Approximations to Optimal Feedback Control Using a Successive Wavelet Collocation Algorithm

Chandeok Park* and Panagiotis Tsiotras†
School of Aerospace Engineering
Georgia Institute of Technology, Atlanta, GA, 30332-0150

Abstract

Wavelets, which have many good properties such as time/freqency localization and compact support, are considered for solving the Hamilton-Jacobi-Bellman (HJB) equation as appears in optimal control problems. Specifically, we propose a Successive Wavelet Collocation Algorithm (SWCA) that uses interpolating wavelets in a collocation scheme to iteratively solve the Generalized-Hamilton-Jacobi-Bellman (GHJB) equation and the corresponding optimal control law. Numerical examples illustrate the proposed approach.

1 Introduction

For general nonlinear systems with arbitrary cost functionals, a solution to Hamilton-Jacobi-Bellman (HJB) equation yields the optimal value function in terms of the state variable; the optimal control law is then given in a state feedback form. However, the HJE is notoriously hard to deal with. It is a nonlinear (quadratic) first-order pde which has no closed-form solution except in very simple formulations. It also often exhibits non-smooth solutions. As a result, analytical solutions as extremely difficult to find.

A recent breakthrough has been achieved by Beard et al [1] who proposed an iterative scheme for solving the HJB equation. The scheme iterates between solutions of the pre-Hamiltonian – which satisfies the so-called Generalized Hamilton-Jacobi-Bellman (GHJB) equation – and the control law derived from the pre-Hamiltonian. This iteration defines a contraction on a set of admissible controls and the optimal control is the fixed point of the contraction, if it exists. The GHJB is solved in [1] by Galerkin's spectral method. This approach – coined the Successive Galerkin Approximation (SGA) algorithm by the authors – seems to provide efficient solutions for a series of practical problems [7, 2]. The computational complexity is still high however, although it may be decreased by using the structure of the SGA algorithm [2]. Another attempt to reduce the computational load of the SGA method has been proposed recently by Curtis and Beard [5]. In that article, the authors devised a collocation method for solving the GHJB locally. Their idea is based on the observation that the optimal control is only a function of the local/current state; the GHJB equation is thus solved (approximately) only at a set of discrete points around the current state.

In the present and a companion paper [8] we explore the use of wavelets as basis functions for approximating the solution to the GHJB equation. In [8] the anti-derivatives of wavelets [11] have been used as basis in a Galerkin projection framework. In the current paper we use a wavelet collocation method for obtaining the optimal cost and corresponding optimal control law. Our approached is based on the results of Bertoluzza et al [3]. Using the autocorrelation function of the Daubechies wavelets (interpolating wavelets) as in [3] we propose herein a Successive Wavelet Collocation Algorithm (SWCA) which solves the GHJB equation at a number of discrete points. In that respect, our approach can be best described as complimentary to the one of Curtis and Beard [5]. The use of the interpolating wavelets has certain advantages over other collocation methods including inherent adaptivity, high recursiveness of computations, and the induced multiresolution decomposition of the domain of interest.

Since we will deal with partial differential equations of first order, we need to work with function which belong to the Sobolev space $H^1(\Omega)$. Given the open set $\Omega \subset \mathbb{R}$, the Sobolev space $H^1(\Omega)$ is defined as

$$H^1(\Omega) \triangleq \left\{ u \in \mathcal{L}^2(\Omega) \mid \int_\Omega |\hat{u}(\omega)|^2 (1 + |\omega|^2) \, \mathrm{d}\omega \leq \infty \right\}$$

where $\hat{u}$ is the Fourier transform of u

$$\hat{u}(\omega) \triangleq \int_\Omega e^{j\omega x} u(x) \, \mathrm{d}x$$

The space $H^1(\Omega)$ is equipped with the standard norm and seminorm

$$\|u\|^2_{1,\Omega} \triangleq \sum_{k=0}^{1} \int_\Omega |u^{(k)}(x)|^2 \, \mathrm{d}x$$

$$|u|^2_{1,\Omega} \triangleq \int_\Omega |u^{(1)}(x)|^2 \, \mathrm{d}x.$$

*Graduate student, currently at the Department of Aerospace Engineering, The University of Michigan, Email: chandeok@umich.edu

†Associate Professor, Email: p.tsiotras@ae.gatech.edu, Tel: (404) 894-9526, Fax: (404) 894-2760. Corresponding author.

2 Interpolating Wavelets

The family of Daubechies' wavelets [6] is one of the well-known sets of functions that generate a Multi-Resolution Analysis (MRA) of $\mathcal{L}_2(\mathbb{R})$; see [4] for a comprehensive discussion of MRA's and the wavelet basis functions for $\mathcal{L}_2(\mathbb{R})$. The Daubechies wavelets have compact support given by supp $\psi = [0, L] = [0, 2p - 1]$ where p is the order of the wavelet. Starting with any of the compactly supported Daubechies wavelets we construct a new MRA by taking the auto-correlation function of the scaling function ϕ. To this end, one defines the interpolating wavelet θ by

$$\theta(x) \triangleq \phi(\cdot) \star \phi(-\cdot)(x) \triangleq \int_{-\infty}^{+\infty} \phi(y)\phi(y - x)\,\mathrm{d}y \qquad (1)$$

The main feature of θ is its interpolation property or pointwise orthogonality, that is,

$$\theta(n - k) = \delta_{nk}, \quad \forall\ n, k \in \mathbb{Z},$$

which is evident from the orthogonality of the system $\{\phi(x - n),\ n \in \mathbb{Z}\}$. Figure 1 depicts the interpolating wavelet resulting from the Daubechies wavelet of order $p = 3$.

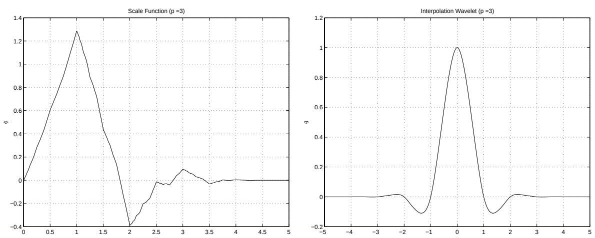

Figure 1: Daubechies' scaling function and the corresponding interpolating wavelet ($p = 3$).

Defining now the space $\tilde{\mathcal{V}}_j$ as $\tilde{V}_j \triangleq \text{span}\{\theta_{jk} \triangleq 2^{j/2}\theta(2^j x - k),\ j, k \in \mathbb{Z}\}$ we have a new MRA of $\mathbb{R}$; see [3]. Let us now define the interpolation operator

$$I_j u(x) \triangleq \sum_k u(x_{jk})\theta(2^j x - k), \quad \forall u \in H^1(\Omega) \qquad (2)$$

where $x_{jk} = 2^{-j}k,\ j, k \in \mathbb{Z}$. In such a framework it is evident the one-to-one correspondence between interpolating wavelets and dyadic points. That is, each function $\theta(2^j x - k)$ is associated with the point $x_{jk} = 2^{-j}k$ about which it is symmetric; see Fig. 1. In [3] it is proved that such an interpolation operator approximates a function with the same order of accuracy as from the $\mathcal{L}^2$ projection.

2.1. Boundary Wavelets

Consider, for convenience the interval of interest $\Omega = (0, 1)$. The simplest collocation scheme comes from the choice of uniform dyadic points, i.e.,

$$x_{jk} = 2^{-j}k, \quad j \in \mathbb{N} \cup \{0\},\ \{k \in \mathbb{Z},\ 0 \le k \le 2^j\}$$

We take the functions $\theta_{jk}(x) = \theta(2^j x - k)$ corresponding to each dyadic point. Specifically, we look for an approximation of a function u onto $\tilde{\mathcal{V}}_j$ of the form

$$u(x) \simeq I_j u(x) = \sum_{k=0}^{2^j} u(x_{jk})\theta(2^j x - k)$$

which is consistent with the given functions at each collocation point and satisfies the boundary conditions. However, it is easily verified that when $u(x) = \text{const.}$,

$$\|u - \sum_{0}^{2^j} u(x_{jk})\theta(2^j x - k)\|_{1,(0,1)} = \text{const.}$$

which implies that the space introduced so far is not sufficient. In order to have constant functions interpolated exactly and thus to be compatible with the space of interest, we modify the bases $\theta(2^j x)$ and $\theta(2^j x - 2^j)$ at the boundary points as follows[1].

$$\tilde{\theta}_{j0}(x) \triangleq \sum_{k \le 0} \theta(2^j x - k), \quad \tilde{\theta}_{j,2^j}(x) \triangleq \sum_{k \ge 2^j} \theta(2^j x - k)$$

The following properties are easily verifiable:

$$\begin{array}{ll} \tilde{\theta}_{j0}(0) = 1, & \tilde{\theta}_{j0}(x_k) = 0, \quad k > 0 \\ \tilde{\theta}_{j,2^j}(1) = 1, & \tilde{\theta}_{j,2^j}(x_k) = 0, \quad k < 2^j. \end{array}$$

Now, with a new interpolation operator defined as[2]

$$\begin{aligned} \tilde{I}_j u \ &\triangleq\ u(0)\tilde{\theta}_{j0} + \sum_{k=1}^{2^j-1} u(x_{jk})\theta(2^j x - k) + u(1)\tilde{\theta}_{j,2^j} \\ &=\ u(0)\sum_{k \le 0}\theta(2^j x - k) + \sum_{k=1}^{2^j-1} u(x_{jk})\theta(2^j x - k) \\ &\quad + u(1)\sum_{k \ge 2^j}\theta(2^j x - k) \end{aligned}$$

We have $\|c - \tilde{I}_j c\|_{1,(0,1)} = 0$. Finally, the following estimation lemma holds [3, Lemma 3.1].

Lemma 1 *Let $u \in H^{3/2+\epsilon}(0, 1)$. Then*

$$\|u - \tilde{I}_j u\|_{1,(0,1)} \le C 2^{-j/2}.$$

where C is a constant which depends only on u.

3 Optimal Feedback Control

Assume that the system to be controlled is given by the nonlinear differential equation

$$\dot{x} = f(x) + g(x)u$$

[1] These are said to be the boundary wavelets [3]. They are constructed in order to be compatible with the corresponding function space.

[2] This is the low order collocation scheme devised by Bertoluzza et al [3].

where $x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n\times m}, u : \mathbb{R}^n \to \mathbb{R}^m$. The control objective is to minimize the cost functional

$$V(x;u) = \int_0^\infty \left(\ell(x(t)) + \|u(x(t))\|_R^2\right) \mathrm{d}t$$

where the system with output $\sqrt{\ell(x)}$ is zero state observable, $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix, and $\|u\|_R^2 \triangleq u^T R u$.

Standard results from optimal control theory provide the optimal controller in *feedback form* as

$$u^*(x) = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V^*(x)}{\partial x}, \tag{3}$$

where V^* is the solution to the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} &\frac{\partial V^{*T}(x)}{\partial x} f(x) + \ell(x) \\ &- \frac{1}{4} \frac{\partial V^{*T}(x)}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V^*(x)}{\partial x} = 0 \end{aligned} \tag{4}$$

with boundary condition $V^*(0) = 0$.

It is difficult to solve the nonlinear HJB equation and compute $V^*(x)$ and $u^*(x)$ simultaneously. Rather, we use the approach of [10] and [1] and solve iteratively the following pair of equations, instead

$$\frac{\partial V^{(i)T}}{\partial x} \left(f(x) + g(x) u^{(i)}(x)\right) + \ell(x) + \|u^{(i)}(x)\|_R^2 = 0 \tag{5}$$

where $V^{(i)}(0) = 0$, and

$$u^{(i+1)}(x) = -\frac{1}{2} R^{-1} g(x)^T \frac{\partial V^{(i)}}{\partial x} \tag{6}$$

The linear pde (5) is called the Generalized Hamilton-Jacobi Equation (GHJE) in [1]. If an initial stabilizing controller $u^0(x)$ is known, one can improve its performance by iteratively solving (5)-(6) and approximate the solution to HJB equation and the optimal control law as close as possible.

4 Successive Wavelet Collocation Algorithm (SWCA)

As a simple illustration of the proposed approach, consider the following one-dimensional optimal feedback control problem:

$$\min_{u\in\mathbb{R}} V(x;u) = \int_0^\infty \left(x^2(t) + R u^2(x(t))\right) dt \tag{7}$$

subject to the dynamics

$$\dot{x} = f(x) + g(x) u, \quad x(0) = x_0, \tag{8}$$

For convenience, and without loss of generality, the domain of interest is chosen to be $\Omega = (0, 1)$. Although the desired equilibrium is a boundary point of the domain, this choice simplifies the imposition of the boundary condition $V(0) = 0$. To cover a complete neighborhood of the origin, the process described below has to be repeated for the interval $\Omega = (-1, 0)$ as well.

For system (8) with cost (7) the pair of the GHJB equation and the control law and the value function are given by (5) and (6). Suppose that at the ith step of the process we have found a stabilizing control $u^{(i)}$ which is not optimal. Then, $u^{(i)}$ can be projected onto the j-th level wavelet subspace $\tilde{\mathcal{V}}_j$ as follows.

$$\begin{aligned} u^{(i)}(x) &\simeq \tilde{I}_j u^{(i)}(x) \\ &\triangleq u^{(i)}(0)\tilde{\theta}_{j0} + \sum_{k=1}^{2^j-1} u^{(i)}(x_{jk})\theta(2^j x - k) \\ &\quad + u^{(i)}(1)\tilde{\theta}_{j,2^j} \end{aligned}$$

Similar projections are applied to $f(x)$, $g(x)$, and $\ell(x)$. Now, assume that the solution to the GHJB at the j-th resolution level, $V^{(i)}(x)$, can be approximated by

$$\begin{aligned} V^{(i)}(x) &\simeq \tilde{I}_j V^{(i)}(x) \\ &\triangleq V^{(i)}(0)\tilde{\theta}_{j0} + \sum_{k=1}^{2^j-1} V^{(0)}(x_{jk})\theta(2^j x - k) \\ &\quad + V^{(i)}(1)\tilde{\theta}_{j,2^j}. \end{aligned}$$

Introduction of the previous projections results to the evaluation of the GHJB equation at each dyadic point, that is,

$$\begin{aligned} \frac{\mathrm{d}V^{(i)}(x_n)}{\mathrm{d}x} &\left(f(x_n) + g(x_n) u^{(i)}(x_n)\right) \\ &+ x_n^2 + R u^{(i)2}(x_n) = 0 \end{aligned} \tag{9}$$

for all x_n with $n \in \{n \in \mathbb{Z}, 0 \le n \le 2^j\}$.

The unknowns in (9) are $V^{(i)}(x_n)$, i.e., the values of $V^{(i)}(x)$ at the dyadic points. They are computed by solving the linear system of equations

$$\begin{aligned} & V^{(i)}(0) \sum_{k\le 0} \left[2^j \theta'(n-k)(f(x_n) + g(x_n)u^{(i)}(x_n))\right] \\ +\ & \sum_{k=1}^{2^j-1} V^{(i)}(x_k) \left[2^j \theta'(n-k)(f(x_n) + g(x_n)u^{(i)}(x_n))\right] \\ +\ & V^{(i)}(1) \sum_{k\ge 2^j} \left[2^j \theta'(n-k)(f(x_n) + g(x_n)u^{(i)}(x_n))\right] \\ =\ & -\left(x_n^2 + R u^{(i)2}(x_n)\right) \triangleq K. \end{aligned} \tag{10}$$

where θ' denotes the derivative of θ. This can be computed by differentiating the convolution operator. It is clear that

$$\theta^{(s)}(x) = (-1)^{s-l} (\phi^{(l)} \star \phi^{(s-l)}(-\cdot))(x), \quad l, s \in \mathbb{Z},\ 0 \le l \le s$$

For integer values of the independent variable this results to

$$\theta^{(s)}(k) = (-1)^{(s-l)} \int \phi^{(l)}(x) \star \phi^{(s-l)}(x-k)dx, \quad k \in \mathbb{Z} \quad (11)$$

Combining now (10) and (11) we obtain the following system of 2^j+1 linear algebraic equations with the same number of unknowns.

$$MV^{(i)} = K$$

where,

$$M_{nk} = \begin{cases} \sum_{k\leq 0}(f(x_n) + g(x_n)u^{(i)}(x_n)) \int \phi_{jn}\phi'_{jk}, \ k = 0 \\ (f(x_n) + g(x_n)u^{(i)}(x_n)) \int \phi_{jn}\phi'_{jk}, \quad 1 \leq k \leq 2^j - 1 \\ \sum_{k\geq 2^j}(f(x_n) + g(x_n)u^{(i)}(x_n)) \int \phi_{jn}\phi'_{jk}, k = 2^j \end{cases}$$

and where $V^{(i)} = [V^{(i)}(0),\ V^{(i)}(x_1), \cdots, V^{(i)}(x_{2^j})]^T$ and $K = [K(0),\ K(x_1), \cdots, K(x_{2^j})]^T$.

The differentiation of $\phi_{jk}(x)$ as well as the evaluations of the integrals in the stiffness matrix M can be done efficiently, using connection coefficients; see, for instance, [3, 9] and the brief discussion in the Appendix.

5 Numerical Examples

Example 1 Consider a simple example with the following data: $R = 1,\ f(x) = x,\ g(x) = 1$. The plots in Figs. 2 show the monotonic convergence of the value functions and the corresponding control to the optimal one when using Daubechies wavelets with $p = 3$ and at resolution level $J = 3$.

Example 2 We consider a nonlinear example with the following data: $R = 1,\ f(x) = xe^{-x},\ g(x) = e^{-x}$. For this example, the solution to the HJB equation can be found analytically to be

$$V^*(x) = 2(1 + \sqrt{2})(xe^{-x} - e^x + 1)$$

with the corresponding optimal control is $u^*(x) = -(1 + \sqrt{2})x$. Applying the SWCA we get the results shown in Figs. 3 and 4. These figures show the monotonic convergence of the value functions and the corresponding control to the optimal one when using Daubechies wavelets with $p = 3$ and at resolution levels $J = 3$ and $J = 4$.

6 Conclusions

A collocation method that uses interpolating wavelets has been proposed for solving the Generalized Hamilton-Jacobi Bellman (GHJB) equation which arises from an iterative algorithm applied to the HJB equation. Preliminary results show that SWCA is effective for solving such problems. Difficulties still exist for multi-dimensional problems as the complexity of the algorithm is directly related to the number of collocation

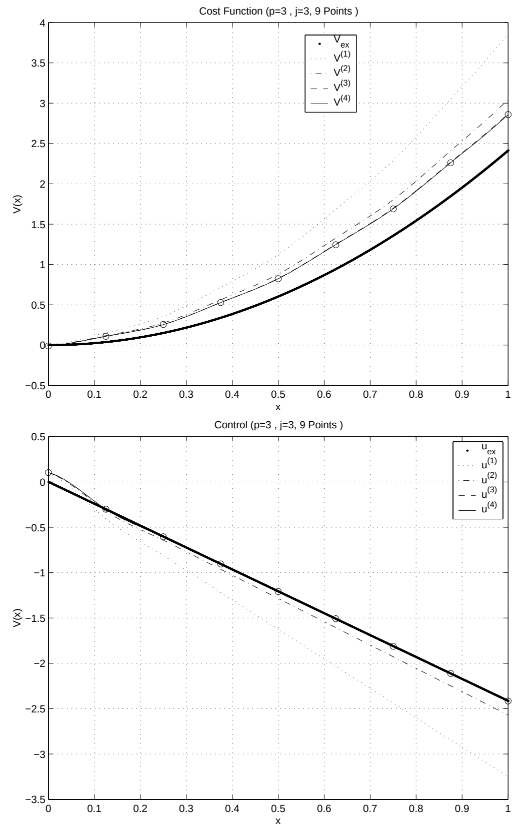

Figure 2: Value function and control for Example 1 via the SWCA ($p = 3,\ J = 3$).

points. Owing to the inherent adaptivity and multiresolution properties of wavelets, however, it is expected that the numerical complexity can be reduced by having non-uniform grid; wavelets are particularly suited in that respect. Nonuniform grids can be achieved by keeping only the wavelet coefficients above a certain threshold at the dyadic points, at each resolution level separately for each system state. These issues will be investigated in forthcoming papers.

Acknowledgement: Support for this work was provided by the National Science Foundation under award no. CMS-0084954.

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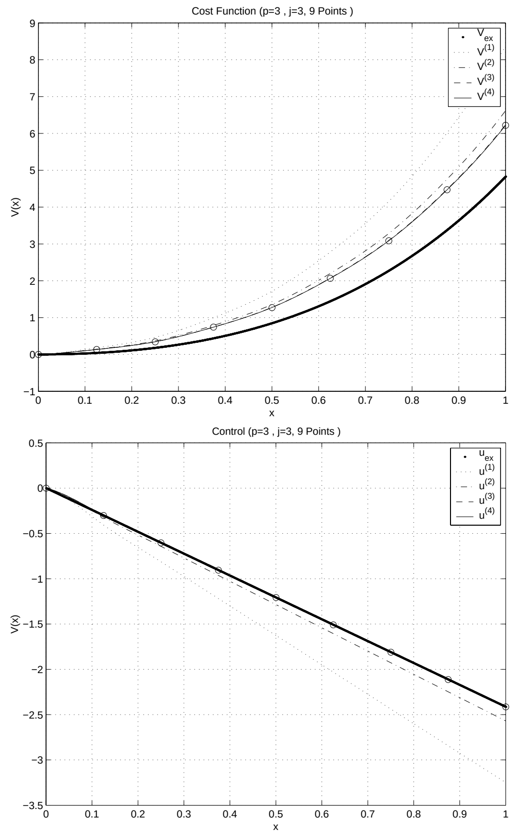

Figure 3: Value function and control for Example 2 via the SWCA ($p = 3,\ J = 3$).

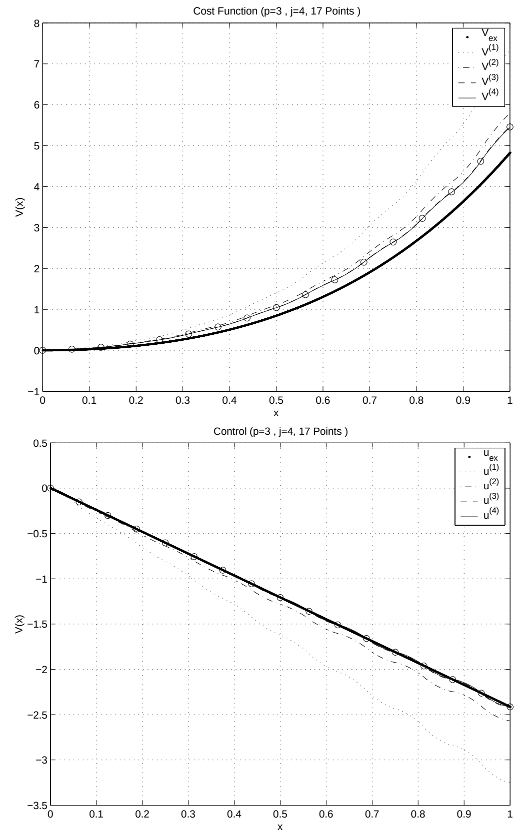

Figure 4: Value function and control for Example 2 via the SWCA ($p = 3,\ J = 4$).

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Appendix

We provide a brief overview of the wavelet connection coefficients; they allow fast computation of integration and differentiation of a wavelet system. The material in this appendix is mainly taken from [9].

Connection Coefficients: The multiresolution property of a wavelet system implies that we can write the expansion of the derivatives of the scaling and wavelet functions in a wavelet series as follows

$$\begin{aligned} \phi_l'(x) &= \sum_n \Gamma_l^n \phi_n(x) + \sum_{im} \Gamma_l^{i,m} \psi_{im}(x) \\ \psi_{jk}'(x) &= \sum_n \Gamma_{jk}^n \phi_n(x) + \sum_{i,m} \Gamma_{jk}^{im} \psi_{im}(x) \end{aligned}$$

The Γ_l^n, Γ_l^{im}, Γ_{jk}^n and Γjk^{im} are uniquely defined expansion coefficients and are called *connection coeffi-*

cients given by

$$\Gamma_l^n = \int_{\mathbb{R}} \phi_l'(x)\phi_n(x)\,\mathrm{d}x, \quad \Gamma_l^{im} = \int_{\mathbb{R}} \phi_l'(x)\psi_{im}(x)\,\mathrm{d}x$$
$$\Gamma_{jk}^n = \int_{\mathbb{R}} \psi_{jk}'(x)\phi_n(x)\,\mathrm{d}x \quad \Gamma_{jk}^{im} = \int_{\mathbb{R}} \psi_{jk}'(x)\psi_{im}(x)\,\mathrm{d}x \tag{12}$$

These equations give the connection coefficients as integrals. Calculating those integrals numerically would bring additional errors to our approximations. Fortunately, the connection coefficients can be calculated exactly through recursive algorithms which depend only on the given wavelet system.

Theorem 1 ([9, Th. 10.1]) *Let the connection coefficients Γ_l^n, Γ_l^{im}, Γ_{jk}^n and Γ_{jk}^{im} be defined as in (12). Then the following equalities hold:*

$$\begin{aligned}
\Gamma_l^n &= -\Gamma_n^l, \quad \Gamma_l^{jk} = -\Gamma_{jk}^l, \quad \Gamma_{jk}^{im} = -\Gamma_{im}^{jk} \\
\Gamma_l^n &= \Gamma_0^{n-l} = -\Gamma_{n-l}^0 \\
\Gamma_l^n &= -\Gamma_0^{l-n} = \Gamma_{l-n}^0 \\
\Gamma_l^{jk} &= \Gamma_0^{j,k-2^j l} = -\Gamma_{j,k-2^j l}^0 \\
\Gamma_{jk}^l &= \Gamma_{j,k-2^j l}^0 = -\Gamma_0^{j,k-2^j l} \\
\Gamma_{im}^{jk} &= 2^i\Gamma_{0,0}^{j-i,k-2^{j-i}m} = -2^i\Gamma_{j-i,k-2^{j-i}m}^{0,0} \quad \text{if} \quad j \ge i
\end{aligned}$$

Theorem 2 ([9, Th. 10.2]) If $j > 0$*, then*

$$\Gamma_0^{0k} = \textstyle\sum_{n,l} a_n b_l \Gamma_n^{2k+l}, \quad \Gamma_0^{jk} = \sqrt{2}\sum_n a_n \Gamma_n^{j-1,k}$$
$$\Gamma_{00}^{jk} = \sqrt{2}\textstyle\sum_l b_l \Gamma_l^{j-1,k}$$

From Theorem 1 and Theorem 2, we see that all the connection coefficients can be calculated from Γ_0^k for $k = 1, \ldots, 2p-2$. Those particular coefficients are called the *fundamental connection coefficients*. In order to calculate them, we need to introduce the autocorrelation coefficients γ_k defined as follows:

$$\gamma_k = \sum_n a_n a_{n+k}.$$

Then if $j \ge 0$ and for all $l \in \{-2p+2, \ldots, 2p-2\}$, it holds that

$$\Gamma_0^l = \sum_k \gamma_{k-2l}\Gamma_0^k.$$

The following relations result from the moment properties of the wavelets.

Theorem 3 ([9, Th. 10.4]) *If $\int t^j \psi(t)\,\mathrm{d}t = 0$ for $j \in \{0, \ldots, L\}$, then*

$$\sum_l l_\alpha \Gamma_0^l = 0, \quad \text{for} \quad 0 \le \alpha \le 2L{+}2, \ \alpha \ne 1, \quad \sum_l l\Gamma_0^l = -1$$

$$\sum_{jk} k^\alpha \Gamma_0^{jk} = 0 \quad \text{for} \quad 0 \le \alpha \le L$$

Using the skew symmetry of connection coefficients, i.e., $\Gamma_0^k = -\Gamma_0^{-k}$, one may show that the following holds

$$\Gamma_0^l = \sum_{k=1}^{2p-2} (\gamma_{k-2l} - \gamma_{-k-2l})\Gamma_0^k \quad \text{for all} \quad l \in \{1, \ldots, 2p-2\}$$

and $\sum_{k=1}^{2p-2} 2k\Gamma_0^k = -1$.

Wavelet Differentiation: For a sufficiently large J a function $f(x)$ is approximated by the scaling function expansion at that level, using values of the function at the lattice as coefficients for the wavelet expansion, i.e.,

$$f(x) \simeq P_J f(x) = \sum_k 2^{-J/2} f(x_k)\phi_{Jk}(x)$$

where $\phi_{Jk} = 2^{J/2}\phi(2^J x - k)$ and $x_k = 2^{-J}k$. Following the derivation in [9], we obtain

$$\begin{aligned}
f'(x) \simeq \frac{\mathrm{d}}{\mathrm{d}x} P_J(f)(x) &= \sum_l 2^{-J/2}\phi_{Jl}(x) \sum_k f(x_k) 2^J \Gamma_k^l \\
&= 2^J \sum_l \phi(2^J x - l) \sum_k f(x_k)\Gamma_k^l
\end{aligned}$$

Substitution of $\phi_{jk}(x)$ into the above rule of differentiation results in

$$\phi_{jk}'(x) = 2^{J/2} \sum_l \phi_{Jl}(x) \sum_m \phi_{jk}(x_m)\Gamma_m^l$$

Using orthonormality, we obtain the following expressions for the integrals in the stiffness matrix of Section 4

$$\begin{aligned}
&\int_{\mathbb{R}} \phi_{jn}'(x)\phi_{jk}(x)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}} 2^{J/2} \left(\sum_l \phi_{Jl}(x) \sum_m \phi_{jn}(x_m)\Gamma_m^l \right) \phi_{jk}(x)\,\mathrm{d}x \\
&= 2^{j/2} \sum_m \phi_{jn}(x_m)\Gamma_m^l \qquad (\text{if} \quad J = j)
\end{aligned}$$

and $\int_{\mathbb{R}} \phi_{jn}(x)\phi_{jk}(x)\,\mathrm{d}x = \delta_{nk}$.