mathbb{S}^1$. Furthermore, we assume that the set of admissible control inputs is given by $\mathcal{U} \triangleq \{u \in \mathfrak{F}_{[0,T]} : u(t) \in U, \text{ for all } t \in [0,T]\}$, where $\mathfrak{F}_{[0,T]}$ is the set of all measurable functions on $[0,T]$ and $U = [-1, 1]$ is the corresponding input value set.

In the absence of winds/currents the system described by (1), known as the Dubins vehicle, with input value set U is completely controllable [6]. In the presence of a nonzero wind/current velocity field $w = (w_x, w_y)$, however, controllability is not ensured. For example, if the magnitude of the wind speed w is greater than one (the magnitude of the normalized forward velocity $(\cos\theta, \sin\theta)$ of the vehicle), then the set of unreachable configurations will be non-empty. Next, we investigate the controllability of the system described by (1) for the case of a constant wind velocity field.

A. Controllability for the Case of Constant Wind

First, we show that a necessary and sufficient condition for the system (1) to be completely controllable is that the magnitude of w, $\nu \triangleq \|(w_x, w_y)\|_2$ is less than one. For our analysis, we assume that the vehicle starts from point O, with zero heading and final configuration corresponding to a point B and heading θ_f as shown in Fig. 1(a). The wind velocity is constant and equal to $\nu\vec{\mathsf{e}}_w$, where $\vec{\mathsf{e}}_w$ is the unit vector that determines the direction of the wind velocity vector. Alternatively, we can interpret the steering problem from O with $\theta = 0$ to B with $\theta = \theta_\mathsf{f}$ (fixed end point, free final time) as a moving-target problem as suggested in [10]. In particular, equations (1) can be written as

$$\dot{\chi} = \cos\theta, \quad \dot{\zeta} = \sin\theta, \quad \dot{\vartheta} = u/\rho, \qquad (2)$$

where $\chi \triangleq x - w_x t,\ \zeta \triangleq y - w_y t,\ \vartheta \triangleq \theta$.

The system described by (1) reaches B with θ_f at some time $T > 0$ if and only if the system (2), that is, the standard Dubins vehicle in the absence of winds, intercepts at angle θ_f and at time T a particle that moves along the half-line $\epsilon = \{\vec{\mathsf{p}} \in \mathbb{R}^2 : \vec{\mathsf{p}} = \vec{\mathsf{r}} - \nu t\vec{\mathsf{e}}_w, t \geq 0\}$ with constant speed ν. Let us denote with $\vec{\mathsf{R}}(t)$ and $\vec{\mathsf{p}}(t)$ the position vectors at time t of the Dubins vehicle and the particle respectively. It follows that the controllability of the system described by (1) is equivalent to the existence of a time $T > 0$ such that

$$\vec{\mathsf{R}}(T) = \vec{\mathsf{r}} - \nu T\vec{\mathsf{e}}_w, \qquad (3)$$

where $\vec{\mathsf{R}}(T) = \overrightarrow{\mathsf{OC}}$ and $\mathsf{r} = \overrightarrow{\mathsf{OB}}$ as shown in Fig. 1(a). Next, we show that $\nu < 1$ is a necessary and sufficient condition for the complete controllability of the system.

To show necessity it suffices to observe that if $\nu \geq 1$, the particle (target) travels along ϵ with speed greater than or equal to the speed of the Dubins vehicle (pursuer) and therefore, there would be boundary configurations for which no intercept can take place.

Conversely, if $\nu < 1$ the Dubins vehicle travels faster than the particle. Furthermore, since the Dubins vehicle is completely controllable, there is some time $t_1 > 0$ such that the vehicle intercepts ϵ with heading $-\phi_w$, where $\phi_w \triangleq \mathrm{Arg}(\vec{\mathsf{e}}_w)$. Henceforth, we allow both the Dubins vehicle and the particle to move along ϵ. At some sufficiently large time t_2, the Dubins vehicle will reach some point $\vec{\mathsf{R}}(t_2)$, sufficiently ahead of the particle, say, a distance d. Then we consider the steering problem from $\vec{\mathsf{R}}(t_2)$, with $\overrightarrow{\mathsf{OC}} \triangleq \vec{\mathsf{R}}(t_2)$, with heading $-\phi_w$ to the same point C but with heading θ_f. If T_d is the minimum time for this steering problem, then the Dubins vehicle will intercept the particle at point $\vec{\mathsf{p}}(T) = \vec{\mathsf{R}}(t_2) = \vec{\mathsf{R}}(T)$ with heading θ_f at time $T \triangleq t_2 + T_d$ when $d = \nu T_d$. Note that T_d depends only on $-\phi_w$ and θ_f. The situation is depicted in Fig. 1(b).

The following proposition follows readily from the previous discussion.

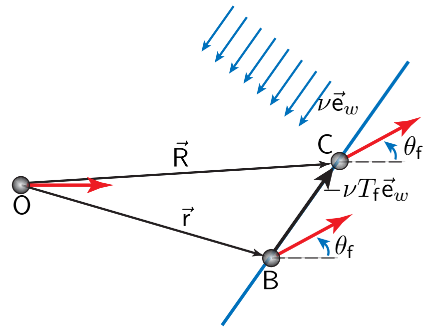
Proposition 1: For constant wind field (w_x, w_y), the system (1) is completely controllable if and only if $\nu < 1$.

Next we briefly discuss the construction of the reachable set for system (1) in the case $\nu \geq 1$. In particular, we adopt an approach similar to Caratheodory's treatment of the controllability of the Zermelo's problem. The vehicle initially at O with $\theta = \theta_0$ can move only along the direction $\overrightarrow{\mathsf{OA}}$, where $\overrightarrow{\mathsf{OA}}$ is the inertial velocity of the vehicle given by $\overrightarrow{\mathsf{OA}} = \overrightarrow{\mathsf{OP}} + \overrightarrow{\mathsf{PA}}$, where $\overrightarrow{\mathsf{OP}} = \nu\vec{\mathsf{e}}_w$ and $\overrightarrow{\mathsf{PA}} = (\cos\theta_0, \sin\theta_0)$. As shown in Fig. 2(a), after some sufficiently small time δt the vehicle driven by some constant control $u \in [-1, 1]$ reaches a point O' with heading $\theta = \theta_0 + u/\rho\delta t$. It follows from Fig. 2(a) that at time $t = \delta t$ the vehicle is constrained to move along a direction that lies within the cone with vertex O' and angle $\widehat{\mathsf{A}^+\mathsf{O}'\mathsf{A}^-}$, where $\mathsf{P}'\mathsf{A}^\pm = (\cos\theta^\pm, \sin\theta^\pm)$ and $\theta^\pm = \theta_0 \pm \delta t/\rho$.

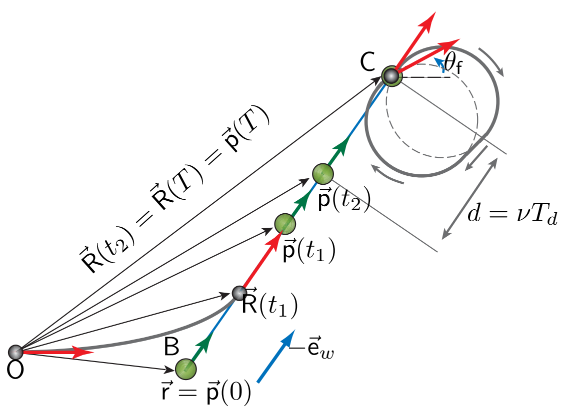
In the standard Zermelo's problem, as shown in Fig. 2(b), the vehicle's inertial velocity $\overrightarrow{\mathsf{OK}_i} = \overrightarrow{\mathsf{OP}} + \overrightarrow{\mathsf{PK}_i}$, $i \in \{1, \ldots, 4\}$, where $\overrightarrow{\mathsf{OP}} = \nu\vec{\mathsf{e}}_w$ and $\overrightarrow{\mathsf{PK}_i}$ is the vehicle's forward velocity, is constrained to lie within the cone with vertex O and angle $\widehat{\mathsf{K}_1\mathsf{O}\mathsf{K}_4}$ for all $\theta \in \mathbb{S}^1$ and for all $t \geq 0$ (for more details see [2]). The reachable set for the Zermelo problem $\mathfrak{R}^{\mathsf{Z}}(0,0)$ and the set of all points $(x_\mathsf{f}, y_\mathsf{f})$ that can be reached from $(0,0,0)$ with free final heading θ_f for the ZMD problem, denoted as $\mathfrak{R}^{\mathsf{ZMD}}(0,0,0)$, are given in Fig.3.

B. Minimum-Time Problem Formulation

To this end, we formulate the following minimum-time problem with fixed initial and terminal conditions.

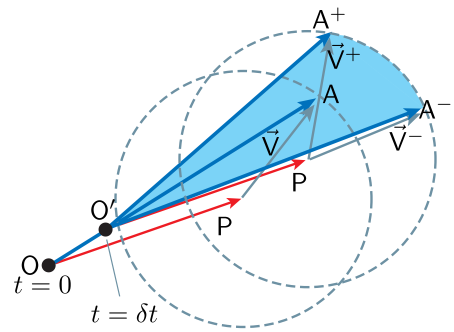


(a) The ZMD can be interpreted as a moving target problem.

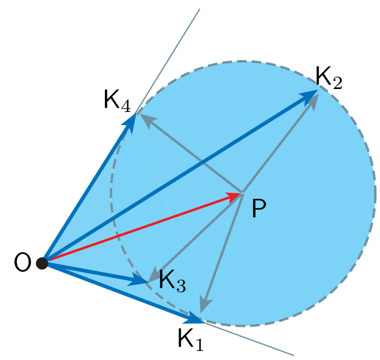


(b) Sufficient condition for controllability.

Fig. 1. The system (1) is completely controllable if and only if $\nu < 1$.



(a) Reachability analysis for the ZMD problem when $\nu \geq 1$.



(b) Reachability analysis for the Zermelo's problem when $\nu \geq 1$.

Fig. 2. Reachability analysis for the ZMD and the standard Zermelo's problems when $\nu \geq 1$.

Problem 1 (ZMD): Given the system described by equations (1) determine the control input $u^* \in \mathcal{U}$ such that

1) The control u^* minimizes the cost functional $J(u) \triangleq T_f$, where T_f is the free final time.
2) The trajectory $x^* : [0, T_f] \mapsto \mathcal{M}$ generated by the control u^* satisfies the boundary conditions

$$x^*(0) = (0, 0, 0), \; x^*(T_f) = (x_f, y_f, \theta_f), \tag{4}$$

III. Existence of Optimal Solutions

To show existence of an optimal solution to Problem 1 we apply Filippov's general theorem on minimum-time problems with prescribed initial and terminal states [17], [18]. In particular, the right hand side of (1) defines a vector field $f(x, u)$, which is continuous in u and continuously differentiable in x, and the input value set $U = [-1, 1]$ is convex and compact. Furthermore, since the vector field is affine in the control, and the input value set $U = [-1, 1]$ is convex, it follows that given any $t \geq 0$ and $x \in \mathcal{M}$ the image set of the vector field defined by the right hand sides of (1) is convex for all $u \in \mathcal{U}$. It follows by virtue of the triangle and Cauchy-Schwartz inequalities that

$$|\langle x, f(x, u)\rangle| \leq (1+\nu)\|(x, y)\| + |\theta|/\rho. \tag{5}$$

Furthermore, by using the inequality $2\|x\| \leq 1 + \|x\|^2$ and the fact $\|(x, y)\| + |\theta| \leq \sqrt{2}\|x\|$ it follows

$$|\langle x, f(x, u)\rangle| \leq c\left(1 + \|x\|^2\right), \; c = \sqrt{2}\max\{1+\nu, \frac{1}{\rho}\}. \tag{6}$$

Thus, all conditions of Filippov's theorem are satisfied and therefore we have the following two propositions.

Proposition 2: Given two arbitrary configurations x_0 and x_f in $\mathcal{M}$, existence of a feasible path from x_0 to x_f implies the existence of a minimum-time path as well.

Corollary 1: The minimum-time Problem 1 for $\nu < 1$ always has a solution.

IV. Optimal Control Analysis

In order to characterize the extremals of Problem 1 we carry out a standard optimal control analysis based on PMP arguments [19]. To this end, consider the Hamiltonian $\mathcal{H} : \mathcal{M} \times \mathbb{R}^3 \times U \mapsto \mathbb{R}$ of Problem 1 as follows

$$\mathcal{H}(x, p, u) = p_0 + p_1 \cos\theta + p_2 \sin\theta + p_3 u/\rho, \tag{7}$$

where p_0 is some scalar and $p : [0, T_f] \mapsto \mathbb{R}^3$. From PMP it follows that if x^* is a time-optimal trajectory generated by the control $u^* \in \mathcal{U}$, then there exists a scalar $p_0^* \in \{0, 1\}$ and an absolutely continuous function $p^* : [0, T_f] \mapsto \mathbb{R}^3$, where $p^* = (p_1^*, p_2^*, p_3^*)$, known as the costate, such that

1) $\|p^*(t)\| + |p_0^*|$ never vanishes,

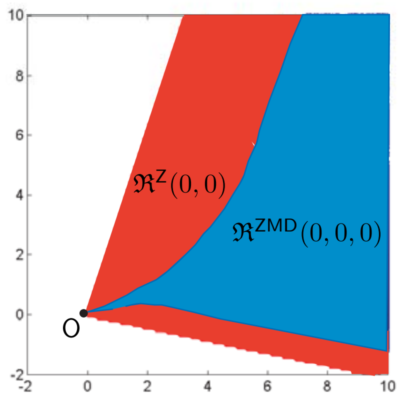


Fig. 3. Reachable sets for the ZMD and the standard Zermelo's problems when $\nu \geq 1$. Note that $\mathfrak{R}^{\mathsf{ZMD}}(0,0,0) \subset \mathfrak{R}^{\mathsf{Z}}(0,0)$.

2) $\mathsf{p}^*(t)$ satisfies, for almost all $t \in [0, T_{\mathsf{f}}]$, the canonical equation $\dot{\mathsf{p}}^* = -\partial\mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, u^*)/\partial\mathsf{x}$ or, equivalently,

$$\dot{p}_1^* = 0,\ \dot{p}_2^* = 0,\ \dot{p}_3^* = p_1^* \sin\theta^* - p_2^* \cos\theta^*, \quad (8)$$

3) $\mathsf{p}(T_{\mathsf{f}})$ satisfies the transversality condition associated with the free final-time Problem 1

$$\mathcal{H}(\mathsf{x}^*(T_{\mathsf{f}}), \mathsf{p}^*(T_{\mathsf{f}}), u^*(T_{\mathsf{f}})) = 0. \quad (9)$$

Because the Hamiltonian does not depend explicitly on time, it follows from (9) that $\mathcal{H}(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)) = 0$, for almost all $t \in [0, T_{\mathsf{f}}]$.

Furthermore, the optimal control u^* necessarily minimizes the Hamiltonian evaluated along the optimal trajectory x^* and the corresponding costate vector p^*. Thus,

$$\mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, u^*) = \min_{v\in[-1,1]} \mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, v), \ \text{ a.e. } t \in [0, T_{\mathsf{f}}]. \quad (10)$$

It follows from (10) that $u^* = -\mathrm{sgn}(p_3^*)$ when $p_3^* \neq 0$ and $u^* \in [-1, 1]$ otherwise. Thus, a solution-trajectory for Problem 1 corresponds necessarily to either a regular bang-path (concatenations of bang arcs) or a composite path that is a concatenation of bang and singular arcs. We denote a bang and a singular arc as C_τ and S_τ, respectively, where τ denotes the time of motion along each arc. Next, we state without proof Theorem 1, which deals with the structure of minimum-time paths of Problem 1. Elements of the proof of Theorem 1 can be found in [9]. Due to space limitations, a complete treatment on the characterization of the minimum-time paths for Problem 1 will not be presented in this work.

Theorem 1: Any optimal solution to Problem 1 corresponds necessarily to paths of the form

1) $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$, with $\alpha \in [0, 2\pi\rho)$ and $\beta \in [0, \pi\rho)$,
2) $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$, with $\alpha \in [0, 2\pi\rho)$ and $\beta \in [\pi, 2\pi\rho)$,
3) $\mathrm{C}_\alpha \mathrm{S}_\beta \mathrm{C}_\gamma$, with $\alpha \in [0, 2\pi\rho)$ and $\beta \in [0, \infty)$,

where $\gamma = T_{\mathsf{f}} - \alpha - \beta$.

Note that the first type of paths does not correspond to optimal paths for the standard MD problem. Furthermore, since a bang arc corresponds to either $u = +1$ or $u = -1$, the path types of Theorem 1 define a family of eight paths, denoted as $\mathfrak{T}$, that depends on two parameters, namely α and β. Since for the system (1) the trajectory determines the control, we can associate uniquely to each element of the family $\mathfrak{T}$ a family of eight control sequences $\mathfrak{U}$ composed of piecewise constant control laws with at most two switches.

From this point on, we shall denote bang arcs that correspond to $u^* = +1$ and $u^* = -1$ as L and R respectively. If the bang arc is the second arc of a $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$ with $\beta \in [0, \pi\rho)$ we shall write l_β and r_β for $u^* = +1$ and $u^* = -1$ respectively.

V. Reachability Analysis and the Inverse Problem

In this section we carry out the reachability analysis for the system (1). In light of Propositions 1 and 3, we henceforth assume that $\nu < 1$. Using an approach similar to the one in Bui et al [12], we first construct the reachable set for each extremal control sequence $u_k \in \mathfrak{U}^*$, where $k \in \mathfrak{T} \triangleq \{\mathsf{LSL}, \mathsf{LSR}, \mathsf{LRL}, \mathsf{LrL}, \mathsf{RSR}, \mathsf{RSL}, \mathsf{RLR}, \mathsf{RlR}\}$. In particular, for each $u_k \in \mathfrak{U}^*$, and given the total time of motion $t^k \in [0, \infty)$, we integrate equations (1) from $t = 0$ to $t = t^k$ with $(x(0), y(0), \theta(0)) = (0, 0, 0)$; we denote the corresponding solution as $\varphi^k : [0, \infty) \mapsto \mathcal{M}$, where $\varphi^k(t) = (x^k(t), y^k(t), \theta^k(t))$. By virtue of Proposition 1, for each $k \in \mathfrak{T}$ the solution φ^k depends on two parameters, namely $\hat{\alpha}$ and $\hat{\beta}$. In particular, $\hat{\alpha}$ equals the change of vehicle heading α/ρ along the first bang arc; we write $\alpha(\hat{\alpha}) = \hat{\alpha}\rho$. Furthermore, $\hat{\beta}$ is either the change of vehicle heading β/ρ along the second bang arc for a $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$ path or the time β for which the control is zero for a $\mathrm{C}_\alpha \mathrm{S}_\beta \mathrm{C}_\gamma$ path; we write $\beta(\hat{\beta}) = \{\hat{\beta}\rho, \hat{\beta}\}$ for a $\mathrm{C}_\alpha \mathrm{C}_\beta \mathrm{C}_\gamma$ and a $\mathrm{C}_\alpha \mathrm{S}_\beta \mathrm{C}_\gamma$ path respectively. Note that given the total time t^k, the time of motion along the third segment γ is uniquely defined by $\gamma = t^k - \alpha(\hat{\alpha}) - \beta(\hat{\beta})$, with $t^k \geq \alpha + \beta$, for all types of admissible paths. Using Propositions 1, we can readily obtain for each k the intervals $\mathcal{I}_{\hat{\alpha}}^k$ and $\mathcal{I}_{\hat{\beta}}^k$ on which the parameters $\hat{\alpha}$ and $\hat{\beta}$ belong to. The Cartesian product of $\mathcal{I}_{\hat{\alpha}}^k$ and $\mathcal{I}_{\hat{\beta}}^k$ determines the parameter space $\mathfrak{P}_k \triangleq \{(\hat{\alpha}, \hat{\beta}) : \hat{\alpha} \in \mathcal{I}_{\hat{\alpha}}^k, \hat{\beta} \in \mathcal{I}_{\hat{\beta}}^k\}$ of the problem. Let P_θ be the set of all configurations $(x, y, \theta) \in \mathcal{M}$ with $\theta = \theta_{\mathsf{f}}$. We define the projection $\Pi_\theta : \mathcal{M} \mapsto P_\theta$ as

$$\Pi_\theta\left((x^k, y^k, \theta^k)\right) \triangleq \begin{cases} (x^k, y^k), & \text{if } \theta^k = \theta, \\ \emptyset, & \text{otherwise.} \end{cases} \quad (11)$$

The reachable sets $\mathfrak{R}_{k,\theta} \subseteq P_\theta$ for the control sequence $u_k \in \mathfrak{U}^*$ is thus given by

$$\mathfrak{R}_{k,\theta} \triangleq \bigcup_{\substack{(\hat{\alpha},\hat{\beta})\in\mathfrak{P}^k \\ t^k \geq \alpha(\hat{\alpha})+\beta(\hat{\beta})}} \Pi_\theta\left(\varphi^k(t^k; \hat{\alpha}, \hat{\beta})\right). \quad (12)$$

Finally, we denote as $\mathfrak{P}_{\mathrm{LSL},\theta}$ the parameter space that corresponds to the reachable set $\mathfrak{R}_{\mathrm{LSL},\theta}$.

A. $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$ *Paths*

In order to construct the reachable set that corresponds to $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$ paths, we determine the coordinates $(x_{\mathsf{f}}, y_{\mathsf{f}})$ of all positions in the P_θ plane that can be reached from $(0,0,0)$. After integrating equations (1) from $t=0$ to $t=\alpha$ with $u=+1$, from $t=\alpha$ to $t=\alpha+\beta$ with $u=0$ and from $t=\alpha+\beta$ to $t=T_{\mathsf{f}}$ with $u=+1$, it follows that

$$x_{\mathsf{f}} = \rho\sin\theta + \hat{\beta}\cos\hat{\alpha} + w_x T_{\mathsf{f}}, \tag{13}$$

$$y_{\mathsf{f}} = \rho(1-\cos\theta) + \hat{\beta}\sin\hat{\alpha} + w_y T_{\mathsf{f}}, \tag{14}$$

where $T_{\mathsf{f}} = \rho(\hat{\alpha}+\hat{\gamma}) + \hat{\beta},\ \hat{\gamma} = (\theta-\hat{\alpha}) \bmod 2\pi$. Taking the union of $(x_{\mathsf{f}}, y_{\mathsf{f}})$ for all $\hat{\alpha}\in[0,2\pi)$ and $\hat{\beta}\in[0,\infty)$ we construct the reachable set $\mathfrak{R}_{\mathrm{LSL},\theta}$. The reachable set $\mathfrak{R}_{\mathrm{LSL},\theta}$ along with the contours of the cost function (minimum time T_{f}) are depicted in Fig. 4(a).

Conversely, given a point $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{LSL},\theta}$ we can determine $(\hat{\alpha},\hat{\beta}) \in \mathfrak{P}_{\mathrm{LSL},\theta}$ (*inverse problem*). In particular, after straightforward algebraic manipulation it follows that $\hat{\beta}$ satisfies the following quadratic equation (decoupled from $\hat{\alpha}$)

$$(1-\nu^2)\hat{\beta}^2 + 2(Aw_x + Bw_y)\hat{\beta} - (A^2+B^2) = 0, \tag{15}$$

where

$$A = x_{\mathsf{f}} - \rho\sin\theta - w_x\rho\hat{\theta}, B = y_{\mathsf{f}} + \rho(\cos\theta - 1) - w_y\rho\hat{\theta}$$

and $\hat{\theta}=\theta$ when $\hat{\alpha}\le\theta_{\mathsf{f}}$ and $\hat{\theta} = 2\pi+\theta$ otherwise.

If $\hat{\beta}\in[0,2\pi)$ is one solution of (15) then $\hat{\alpha}$ is determined with back substitution in (13) and (14) when $\hat{\beta}\neq 0$ whereas $\hat{\alpha} = \theta \bmod 2\pi$ otherwise. If there exist two solutions of (15) in the interval $[0,2\pi)$, we repeat the previous steps and determine a second pair of parameters $(\hat{\alpha},\hat{\beta})$ and the corresponding final time $T_{\mathsf{f}}(\hat{\alpha},\hat{\beta})$. Finally, we associate the configuration $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in P_\theta$ with the pair $(\hat{\alpha},\hat{\beta})$ that gives the minimum total time T_{f}. In this way, we associate uniquely, the configuration $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in P_\theta$ with a pair $(\hat{\alpha},\hat{\beta}) \in \mathfrak{P}_{\mathrm{LSL},\theta}$.

B. $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{R}_\gamma$ *Paths*

Similarly to the reachability analysis of $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$ paths, it follows that the coordinates $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{LSR},\theta}$ are given by

$$x_{\mathsf{f}} = 2\rho\sin\hat{\alpha} + \hat{\beta}\cos\hat{\alpha} - \rho\sin\theta + w_x T_{\mathsf{f}}, \tag{16}$$

$$y_{\mathsf{f}} = \rho(1+\cos\theta) - 2\rho\cos\hat{\alpha} + \hat{\beta}\sin\alpha\rho + w_y T_{\mathsf{f}}, \tag{17}$$

where $T_{\mathsf{f}} = \rho(\hat{\alpha}+\hat{\gamma}) + \hat{\beta},\ \hat{\gamma} = (\hat{\alpha}-\theta) \bmod 2\pi$. The reachable set $\mathfrak{R}_{\mathrm{LSR},\theta}$ and the contours of the minimum time T_{f} are depicted in Fig. 4(b).

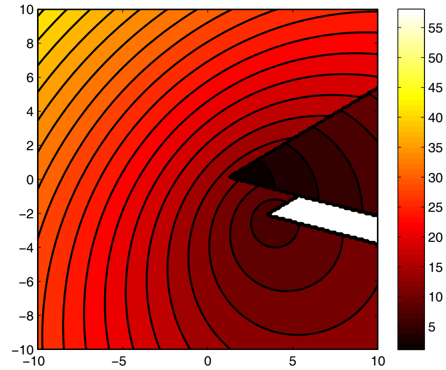
Given a point $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{LSR},\theta}$ we can determine the pair of parameters $(\hat{\alpha},\hat{\beta}) \in \mathfrak{P}_{\mathrm{LSR},\theta}$. In particular, it follows that $\hat{\alpha}$ satisfies the following transcendental equation (decoupled from $\hat{\beta}$)

$$D(\hat{\alpha})\sin\hat{\alpha} + E(\hat{\alpha})\cos\hat{\alpha} = Bw_x - Aw_y + 2\rho, \tag{18}$$

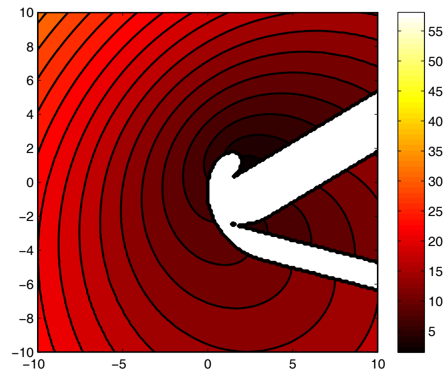
where,

$$A = x_{\mathsf{f}} + \rho\sin\theta + w_x\rho\hat{\theta}, B = y_{\mathsf{f}} - \rho(\cos\theta+1) + w_y\rho\hat{\theta},$$
$$D(\hat{\alpha}) = A - 2\rho(w_y + w_x\hat{\alpha}), E(\hat{\alpha}) = -B - 2\rho(w_x - w_y\hat{\alpha}),$$

and $\hat{\theta}=\theta$ when $\hat{\alpha}\ge\theta$ and $\hat{\theta}=2\pi+\theta$ otherwise. Since $\hat{\alpha}\in[0,2\pi)$ necessarily, it follows that equation (18) has a finite number of solutions. For each $\hat{\alpha}\in[0,2\pi)$, we can uniquely determine $\hat{\beta}$ with back substitution in (16) and (17).



(a) Reachable set $\mathfrak{R}_{\mathrm{LSL},\theta}$.



(b) Reachable set $\mathfrak{R}_{\mathrm{LSR},\theta}$.

Fig. 4. Reachable sets $\mathfrak{R}_{k,\theta}$.

C. $\mathrm{L}_\alpha\mathrm{R}_\beta\mathrm{L}_\gamma$ *and* $\mathrm{L}_\alpha\mathrm{r}_\beta\mathrm{L}_\gamma$ *Paths*

Similarly to the reachability analysis of $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$ paths, it follows that the coordinates $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{LSR},\theta}$ are given by

$$x_{\mathsf{f}} = 2\rho(\sin\hat{\alpha} + \sin(\hat{\beta}-\hat{\alpha})) + \rho\sin\theta + w_x\rho T_{\mathsf{f}}, \tag{19}$$

$$y_{\mathsf{f}} = \rho(1-\cos\theta) - 2\rho(\cos\hat{\alpha} - \cos(\hat{\beta}-\hat{\alpha})) + w_y\rho T_{\mathsf{f}}, \tag{20}$$

where, $T_{\mathsf{f}} = \rho(\hat{\alpha}+\hat{\beta}+\gamma),\ \gamma = (\theta-\hat{\alpha}+\hat{\beta}) \bmod 2\pi$. The reachable sets $\mathfrak{R}_{\mathrm{LrL},\theta}$ and $\mathfrak{R}_{\mathrm{LRL},\theta}$ along with the contours of the cost function (minimum time T_{f}) when $\beta\in[0,\pi)$ and $\beta\in[\pi,2\pi)$ are depicted in Fig. 5(a) and Fig. 5(b) respectively.

Given now a point $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{LRL},\theta}$ or $\mathfrak{R}_{\mathrm{LrL},\theta}$ we can determine the corresponding pair of parameters $(\hat{\alpha},\hat{\beta})$. In particular, it follows after some algebraic manipulation that $\hat{\beta}$ satisfies the following transcendental equation (decoupled from $\hat{\alpha}$)

$$K(\hat{\beta}) + L(\hat{\beta}) + 8\rho^2(\cos\hat{\beta} - 1) = 0, \tag{21}$$

where

$$K(\hat{\beta}) = A^2 + B^2 + 4\rho^2\nu^2\hat{\beta}^2,\ L(\hat{\beta}) = 4\rho\hat{\beta}(Bw_y - Aw_x),$$
$$A = x_{\mathsf{f}} - \rho\sin\theta - w_x\rho\hat{\theta}, B = -y_{\mathsf{f}} + \rho(1-\cos\theta) + 2w_y\rho,$$

and $\hat{\theta} = \theta$ when $0 \le \theta - \hat{\alpha} + \hat{\beta} < 2\pi$, $\hat{\theta} = -2\pi + \theta$ when $2\pi \le \theta - \hat{\alpha} + \hat{\beta} < 4\pi$ and $\hat{\theta} = 2\pi + \theta$ when $-2\pi \le \theta - \hat{\alpha} + \hat{\beta} < 0$.

Given $\hat{\beta} \in [0, 2\pi)$, it follows after some algebraic manipulation that $\hat{\alpha}$ satisfies

$$\begin{bmatrix} M(\hat{\beta}) & N(\hat{\beta}) \\ -N(\hat{\beta}) & M(\hat{\beta}) \end{bmatrix} \begin{bmatrix} \sin\hat{\alpha} \\ \cos\hat{\alpha} \end{bmatrix} = 2\rho \begin{bmatrix} 1-\cos\hat{\beta} \\ \sin\hat{\beta} \end{bmatrix}, \tag{22}$$

where

$$M(\hat{\beta}) = A - 2\hat{\beta}\rho w_x,\ N(\hat{\beta}) = B + 2\hat{\beta}\rho w_y.$$

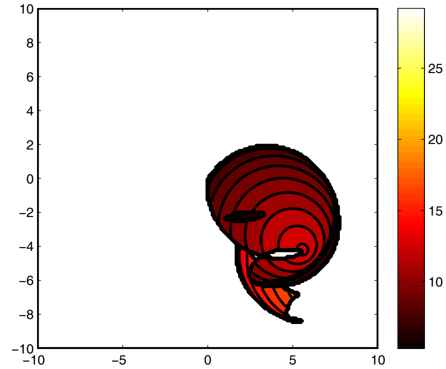
The reachable sets for the other path types, namely $\mathrm{R}_\alpha\mathrm{S}_\beta\mathrm{R}_\gamma$, $\mathrm{R}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$, $\mathrm{R}_\alpha\mathrm{L}_\beta\mathrm{R}_\gamma$ and $\mathrm{R}_\alpha\mathrm{l}_\beta\mathrm{R}_\gamma$, can be constructed in a similar fashion. It is interesting to note that the parameter vector $(\hat{\alpha}, \hat{\beta})$ is given in closed form only for $\mathrm{L}_\alpha\mathrm{S}_\beta\mathrm{L}_\gamma$ and $\mathrm{R}_\alpha\mathrm{S}_\beta\mathrm{R}_\gamma$, whereas the determination of the parameter vector for the other six types of paths of Theorem 1 requires the solution of a decoupled transcendental equation. This is a significant departure from the results of the standard MD problem where the parameter vector is always given in a closed form expression (see for example [15], [16]). However, our analysis has allowed us to reduce the inverse problem of determining $(\hat{\alpha}, \hat{\beta})$ as a function of $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in P_\theta$ to a system of equations of triangular form. Thus, the path-synthesis for a specific steering problem reduces to the solution of either a single transcendental or quadratic equation to determine one parameter of the problem whereas the second parameter is computed directly with back substitution.

VI. Time-Optimal Synthesis

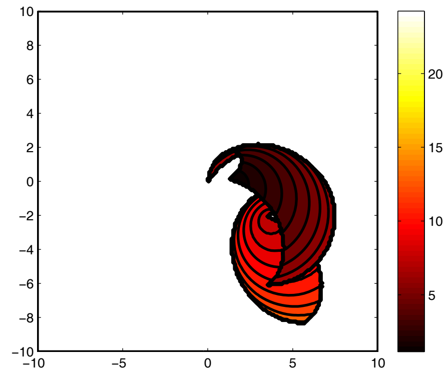
The last step of our analysis deals with the construction of a partition of the plane P_θ, such that, in each element of the partition, a particular control sequence is optimal. In particular, given a point $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in P_\theta$ with $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in \mathfrak{R}_{k,\theta}$, where $k \in \mathcal{K} \subseteq \mathfrak{T}$, then $u_\ell \in \mathfrak{U}^*$ for $\ell \in \mathcal{K}$ is a time-optimal control sequence if and only if the time t^ℓ for which $\Pi_\theta\left(\varphi^\ell(t^\ell; \hat{\alpha}, \hat{\beta})\right) = (x_{\mathsf{f}}, y_{\mathsf{f}})$ satisfies $t^\ell = \min_{k\in\mathcal{K}} t^k$; we write $t^\ell = T_{\mathsf{f}}(x, y; \theta)$. Repeating the process for each $(x_{\mathsf{f}}, y_{\mathsf{f}}) \in P_\theta$ we construct the time-optimal partition of P_θ, that is, we divide P_θ into eight domains, $\mathfrak{R}^*_{k,\theta} \subset \mathfrak{R}_{k,\theta}$ with $k \in \mathfrak{T}$, not necessarily connected, such that any terminal configuration that lies in $\mathfrak{R}^*_{k,\theta}$ can be reached in minimum time by application of the optimal control sequence $u^* = u_k \in \mathfrak{U}^*$. Furthermore, the terminal configurations that correspond to nonempty intersections of the boundaries of two or more domains $\mathfrak{R}^*_{k,\theta}$ can be reached in minimum time with the application of more than one of the eight control sequences; we denote the union of all these nonempty intersections as $\partial\mathfrak{R}^*_\theta$.

The partition of P_θ, for $\theta = \pi/3$ and fixed wind direction $\phi_w = -\pi/4$ are given in Fig. 7 for different values of the wind-speed ν. In particular, as we observe in Fig. 7(a), the partition of P_θ as well as the contours of the cost function (minimum time T_{f}) for $\nu = 0.2$ (Fig 7(a)), are close to the partition of the standard MD problem given in [13]. As the magnitude ν increases, the partition, as well as the contours of the cost function of ZMD and MD, become significantly different. Furthermore, we observe that the extremal path $\mathrm{L}_\alpha\mathrm{r}_\beta\mathrm{L}_\gamma$, which is never an optimal path for the standard MD problem, corresponds to non-negligible portions of the partition as ν increases (Fig 7(c)-7(d)).

The partition of P_θ, for $\theta = \pi/3$ and fixed wind-speed $\nu = 0.5\mathrm{m/s}$ are given in Fig. 8 for different wind directions ϕ_w. Figures 8(a)-8(d) demonstrate how the wind direction can dramatically alter the partition of the ZMD problem compared with the standard MD problem. It is interesting to note that for $\phi_w = -3\pi/4$ (Fig. 8(a)) the extremal path $\mathrm{L}_\alpha\mathrm{r}_\beta\mathrm{L}_\gamma$ corresponds to an optimal solution for the ZMD problem for a significant portion of the partition of P_θ. Furthermore, for each value of ϕ_w some path types are somehow more favorable than others. For example, when $\phi_w = -3\pi/4$, as we see in Fig. 8(a), the $\mathrm{L}_\alpha\mathrm{R}_\beta\mathrm{L}_\gamma$ paths correspond to a significantly larger portion of the partition when compared with the standard MD problem, while the same is true for $\mathrm{L}_\alpha\mathrm{R}_\beta\mathrm{L}_\gamma$ paths when $\phi_w = 3\pi/4$ as we see in Fig. 8(d).



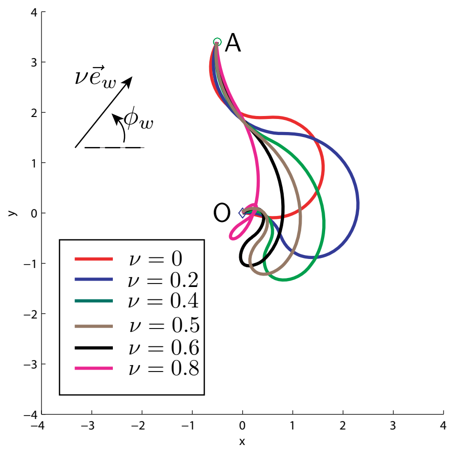
(a) Reachable set $\mathfrak{R}_{\mathrm{LRL},\theta}$.



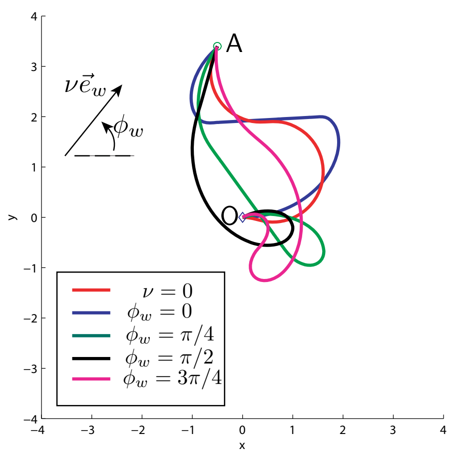
(b) Reachable set $\mathfrak{R}_{\mathrm{LrL},\theta}$.

Fig. 5. Reachable sets $\mathfrak{R}_{k,\theta}$.

Plots of minimum-time paths for the steering problem from $x_0 = (0, 0, 0)$ to $x_f = (-0.5, 3.4, \pi/3)$ for different values of ν and ϕ_w are given in Figures 6(a) and 6(b) respectively. We observe that the geometric characteristics of the paths appear to be more sensitive to variations of the wind direction than wind-speed variations.



(a) Minimum-time paths for different values of ν and $\phi_w = 3\pi/4$.



(b) Minimum-time paths for different values of ϕ_w and $\nu = 0.5$.

Fig. 6. Minimum-time paths for the ZMD problem.

VII. Conclusion

In this article we have addressed the time-optimal synthesis problem for the steering of a vehicle with Dubins' type kinematics operating in the presence of winds/currents. Our analysis reveals that while the structure of the optimal control for both the Zermelo-Markov-Dubins and the Markov-Dubins problems are similar, the time-optimal synthesis problems are significantly differentiated. Future work includes the investigation of the ZMD for more realistic and challenging wind/current velocity models.

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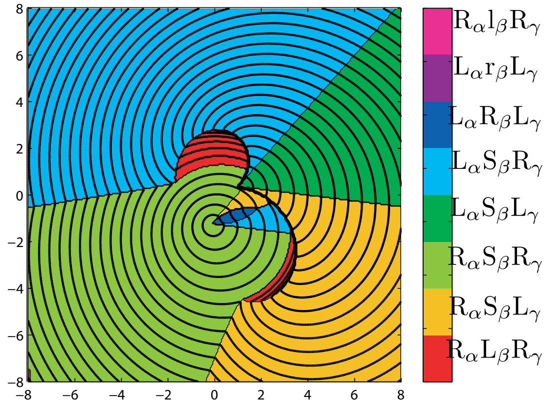
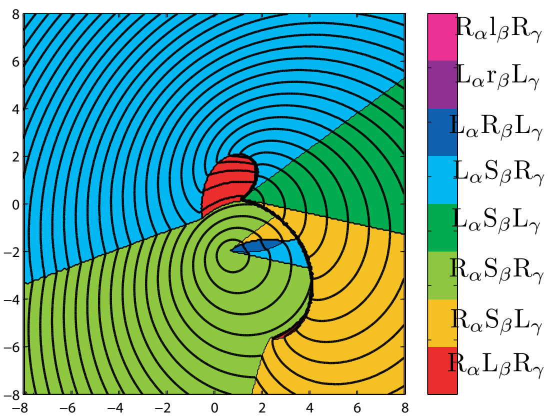
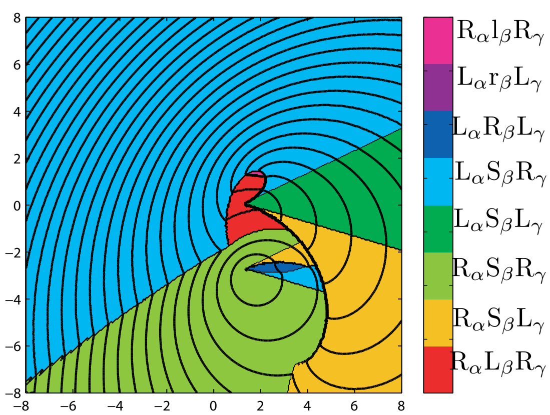


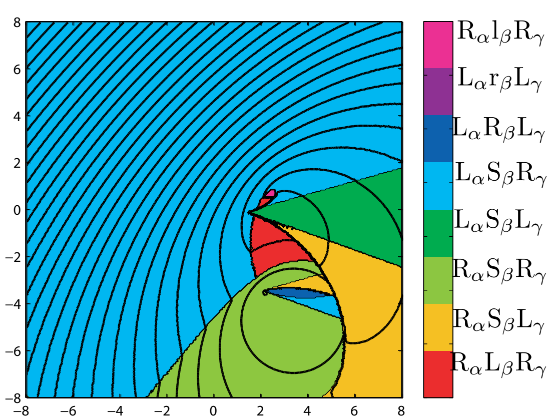
(a) $\nu = 0.2\text{m/s}$



(b) $\nu = 0.4\text{m/s}$

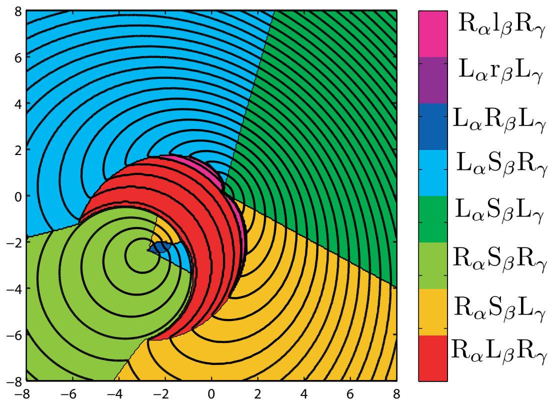


(c) $\nu = 0.6\text{m/s}$

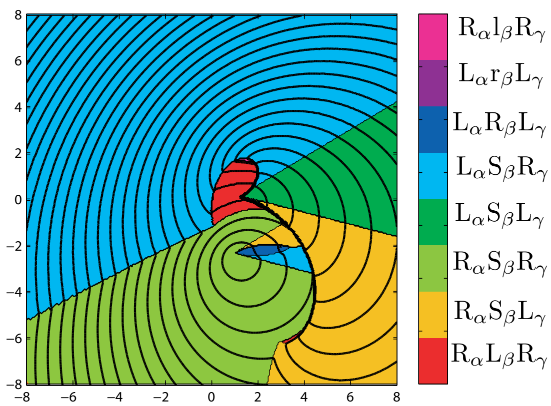


(d) $\nu = 0.8\text{m/s}$

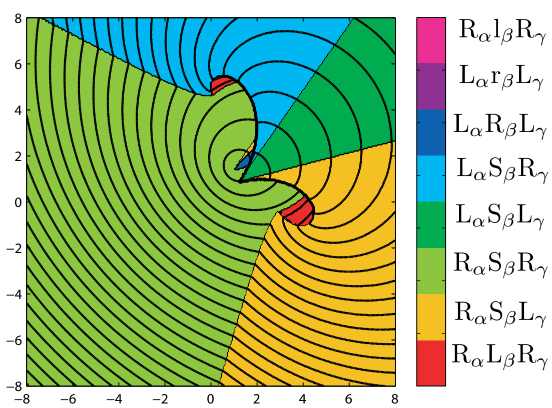
Fig. 7. Partition of $P_{\pi/3}$ and contours of $T_f = T_f(x_f, y_f)$ for different values of the ν and $\phi_w = -\pi/4$.



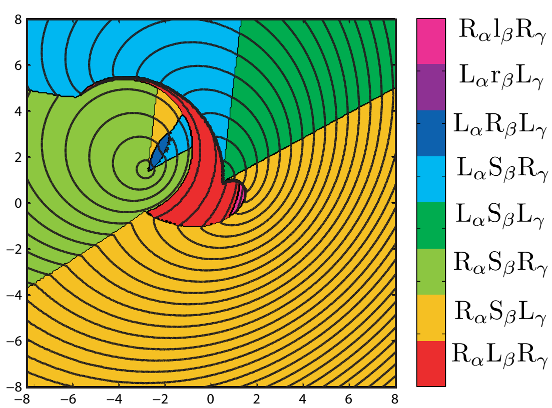
(a) $\phi_w = -3\pi/4$



(b) $\phi_w = -\pi/4$



(c) $\phi_w = \pi/4$



(d) $\phi_w = 3\pi/4$

Fig. 8. Partition of $P_{\pi/3}$ and contours of $T_f = T_f(x_f, y_f)$ for different values of the ϕ_w and $\nu = 0.5\text{m/s}$.