Sub-Optimal Feedback Control Using a Successive Wavelet-Galerkin Algorithm

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Abstract

We present a numerical algorithm for solving the Hamilton-Jacobi Bellman equation using a successive Galerkin-wavelet projection scheme. According to this scheme, the so-called Generalized-Hamilton-Jacobi-Bellman (GHJB) equation is solved iteratively starting from a stabilizing solution. As basis function for the Galerkin projections we consider the anti-derivatives of the well-known Daubechies' wavelets. Wavelets offer several advantages over traditional bases functions such as time-frequency localization and compact support. A numerical example illustrates the proposed approach.

1 Introduction

For general nonlinear systems with arbitrary performance criteria, optimal feedback controllers are computed via the solution to the Hamilton-Jacobi-Bellman (HJB) partial differential equation. The solution to this nonlinear pde provides the optimal cost as a function of the system state. The corresponding optimal control is subsequently derived in terms of an explicit expression of this solution. The difficulty of the approach lies in the fact that, in general, it is difficult to obtain closed-form solutions to the HJB equation and thus, more often than not, one resorts to numerical solutions. Even numerical techniques may not be adequate however due to the high complexity of the problem (especially for multi-state systems) and the possibility of the existence of non-smooth solutions. As a result, several alternative techniques have been proposed in the past that approximate the solution to the HJB equation using perturbation methods [17], feedback linearization [10], state dependent Ricatti equations [6], neural network or other open-loop interpolation methods [14, 12], finite element and finite differences [13, 5] etc.

Recently, Beard et al [1] proposed a Galerkin approximation method for solving the so-called Generalized Hamilton-Jacobi-Bellman (GHJB) equation. They proposed the Successive Galerkin Approximation (SGA) algorithm based on iterating between the solution of the value function that satisfies the pre-Hamiltonian and the optimal control law. Under certain mild conditions the GHJB equation defines a contraction mapping on the set of admissible controls. It is shown in [1] that the optimal control is the fixed point of this contraction, if one exists. Galerkin's spectral method is used at each iteration step in order to find an approximation of the value function (and hence also the associated suboptimal controller) in a space of preselected (typically polynomial) basis functions. In [1] it was proven that the so-generated series of suboptimal control laws converges to the optimal one by choosing a sufficiently large number of bases functions. The approach seems to work well for a series of practical problems [15, 2].

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2 Wavelets as Bases for the SGA

In the Galerkin projection scheme of the SGA algorithm proposed by [1] and [2], the choice of basis functions is crucial. A rich set of polynomial basis functions will ensure a good approximation in a relatively small number of iteration steps. However, the number and form of basis functions has to be selected a priori. If the required accuracy is not achieved, more basis function must be added to the original set and the process must be repeated. This increases the computational burden and often leads to trial and error for selecting a complete set of basis functions, although it is reasonable to start from a set that spans the system dynamics, as was done, for example, in [2]. An additional point of complication arises from the well-known fact that the HJB equation may have non-smooth solutions (such is the case for time-optimal control problems). Such non-smooth solutions are not represented efficiently using polynomial or other smooth basis functions.

To overcome the aforementioned difficulties, in this paper we propose the use wavelets as basis functions in the Galerkin projection scheme of the SGA. Wavelets have certain appealing properties for efficient function approximation including (bi-)orthogonality, time-frequency localization, compact support, etc. Another important aspect is the availability of fast, recursive algorithms for doing computations with wavelets.

Since Mallat developed the main algorithm for the wavelet transform [16], wavelets have played a critical role in the areas of signal processing, data and image compression, modelling of multi-scale phenomena, etc. The advantages of wavelets in solving pde's has been noticed early on [11, 21]. Some of the most recent results in this context have appeared in [7]. Glowinski et al, for example, formulated a Galerkin-wavelet method for various boundary value problems [11]. They applied their method to the heat equation and to Burger's equation. However, their methodology may cause some difficulties. First, Daubechies' wavelets of low order cannot be used due to their lack of sufficient regularity. Second, Dirichlet boundary conditions cannot be applied directly without further modifications. Related results have appeared in [3]. In order to overcome the previous difficulties, Xu and Shann [21] constructed a set of basis functions using the anti-derivatives of wavelets. This set is sufficiently smooth with small support. They applied this set of bases to two-point boundary value problems and obtained numerical results of high consistency. The results presented herein have been motivated to a large extend by these results.

In addition to Galerkin-wavelet methods, Bertoluzza et al [3] studied collocation using interpolating wavelets derived from the auto-correlation function of the Daubechies wavelets for solving pde's. According to the assignment of collocation points several collocation algorithms were devised and were applied to two-point boundary value problems. The implementation of this collocation algorithm for solving the HJB/GHJB equation is described in a companion paper [18].

3 Wavelet Theory Fundamentals

Suppose one is given a sequence $\{\mathcal{V}_j : j \in \mathbb{Z}\}$ of closed subspaces of $\mathcal{L}^2(\mathbb{R})$ with the following properties [8].

1. Nesting: $\mathcal{V}_j \subset \mathcal{V}_{j+1}, \ \ \forall j \in \mathbb{Z}$
2. Closure: $\operatorname{clos}\left(\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j\right) = \mathcal{L}^2(\mathbb{R})$
3. Shrinking: $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$
4. Multiresolution: $f(x) \in \mathcal{V}_j \Leftrightarrow f(2x) \in \mathcal{V}_{j+1}, \ \ \forall j \in \mathbb{Z}$
5. Shifting: $f(x) \in \mathcal{V}_j \Leftrightarrow f(x - 2^{-j}k) \in \mathcal{V}_j, \ \ \forall j \in \mathbb{Z}$.
6. There exists a *scaling function* $\phi \in \mathcal{V}_0$ such that the integer shifts of ϕ form an orthonormal basis for $\mathcal{V}_0$, i.e.,

$$\mathcal{V}_0 = \operatorname{span}\{\phi(x-k), \ k \in \mathbb{Z}\}$$

The scaling function ϕ can be used to construct the *wavelet* ψ such that

$$\mathcal{W}_j = \operatorname{span}\{2^{j/2}\psi(2^j x - k), \ k \in \mathbb{Z}\}.$$

Then $\mathcal{L}^2(\mathbb{R})$ can be decomposed as

$$\mathcal{L}^2(\mathbb{R}) = \mathcal{V}_0 \bigoplus_{j=0}^{+\infty} \mathcal{W}_j = \bigoplus_{j=-\infty}^{+\infty} \mathcal{W}_j \tag{1}$$

where $\mathcal{W}_j$ is the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. Notice, in particular, that $\mathcal{V}_0 = \bigoplus_{j=-\infty}^{-1} \mathcal{W}_j$. Equation (1) is said to provide a Multi-Resolution Analysis (MRA) of $\mathcal{L}_2(\mathbb{R})$. In case the scaling function ϕ and the wavelet ψ have compact support, we have a compactly supported MRA.

The family of Daubechies' wavelets [8] is one of the well-known sets of functions that generate an MRA of $\mathcal{L}_2(\mathbb{R})$ by satisfying all of the above properties. The higher the order of the Daubechies' wavelet, the smoother the scaling function and the associated wavelet. The width of compact support of the Daubechies wavelet of order p will be denoted by $[0, L] = [0, 2p-1]$. Differentiability requires $p \geq 3$. As a matter of fact, the smoothness properties of the wavelets is related to the support of the function and for the Daubechies family it increases approximately linearly with the wavelet order[1]. Differentiability of the wavelet is important since it determines the approximation error; it is ultimately related to the wavelet moments. In fact, the following is true [4, Th. 20].

Theorem 1 *The following are equivalent:*

(i) The first m moments of the wavelet ψ are zero, i.e.,

$$\int x^\ell \psi(x)\,\mathrm{d}x = 0, \qquad \ell = 0, 1, \ldots, m-1$$

(ii) All polynomials of degree up to $m-1$ can be expressed as linear combinations of shifted scaling functions at any scale.

Using this result we can ensure that the approximation space includes the polynomials up to a certain degree by choosing to work with a wavelet having a sufficient number of zero moments.

4 Function Spaces and Frames

The original work of Daubechies involves wavelets defined over the whole real line. For pde's over a finite domain with boundary conditions one needs to construct MRA's over bounded, open intervals. Without loss of generality, we henceforth assume that the domain of interest is $\Omega = (0, L) = (0, 2p-1)$. We restrict our attention to approximating functions that provide solutions to pde's and we thus focus on the following $\mathcal{L}_2$-Sobolev space $H^s(\Omega)$, defined as

$$H^s(\Omega) \quad \triangleq \quad \left\{ u \in \mathcal{L}^2(\Omega) \mid \int_\Omega |\hat{u}(\omega)|^2 (1+|\omega|^2)^s \,\mathrm{d}\omega \leq \infty \right\}$$

[1] It can be shown that $\phi \in \mathrm{C}^{\alpha(p)}$; for $p=2$, $\alpha(2) \approx 0.55$ for $p=3$, $\alpha(3) \approx 1.088$ while for large p, $\alpha(p) \approx 0.3485p$.

where $\hat{u}$ is the Fourier transform of u

$$\hat{u}(\omega) \triangleq \int_\Omega e^{j\omega x} u(x)\,\mathrm{d}x$$

Sufficient smoothness of the functions belonging to the space $H^s(\Omega)$ is ensured by the Sobolev embedding theorem [20] which states that $H^s(\Omega) \subset \mathrm{C}^{s'}(\Omega)$ for $s \leq s' + \frac{1}{2}$. In this paper we deal only with the case $s = 1$. For certain problems with zero left-boundary condition we will also need to work with the following subspace of $H^1(\Omega)$ defined by $H^1_*(\Omega) \triangleq \{u \in H^1(\Omega) \mid u(0) = 0\}$.

A first step in constructing a basis for the spaces $H^1(\Omega)$ and $H^1_*(\Omega)$ is the construction of a frame.

Definition 1 *Let $\{\phi_n\}_{n=1}^\infty$ be a subset of a Banach space $(X, \|\cdot\|_X)$ and let* $\mathrm{span}\{\phi_n | 1 \leq n < \infty\}$ *be the set of all elements $\sum \alpha_n \phi_n$ ($\alpha_n \in \mathbb{R}$) which converge (strongly) in X. Then $\{\phi_n\}$ is said to be a frame of X if* $\mathrm{span}\{\phi_n\} = X$.

Note that a frame is not necessarily a basis. Specifically, linear independence is not required. We next define the following two-index set of functions obtained from the scaling function ϕ via translations and dilations as follows

$$\begin{aligned} \psi_{jk}(x) &\triangleq \begin{cases} \phi(x-k), & \text{for } j = -1 \\ 2^{j/2}\psi(2^j x - k), & \text{for } j \geq 0 \end{cases} \\ \phi_{jk}(x) &\triangleq 2^{j/2}\phi(2^j x - k). \end{aligned} \tag{2}$$

The main result from wavelet theory states that the set of wavelets in (2) forms a frame for $\mathcal{L}_2(\mathbb{R})$.

Since the domain of interest is Ω rather than $\mathbb{R}$, we desire a frame for the former. First, it is clear that $\psi_{jk}|_\Omega$ form a frame for $\mathcal{L}_2(\Omega)$. In fact, the functions $\{\psi_{jk} \mid \mathrm{supp}\,\psi_{jk} \cap \Omega \neq \emptyset\}$ form a frame for $\mathcal{L}^2(\Omega)$; see Eq. (1). This is elaborated upon by the following lemma.

Lemma 1 ([21]) *The set of functions defined as $\{\psi_{jk} \mid j \geq -1,\, k \in I_j\}$ where*

$$I_j = \{k \in \mathbb{Z} \mid 1 - L \leq k \leq 2^{\hat{j}} L - 1\}, \quad \hat{j} = \max\{0, j\}$$

forms a frame for $\mathcal{L}^2(\Omega)$.

In order to apply Galerkin projections, we need to construct a *basis* for $\mathcal{L}_2(\Omega)$ (more precisely for $H^1_*(\Omega)$). Such a basis can be constructed from Lemma 1 by removing the redundant elements from the frame. Although in this manner one constructs a basis form the Daubechies' wavelets themselves, this option may not always be desirable. Specifically, the lack of regularity of low order Daubechies wavelets renders necessary the use of higher order wavelets which have larger supports. Furthermore, one should follow rather complex modifications to treat boundary conditions. In order to avoid these difficulties Xu and Shann [21] introduced the anti-derivatives of the Daubechies' wavelets as bases. By construction, these are smooth even for low-order wavelets.

5 Anti-derivatives of Wavelets as Basis

The elements of this section are taken from [21]; we refer the reader to this article for the details of the construction of the basis. We remind the reader that the interval of interest is $\Omega = (0, L)$; any other interval can be treated similarly by proper scaling. According to [21], one starts by defining the functions

$$\begin{aligned} \Psi_{jk}(x) &\triangleq \int_0^x \psi_{jk}(s)\,\mathrm{d}s, \quad 0 \leq x \leq L \\ \Phi_{jk}(x) &\triangleq \int_0^x \phi_{jk}(s)\,\mathrm{d}s, \quad 0 \leq x \leq L. \end{aligned}$$

For every $J \geq 0$ define the following finite-dimensional subspace of $H^1_*(\Omega)$

$$\begin{aligned} S_J &= \mathrm{span}\,\{\Psi_{jk} \mid -1 \leq j < J,\ k \in I_j\} \\ &= \mathrm{span}\{\Phi_{Jk} \mid k \in I_J\} \end{aligned} \tag{3}$$

The spaces S_J will be the approximation spaces in the Galerkin projection scheme. In order to get a basis for S_J we need to eliminate the redundant, linearly dependent elements from the frame.

Lemma 2 (Basis for S_J, [21]) *Let the index set D_j be defined such that*

$$k \in D_j \Leftrightarrow \begin{cases} 1 - L \leq k \leq L - 1, & \text{if} \quad j = -1 \\ p - L \leq k \leq 2^j L - p, & \text{if} \quad j \geq 0. \end{cases}$$

Then $\{\Psi_{jk} \mid -1 \leq j < J,\ k \in D_j\}$ and $\{\Phi_{Jk} \mid k \in I_J\}$ are bases of the finite-dimensional subspace $S_J \subset H^1_*(\Omega)$.

For illustration purposes, Figs. 1 and 2 show the Daubechies wavelets of order $p = 2$ and their associated anti-derivatives.

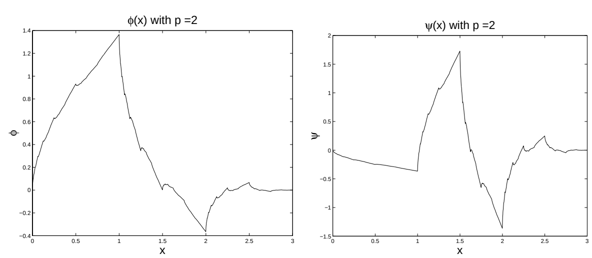

Figure 1: Daubechies scaling function ϕ and wavelet ψ ($p = 2$).

We are now ready to propose a numerical solution to the optimal feedback control problem.

6 Optimal Feedback Control

Consider a system modelled by a nonlinear differential equation which is affine in the control

$$\dot{x} = f(x) + g(x)u \tag{4}$$

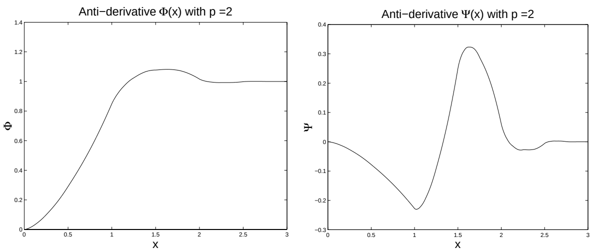

Figure 2: Anti-derivatives of ϕ and ψ ($p = 2$).

where $x \in \mathbb{R}^n, f : \mathbb{R}^n \to \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^{n \times m}, u : \mathbb{R}^n \to \mathbb{R}^m$. Given a control set $\mathcal{U}$, our objective is find the control action $u \in \mathcal{U}$ so as to minimize the cost functional

$$V(x; u) = \int_0^\infty \{\ell(x(t)) + \|u(x(t))\|_R^2\} \, \mathrm{d}t$$

where $\|u\|_R^2 \triangleq u^T R u$ and $R \in \mathbb{R}^{m \times m}$ is a positive definite matrix. That is, $V(x^*; u^*) \leq V(x, u)$ for all $u \in \mathcal{U}$ and $x^*(t)$ is the solution of $\dot{x} = f(x) + g(x)u^*(t)$. If we assume that the system (4) with output $y \triangleq \sqrt{\ell(x)}$ is zero state observable, and $\mathcal{U} = \mathbb{R}^m$, standard results from optimal control theory provide the optimal controller in *feedback form* as

$$u^*(x) = -\frac{1}{2} R^{-1} g^T(x) \frac{\partial V^*(x)}{\partial x}, \tag{5}$$

where for simplicity the asterisk has been dropped from the state x, and where V^* is the solution to the following Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} &\frac{\partial V^{*T}(x)}{\partial x} f(x) + \ell(x) \\ &- \frac{1}{4} \frac{\partial V^{*T}(x)}{\partial x} g(x) R^{-1} g(x)^T \frac{\partial V^*(x)}{\partial x} = 0 \end{aligned} \tag{6}$$

with boundary condition $V^*(0) = 0$. In general, it is difficult to solve the nonlinear pde in (6) in order to compute $V^*(x)$ and subsequently $u^*(x)$ from (5). The algorithm proposed in [1] suggests iterating between the following two *linear* equations instead

$$\frac{\partial V^{(i)T}}{\partial x} \left(f(x) + g(x) u^{(i)}(x) \right) + \ell(x) + \|u^{(i)}(x)\|_R^2 = 0 \tag{7}$$

with initial condition $V^{(i)}(0) = 0$, and

$$u^{(i+1)}(x) = -\frac{1}{2} R^{-1} g(x)^T \frac{\partial V^{(i)}(x)}{\partial x} \tag{8}$$

Equation (7) is called the *Generalized Hamilton-Jacobi Bellman* (GHJB) equation in [1].

It was proven in [1] that, under mild assumptions, the iteration between the GHJB (7) and the control (8) converges to the solution to the original HJB equation (6). At the very least, and if an initial stabilizing control $u^0(x)$ is found one can improve the performance of this controller iteratively using (7)-(8) and approximate the optimal controller as close as possible. Moreover, at each iteration step the controller $u^{(i)}$ is stabilizing.

7 The Successive Wavelet-Galerkin Algorithm (SWGA)

Using the previous results, a numerical solution to the HJB equation is proposed as follows. At each step of the SGA algorithm we find an approximate solution to the GHJB equation by projecting on the subspace S_J for some scaling index $J \geq 0$; see Lemma 2. For J sufficiently large, the approximation to the GHJB on S_J, say $V^{(i,J)}$ along with the corresponding control $u^{(i,J)}$, will approach the optimal solution V^* and u^* as $i \to \infty$. We do not provide any error estimates of the algorithm in this paper as the relevant conditions for convergence along with approximation bounds can be found in [1] and [21].

To keep the notation and the main ideas as simple as possible, we consider the following one-dimensional optimal feedback control problem.

$$\min_{u \in \mathbb{R}} V(x; u) = \int_0^\infty \{x^2(t) + R\, u^2(x(t))\} \, \mathrm{d}t$$

subject to the dynamics

$$\dot{x} = f(x) + g(x) u(x), \quad x(0) = x_0, \tag{9}$$

For convenience, and without loss of generality, the domain of interest is chosen to be $(0, 1)$. We will solve this problem on the domain $\Omega = (0, L)$ ($L \geq 1$) instead, and we will then take the restriction of the solution to $(0, 1)$. A word of caution at this point: Although the desired equilibrium is a boundary point of the domain, this choice simplifies the imposition of the boundary condition $V(0) = 0$. To cover a complete neighborhood of the origin, the following procedure has to be repeated for the interval $(-1, 0)$.

The pair of GHJB equation and the control law is expressed in the form

$$\frac{\partial V^{(i)}(x)}{\partial x} \left(f(x) + g(x) u^{(i)}(x) \right) + x^2 + R u^{(i)2}(x) = 0 \tag{10}$$

with initial condition $V^{(i)}(0) = 0$ and

$$u^{(i+1)}(x) = -\frac{1}{2R} g(x) \frac{\partial V^{(i)}(x)}{\partial x} \tag{11}$$

It is assumed that there exists an initial stabilizing control $u^{(0)}$ which is not optimal.

The task is to find the sub-optimal cost on a given subspace $S_J \subset H_*^1(\Omega)$ as well as the corresponding control strategy. From the method of weighted residuals [9], the GHJB equation should satisfy

$$\int_0^L \left(\frac{\partial V^{(i)}(x)}{\partial x} \left(f(x) + g(x)u^{(i)}(x) \right) \right) \chi(x)\,\mathrm{d}x = -\int_0^L \left(x^2 + Ru^{(i)2}(x) \right) \chi(x)\,\mathrm{d}x$$

for all members of $\chi(x) \in H^1_*(\Omega)$. Let $\mathcal{X}(x) = [\chi_1(x),\ \chi_2(x)\ \cdots\ \chi_N(x)]^T$ be the basis vector of the finite dimensional subspace S_J with length N and let $V_N^{(i)}(x) = \sum_{k=1}^N c_k\chi_k(x)$ for some set of constants $[c_1\ c_2\ \cdots\ c_N]^T$. Then the Galerkin projection of $V^{(i)}$ on S_J is given by

$$\int_0^L \left(\frac{\partial V_N^{(i)}(x)}{\partial x} \left(f(x) + g(x)u^{(i)}(x) \right) \right) \chi_k(x)\,\mathrm{d}x = -\int_0^L \left(x^2 + Ru^{(i)2}(x) \right) \chi_k(x)\,\mathrm{d}x, \tag{12}$$

This provides N linear algebraic equations for the N unknown coefficients c_k, $(k = 1, 2, \cdots, N)$. Once $V_N^{(i)}(x)$ is obtained, we can improve the control law at the previous step. Repeating this process leads to a sub-optimal cost as well as a sub-optimal control law as close as possible to the optimal ones.

8 Numerical Example

Consider the system (9) with the following parameters: $R = 1$, $f(x) = x$, $g(x) = 1$. For this simple case the optimal value function can be analytically obtained to be $V^*(x) = (1 + \sqrt{2})x^2$ and consequently the optimal control law in feedback form is given by $u^*(x) = -(1 + \sqrt{2})x$.

The GHJB equation and the corresponding control law are given by equations (10) and (11). For this example, we choose (arbitrarily) to start from the stabilizing control law $u^{(0)}(x) = -5x$. Using the SWGA with the Daubechies wavelets of order $p = 2$ and for detail levels $J = 0$ and $J = 1$ provides the results shown in Figs. 3 and 4.

These plots show the monotonic convergence of cost and the controller to the optimal ones.

9 Conclusions

We propose a methodology of solving (approximately) the HJB equation appearing in the formulation of optimal control problems. The approach uses an iterative algorithm for solving the GHJB equation. At each step, the GHJB is solved on a subspace spanned by the anti-derivatives of the well-known Daubechies wavelet functions. The introduction of the anti-derivatives allows to work with differentiable basis functions with

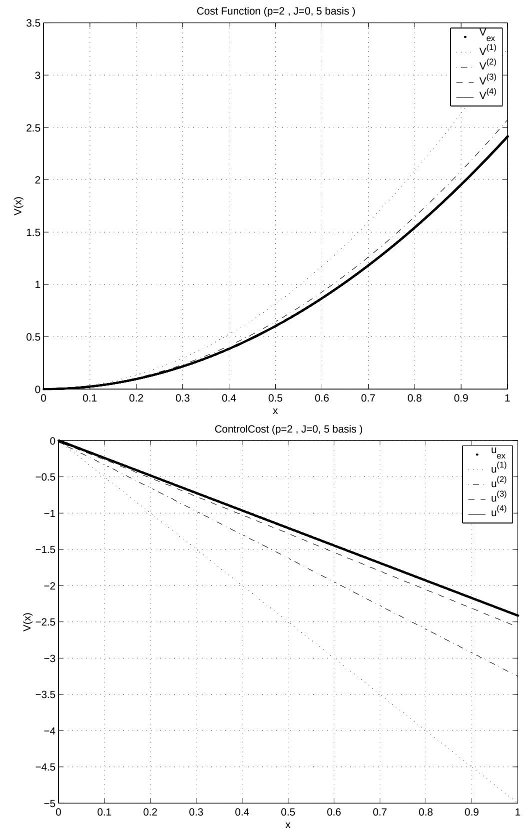

Figure 3: Optimal cost and optimal control via SWGA ($p = 2, J = 0$).

small support. The proposed algorithm has been evaluated and tested on a simple one-dimensional example. Extensions to multi-dimensional problems is a straightforward (using tensor products), albeit rather tedious exercise. The computational complexity can be decreased by using either the structure of the SGA algorithm [2] or the properties of the wavelets themselves [19, Ch. 12].

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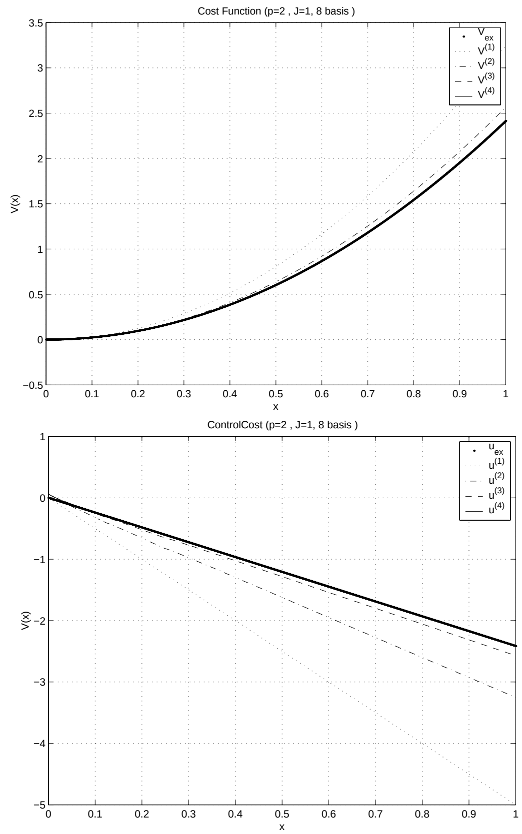

Figure 4: Optimal cost and optimal control via SWGA ($p = 2, J = 1$).

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