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Feasible Trajectory Generation for Underactuated Spacecraft Using Differential Flatness

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We consider the problem of feasible trajectory generation of an underactuated axi-symmetric spacecraft subject to two external torques acting on the plane normal to the symmetry axis. We derive the condition that must be satisfied by an attitude history in order to be a feasible trajectory. We then propose a methodology to generate trajectories satisfying this condition. Our approach makes use of the well-known flatness of the corresponding differential equations. We emphasize the importance of being able to generate these trajectories on-line and with minimal off-line intervention. Feasible trajectories can later be used as reference trajectories for tracking problems for underactuated spacecraft.

Introduction

The problem of stabilization of an underactuated spacecraft has been addressed recently in the literature.[1–7] Several techniques have been proposed both for the axi-symmetric, as well as the non-symmetric case. The stabilization problem for the axi-symmetric case can be considered solved, although there is still work to be done in addressing robustness questions. (See Ref. 8 for a discussion on robustness for the angular velocity stabilization problem of an underactuated rigid body.) The stabilization problem for a non-symmetric spacecraft turned out to be much more challenging, but finally several approaches were proposed with much success.[4,5,9] Nonetheless, these approaches are local and no globally stabilizing control law has been reported in the literature, as far as the author knows.

In a related avenue of research, Ref. 10 addressed and solved the (global) tracking problem for the case of an axi-symmetric spacecraft using two controls. The results in Ref. 10 provide bounded controllers for tracking both 3-axis and 1-axis attitude reorientations. A key assumption made in Ref. 10 (and in most trajectory tracking literature, for that matter) is that the trajectory to be tracked is a *feasible* one. That is, there exists a control input in the space of admissible controls such that, with the correct initial conditions, the output of the system is able to track *exactly* the reference trajectory. This is typically achieved by making the *a priori* assumption that the (reference) trajectory to be tracked is generated by an *exosystem*, which is an exact copy of the plant itself. For the case of left-invertible plants it is relatively easy to find the state and input history that correspond to a given output history. However, even in this case, given an initial and a final point in the output space, it is not clear at all how one can generate an output trajectory that belongs in the system output function space.

In this paper we address the following problem: Given an initial and a final orientation of an underactuated rigid spacecraft (the precise definition of "underactuation" to be given shortly), find an angular velocity history such that, under the influence of this angular velocity history, the body will move from the initial orientation to the final orientation in a given period of time. In order to solve this problem, we first derive a necessary and sufficient condition that must be satisfied by all feasible trajectories. We then use the property of differential flatness,[11,12] satisfied by the system differential equations, in order to derive trajectories in the flat output space. These trajectories are then mapped back to the state space (and eventually to the angular velocity space) to provide feasible trajectories. These feasible trajectories can then be used as reference trajectories for the tracking problem of an axisymmetric spacecraft with two control inputs studied in Ref. 10.

It should be pointed out that although the property of differential flatness for the underactuated rigid body problem has been known for some time[13] no actual results have appeared in the literature dealing specifically with this problem. We also emphasize that we give special attention to the problem of singularity avoidance, often encountered when designing trajectories in the flat output space. Namely, the map from the flat output space to the state space may not be defined at some points. In this paper, we propose a parameterization of the flat outputs such that the transformation to the state and input space remains

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well-defined. This is achieved by taking advantage of the non-uniqueness (multivalued map) of the flat outputs in the flat output space ($S^1 \times S^1$). In addition, the design of trajectories in the flat output space can be completely automated, i.e., without intervention by the user.

A major motivation for insisting for a globally nonsingular, inverse mapping with minimal (or not at all) supervision, is to be able to implement the proposed algorithm on-board the spacecraft. This is a very appealing property of the proposed algorithm since, in general, the design of trajectories in the flat output space (although a completely geometric problem) requires some amount of off-line (i.e., user) intervention. Our philosophy of "minimal user intervention" during the design of the feasible trajectories agrees with the current trend for "smarter" autonomous spacecraft with very little ground-spacecraft communication and control.

As in Refs. 10 and 6, we make use of the kinematic attitude description developed in Refs. 14 and 15. This attitude description allows one to isolate and describe the motion of the (underactuated) symmetry axis of the body using a single complex variable. The whole control design is performed at the kinematics level, (i.e., with the angular velocities assumed to be the control inputs), since it is a straightforward task to "backstep" the kinematic control laws to dynamic ones.[6]

Numerical examples at the end of the paper demonstrate the theoretical results.

Attitude Equations

In this paper we use the attitude description described in Refs. 6,14. According to the results of Ref. 6 the relative orientation between two reference frames can be represented by *two successive rotations*. The first rotation is about the inertial $\hat{\imath}_3$-axis at an angle z. The second rotation is about the unit vector

$$\hat{h} = \left(\frac{w+\bar{w}}{2|w|}\right)\hat{\imath}'_1 + \left(\frac{\mathrm{i}\,(\bar{w}-w)}{2|w|}\right)\hat{\imath}'_2 \tag{1}$$

and has magnitude

$$\theta = \arccos\left(\frac{1-|w|^2}{1+|w|^2}\right) \tag{2}$$

In Eq. (1) $\hat{\mathbf{i}}' = (\hat{\imath}'_1, \hat{\imath}'_2, \hat{\imath}'_3)$ is the intermediate reference frame resulting from the rotation z about the inertial $\hat{\imath}_3$-axis. The situation is depicted in Fig. 1, where (a,b,c) denote the coordinates of the unit vector $\hat{\imath}'_3$ in the body frame, $\hat{\imath}'_3 = a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3$. It can be shown[14] that the location of the body $\hat{b}_3$-axis in the $\hat{\mathbf{i}}'$ frame is also determined by a,b,c from $\hat{b}_3 = -a\hat{\imath}'_1 - b\hat{\imath}'_2 + c\hat{\imath}'_3$ (Fig. 1). With this notation, the complex coordinate w is defined by

$$w = w_1 + \mathrm{i}\,w_2 = \frac{b - \mathrm{i}\,a}{1+c} \tag{3}$$

We note here that in Eqs. (1) and (3) $\mathrm{i} = \sqrt{-1}$, bar denotes the complex conjugate, and $|w|^2 = w\bar{w}$ denotes the absolute value of the complex number $w \in \mathbb{C}$. Conversely, from w one can compute (a,b,c) from

$$a = \mathrm{i}\,\frac{w-\bar{w}}{1+|w|^2}, \quad b = \frac{w+\bar{w}}{1+|w|^2}, \quad c = \frac{1-|w|^2}{1+|w|^2} \tag{4}$$

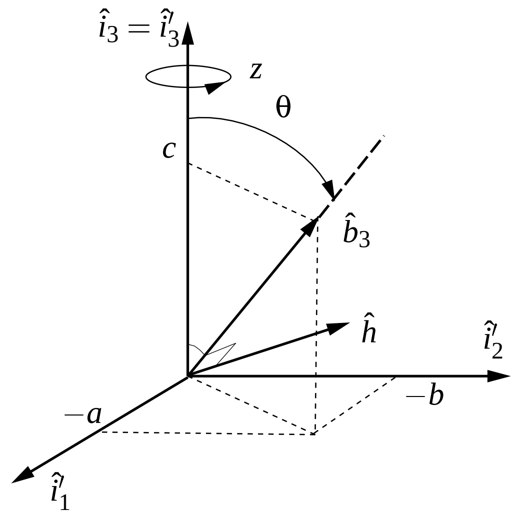


Fig. 1 Attitude description in terms of (w,z) coordinates.

The kinematic differential equations in terms of w and z are given by[6]

$$\dot{w} = -\imath\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{5a}$$
$$\dot{z} = \omega_3 + \mathrm{Im}(\omega\bar{w}) \tag{5b}$$

In this paper we assume that only the angular velocity ω (equivalently, ω_1 and ω_2) can be manipulated. The angular velocity component about the body $\hat{b}_3$-axis ω_3 cannot be changed due to, say, a thruster failure. Specifically, for an axi-symmetric body about the body $\hat{b}_3$-axis with no torque component about this axis, ω_3 remains constant for all $t \geq 0$. In this case, three-axis stabilization and pointing is possible only if, in addition, $\omega_3 \equiv 0$. (Of course, stabilization and inertial pointing of the symmetry axis is still possible.[1,16])

Letting $\omega_3 = 0$, the kinematic equations thus become

$$\dot{w} = \frac{\omega}{2} + \frac{\bar{\omega}}{2}w^2 \tag{6a}$$
$$\dot{z} = \mathrm{Im}(\omega\bar{w}) \tag{6b}$$

Note that the corresponding dynamic equations are simply

$$\dot{\omega} = u \tag{7}$$

where $u = u_1 + \mathrm{i}\,u_2$ and u_i $(i = 1,2)$ is the torque about the ith body axis. Given a desired angular velocity history $\omega_d(t)$, a simple tracking control law at the dynamic level is then given by

$$u = -\kappa(\omega - \omega_d) + \dot{\omega}_d \tag{8}$$

where $\kappa > 0$. The previous equation implies that any acceptable angular velocity history ω_d at the kinematic level must be bounded and must have a bounded derivative.

Feasible trajectory generation

In Ref. 10 the tracking problem for the system in Eq. (6) was addressed. There, it was assumed that the reference trajectory is given as the output of a dynamical system with the same nonlinear structure as the original system. In Ref. 10 this exosystem was called the "virtual" spacecraft. The advantage of this approach was that one could guarantee *a priori* that the trajectories of this exosystem are feasible and perfect tracking can be achieved. That is, given some reference trajectories $w(t)$ and $z(t)$ one could guarantee the existence of an angular velocity command $\omega(t)$ such that Eqs. (6) are satisfied. In general, it is not true that, given some arbitrary functions of time $w(t)$ and $z(t)$, there exists such a command $\omega(t)$.

To see this, consider a given attitude history, expressed in terms of the functions $w(t)$ and $z(t)$ for $t \geq 0$. If these functions correspond to a feasible trajectory, then one could solve Eq. (6) for $\omega(t)$ from

$$\omega = \frac{2}{1-|w|^4}\,(\dot{w} - \tilde{\bar{w}}w^2) \tag{9}$$

at least whenever $|w| \neq 1$. The last equation implies the constraint

$$\dot{z}\,(1-|w|^2) - 2\,\mathrm{Im}(\dot{w}\,\bar{w}) = 0 \tag{10}$$

which, in general, does not hold for arbitrary functions of time $w(t)$ and $z(t)$.

The previous equation is thus a necessary and sufficient condition that must be satisfied by any feasible trajectory $(w(t), z(t))$.

In this section we develop an approach to generate feasible trajectories for the system in Eq. (6). These trajectories, can then be used as reference trajectories for the tracking problem. In particular, given an initial point (w_0, z_0), a final point (w_f, z_f) and a time t_f, we seek time functions $w(t)$ and $z(t)$, defined over the interval $0 \leq t \leq t_f$, such that $(w(0), z(0)) = (w_0, z_0)$, $(w(t_f), z(t_f)) = (w_f, z_f)$ and Eq. (10) is satisfied for all $0 \leq t \leq t_f$. We call such trajectories *feasible* since they ensure the existence of a function $\omega(t)$ such that system in Eq. (6) is satisfied. Such an $\omega(t)$ can be found by Eq. (17) below. The functions $w(t)$ and $z(t)$ are then the solutions of the system (6) with input $\omega(t)$, initial conditions (w_0, z_0), and final conditions (w_f, z_f).

Differential Flatness and Flat Outputs

To solve the feasible trajectory generation problem, we will use the notion of differential flatness.[11,12] Let the system

$$\dot{x} = f(x,u) \tag{11}$$

where $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ are the control variables. This system is differentially flat if one can find outputs $y \in \mathbb{R}^m$ (the same as the number of inputs) of the form

$$y = y(x, u, \dot{u}, \ldots, u^{(p)}) \tag{12}$$

such that all states and inputs of the system can be written as algebraic functions of these flat outputs and their derivatives. In other words, equation (12) can be inverted, such that

$$x = x(y, \dot{y}, \ldots, y^{(q)}) \tag{13}$$

$$u = u(y, \dot{y}, \ldots, y^{(q)}) \tag{14}$$

From the previous equations it becomes evident why flat outputs play a significant role in trajectory generation problems. If the flat output history $y(t)$ is known, then (13) and (14) can be used to compute the corresponding state and input trajectories. Every path in the flat output space is mapped to a feasible trajectory and thus, the trajectory generation problem for flat systems is trivial.

Differentially flat systems are extremely nice since, they are equivalent* to an algebraic system, i.e., a system without dynamics. The downside of this approach is that most (nonlinear) systems are not flat. Also, to date, there does not exist a systematic way for finding the flat outputs† (if they exist), although very often they have intrinsic physical significance. An additional problem may arise if the transformation from the flat output space to the state space is singular. In this paper we address all these problems for the underactuated spacecraft problem, and propose a simple parameterization of trajectories in the flat output space that satisfies all the constraints and avoids any singularities.

Flat Outputs for the Attitude Problem

In this section we show that the system (15) subject to the control inputs ω_1 and ω_2 possesses two flat

*This type of equivalence is called Lie-Bäcklund equivalence and it is quite well-known in physics. Two systems are equivalent in the Lie-Bäcklund sense if any variable of one system may expressed as a function of the variables of the other system and of a finite number of their time derivatives. One system can then be transformed to the other via endogenous feedback. This transformation does not necessarily preserve state dimension. See also Ref. 17.

†Except the case of configuration flat Lagrangian systems with n degrees of freedom and $n-1$ controls, where a complete characterization exists. See, Ref. 18.

outputs. These outputs, denoted below by y_1 and y_2, can be used to solve the feasible trajectory generation problem for the underactuated spacecraft.

From now on, and for clarity of exposition, we switch from complex to real number notation. The kinematic model of an underactuated rigid body is then described by

$$\dot{w}_1 = \frac{1}{2}(1+w_1^2-w_2^2)\,\omega_1 + w_1w_2\omega_2 \quad (15a)$$
$$\dot{w}_2 = \frac{1}{2}(1-w_1^2+w_2^2)\,\omega_2 + w_1w_2\omega_1 \quad (15b)$$
$$\dot{z} = w_1\omega_2 - w_2\omega_1 \quad (15c)$$

or, compactly, by

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{z}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}(1+w_1^2-w_2^2) & w_1w_2\\ w_1w_2 & \frac{1}{2}(1-w_1^2+w_2^2)\\ -w_2 & w_1\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_1\end{bmatrix} = F(w)\begin{bmatrix}\omega_1\\ \omega_1\end{bmatrix} \quad (16)$$

If $[\dot{w}_1\ \dot{w}_2\ \dot{z}]^T$ is in the range of $F(w)$, we can solve the previous equation uniquely for the angular velocities

$$\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \left(F^T(w)F(w)\right)^{-1}F^T(w)\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{z}\end{bmatrix} = \frac{4}{(1+w_1^2+w_2^2)^2}\times \begin{bmatrix}\frac{1}{2}(1+w_1^2-w_2^2) & w_1w_2 & -w_2\\ w_1w_2 & \frac{1}{2}(1-w_1^2+w_2^2) & w_1\end{bmatrix}\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{z}\end{bmatrix} \quad (17)$$

Note that in case $[\dot{w}_1\ \dot{w}_2\ \dot{z}]^T$ is not in the range of $F(w)$, the previous equation solves the minimum distance problem to the range of $F(w)$.

We now return to the characterization of the flat outputs of system (15).

Proposition 1 *The kinematic model of an underactuated rigid body described by Eqs. (15) is differentially flat.*

Proof. Consider the following two functions

$$y_1 = 2\arctan\left(\frac{w_2}{w_1}\right)+z \quad (18a)$$
$$y_2 = z \quad (18b)$$

We claim that these are flat outputs for the system in Eqs. (15).

First note that, trivially, z can be written as a function of y_1 and y_2. Differentiating Eq. (18a) we get

$$\dot{y}_1 = \frac{1-|w|^2}{|w|^2}\dot{y}_2 + \dot{y}_2 = \frac{\dot{y}_2}{|w|^2} \quad (19)$$

or that,

$$|w|^2 = \frac{\dot{y}_2}{\dot{y}_1} \quad (20)$$

Moreover, we have that

$$\arctan\left(\frac{w_2}{w_1}\right) = \frac{y_1-y_2}{2} \quad (21)$$

The previous two equations together imply that

$$w_1 = \sqrt{\frac{\dot{y}_2}{\dot{y}_1}}\cos\left(\frac{y_1-y_2}{2}\right) \quad (22a)$$
$$w_2 = \sqrt{\frac{\dot{y}_2}{\dot{y}_1}}\sin\left(\frac{y_1-y_2}{2}\right) \quad (22b)$$

which, together with equation,

$$z = y_2 \quad (23)$$

provide the desired result.

We have shown that w_1,w_2 and z can be written as algebraic functions of y_1,y_2 and their time derivatives. By virtue of Eq. (17) ω_1 and ω_2 can also be written as functions of y_1,y_2 and their time derivatives. Therefore, y_1 and y_2 are flat outputs for the system in Eqs. (15), as claimed. ■

The initial and final points of the trajectory correspond to the points

$$y_{10} = 2\arctan\left(\frac{w_{20}}{w_{10}}\right)+z_0, \quad y_{20}=z \quad (24a)$$
$$y_{1f} = 2\arctan\left(\frac{w_{2f}}{w_{1f}}\right)+z_f, \quad y_{2f}=z_f \quad (24b)$$

in the y_1-y_2 plane, respectively. We can now construct paths in the y_1-y_2 plane connecting the points (y_{10}, y_{20}) and (y_{1f}, y_{2f}) and map them back to the $w-z$ state space using Eqs. (22) and (18b). We may choose any path we want, as long as $\dot{y}_1\dot{y}_2 \geq 0$. One way to achieve this is as follows: Assume a linear dependence of y_1 with time

$$y_1(t) = y_{10} + \frac{t}{t_f}(y_{1f}-y_{10}) \quad (25)$$

and then parameterize y_2 as a cubic function of y_1

$$y_2 = a_0 + a_1y_1 + a_2y_1^2 + a_3y_1^3 \quad (26)$$

Because the output y_2 is parameterized in terms of y_1, we call y_1 the "independent" flat output. The previous parameterization implies

$$y_2(0) = y_2(y_1(0)) = a_0 + a_1y_1(0) + a_2y_1^2(0) + a_3y_1^3(0) \quad (27a)$$
$$y_2(t_f) = y_2(y_1(t_f)) = a_0 + a_1y_1(t_f) + a_2y_1^2(t_f) + a_3y_1^3(t_f) \quad (27b)$$

From Eq. (20), the boundary conditions at $t=0$ and $t=t_f$ also imply the extra constraints

$$
\begin{aligned}
\frac{dy_2}{dy_1}(0) &= a_1+2a_2y_1(0)+3a_3y_1^2(0) \\
&= w_{10}^2+w_{20}^2=|w(0)|^2 && \text{(28a)} \\
\frac{dy_2}{dy_1}(t_f) &= a_1+2a_2y_1(t_f)+3a_3y_1^2(t_f) \\
&= w_{1f}^2+w_{2f}^2=|w(t_f)|^2 && \text{(28b)}
\end{aligned}
$$

We have a linear system of four equations (27)-(28) in the four unknowns a_0,a_1,a_2,a_3. In order to ensure that $y_2'(y_1)\geq 0$ we take advantage of the ambiguity of the $\arctan(\cdot)$ function in Eqs. (18). First, and without loss of generality we assume that $y_{1f}>y_{10}$ and $y_{2f}\geq y_{20}$. Otherwise, we can add or subtract multiples of 4π to y_{i0} and/or y_{if} $(i=1,2)$ to make sure that the previous inequalities hold.

From Eq. (26) we have that $y_2'(y_1)=0$ whenever

$$
y_1^*=\frac{-a_2\pm\sqrt{a_2^2-3a_1a_3}}{3a_3} \tag{29}
$$

By adding multiples of 4π to y_{2f} one can ensure that $y_1^*\notin(y_{10},y_{1f})$. Since $y_2'(y_1)(0)\geq 0$ it follows that $y_2'(y_1)(t)\geq 0$ for all $t\in[0,t_f]$[‡].

Remark 1 *One can choose many different paths connecting (y_{10},y_{20}) and (y_{1f},y_{2f}) in the y_1-y_2 plane, as long as they satisfy the boundary conditions in Eqs. (27) and (28). A cubic polynomial is the lowest degree polynomial which satisfies the four boundary conditions in Eqs. (27)-(28). Since the cubic polynomial is completely determined by these boundary conditions, there is no extra freedom to satisfy the slope constraint $y_2'(y_1)\geq 0$. One could have chosen a higher order polynomial and use the extra degrees of freedom to satisfy the slope constraint. However, this is a task with no easy analytical answer. Here we have chosen the simplest case of a cubic polynomial, and we have addressed the slope restriction by taking advantage of the fact that multiples or 2π correspond to the same angle (i.e., the same physical orientation of the body).*

Once the time functions $y_1(t)$ and $y_2(t)$ are known from the algorithm above, one can compute $\dot w_1$ and $\dot w_2$ and $\dot z$ from

$$
\begin{aligned}
\dot w_1 &= \frac{1}{2}\left(\frac{\dot y_1}{\dot y_2}\right)^{\frac12}\left(\frac{\ddot y_2}{\dot y_1}-\frac{\dot y_2\ddot y_1}{\dot y_1^2}\right)\cos\left(\frac{y_1-y_2}{2}\right) \\
&\quad -\left(\frac{\dot y_2}{\dot y_1}\right)^{\frac12}\sin\left(\frac{y_1-y_2}{2}\right)\frac{\dot y_1-\dot y_2}{2} && \text{(30a)} \\
\dot w_2 &= \frac{1}{2}\left(\frac{\dot y_1}{\dot y_2}\right)^{\frac12}\left(\frac{\ddot y_2}{\dot y_1}-\frac{\dot y_2\ddot y_1}{\dot y_1^2}\right)\sin\left(\frac{y_1-y_2}{2}\right) \\
&\quad +\left(\frac{\dot y_2}{\dot y_1}\right)^{\frac12}\cos\left(\frac{y_1-y_2}{2}\right)\frac{\dot y_1-\dot y_2}{2} && \text{(30b)} \\
\dot z &= \dot y_2 && \text{(30c)}
\end{aligned}
$$

where $\dot y_1\neq 0$ because of Eq. (25). The previous approach guarantees that the vector $[\dot w_1\ \dot w_2\ \dot z]^T$ is in the range of $F(w)$ for every $t\in[0,t_f]$. Therefore, the corresponding angular velocities can be computed from Eq. (17).

Equations (30) have a potential singularity at $\dot y_2=0$. By the previous discussion it is clear that this can happen only at the boundary points of the interval $[0,t_f]$. Indeed, if either $|w(0)|=0$ or $|w(t_f)|=0$, then from Eq. (20) we must have necessarily $\dot y_2(0)=0$ or $\dot y_2(t_f)=0$, respectively. From Eqs. (30) it is clear that in such a case we need to impose the additional constraint that $\ddot y_2(0)=0$ or $\ddot y_2(t_f)=0$.

Let us first consider the case when $|w(0)|=0$. A simple calculation shows that

$$
\ddot y_2(0)=(2a_2+6a_3y_1(0))\dot y_1^2(0) \tag{31}
$$

We can therefore guarantee that $\ddot y_2(0)=0$ by choosing a function $y_1(t)$ such that $\dot y_1(0)=0$. For instance, we can choose

$$
y_1(t)=y_{10}+\left(\frac{t}{t_f}\right)^2(y_{1f}-y_{10}) \tag{32}
$$

Similarly, for the case when $|w(t_f)|=0$ we can choose the following time parameterization for y_1

$$
y_1(t)=y_{1f}+\left(\frac{t-t_f}{t_f}\right)^2(y_{10}-y_{1f}) \tag{33}
$$

which guarantees that $\dot y_1(t_f)=0$ and hence from Eq. (31) also that $\ddot y_2(t_f)=0$.

Finally, for the case when the initial and final conditions are such that $|w(0)|=|w(t_f)|=0$, we can choose the following (cubic) parameterization of y_1

$$
y_1(t)=2(y_{10}-y_{1f})\left(\frac{t}{t_f}\right)^3-3(y_{10}-y_{1f})\left(\frac{t}{t_f}\right)^2+y_{10} \tag{34}
$$

This expression guarantees that $\dot y_1(0)=\dot y_1(t_f)=0$, hence also that $\ddot y_2(0)=\ddot y_2(t_f)=0$, as required. Notice that the parameterizations of $y_1(t)$ given in Eqs. (32), (33) and (34) ensure that $\dot y_1(t)\neq 0$ for all $t\in(0,t_f)$.

‡A proof of this result is shown in the appendix.

Summarizing, we have shown that a linear parameterization for "independent" flat output $y_1(t)$ along with a cubic parameterization of y_2 in terms of y_1 can be used to solve the trajectory generation problem in the flat output space for the majority of cases. With a linear parameterization of y_1, singularities may occur at the initial and/or final points, if $|w| = 0$ at these points. If this is the case, a quadratic or cubic parameterization of y_1 can be used to circumvent the singularity problem at the boundary points.

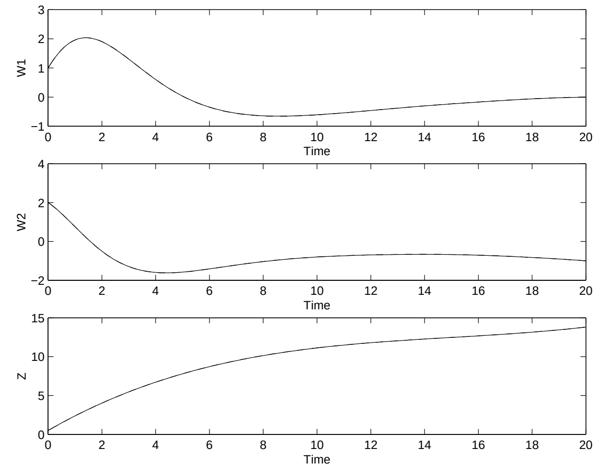
Numerical example

We demonstrate the algorithm for automatic feasible trajectory generation developed in the previous section. We assume that the initial and final conditions are given by $(w_1(0), w_2(0), z(0)) = (1, 2, 0.5)$ and $(w_1(t_f), w_2(t_f), z(t_f)) = (0, -1, 1.25)$, respectively. We also choose $t_f = 20$ sec. The corresponding initial and final points in the $y_1 - y_2$-plane of the flat outputs are calculated by the proposed algorithm as $(y_{10}, y_{20}) = (2.71, 0.5)$ and $(y_{1f}, y_{2f}) = (10.67, 13.81)$. Notice that in this case $y_{1f} = 2\,\mathrm{atan}[w_2(t_f)/w_1(t_f)] + z(t_f) + 4\pi$ and $y_2(t_f) = 4\pi + z(t_f)$. Figure 2(a) shows the trajectories in the $w - z$ space, and Fig. 2(b) shows the corresponding angular velocity history which generates these trajectories. In Fig. 2(a) there are actually plotted two separate sets of trajectories. One set is generated directly from the flat outputs, i.e., from Eqs. (22) and (23), and the other set is generated directly from the dynamical equations (6) subject to the angular velocity history in Fig. 2(b). The two sets are almost exact so there is no visible discrepancy in Fig. 2(a). Figure 3 shows the corresponding path in the flat output space.

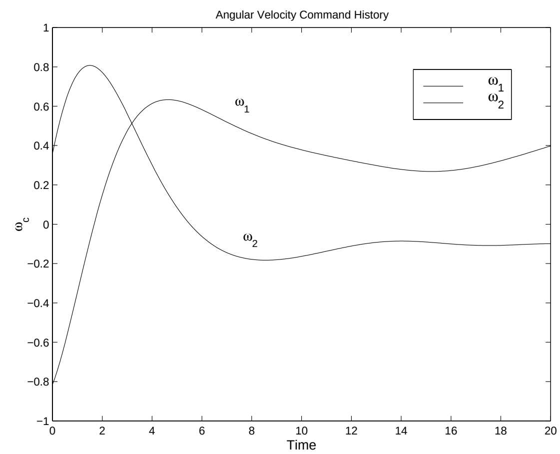
Consider now the case when $(w_1(t_f), w_2(t_f), z(t_f)) = (0, 0, 1.25)$. The initial conditions remain the same as in the previous case. If we use the linear parameterization for y_1 given in Eq. (25) we get the results in Fig. 4. The dashed lines in Fig. 4(a) correspond to the trajectories as given directly by the flat outputs, and the solid lines correspond to the trajectories as given by integrating the system of differential equations using the angular velocity history in Fig. 4(b). Notice that although the trajectory generated by the flat output approach matches very closely the one generated by the dynamical system, the angular velocity history requires large values at the final point. In particular, because of the singularity at that point, we get that $\lim_{t\to t_f} \dot{\omega}(t) = \infty$.

By using the quadratic parameterization of $y_1(t)$ in Eq. (33) we get the results in Fig. 5. The trajectories are essentially the same with the previous ones (since the path in the flat output space remains the same) but the angular velocity history is much better behaved. In particular, the singularity at final time has been eliminated completely.

Finally, we consider the case when $(w_1(0), w_2(0), z(0)) = (0, 0, 1.5)$ and



a) Trajectories in the (w, z) space.



b) Corresponding angular velocity commands.

Fig. 2 Feasible trajectory generation.

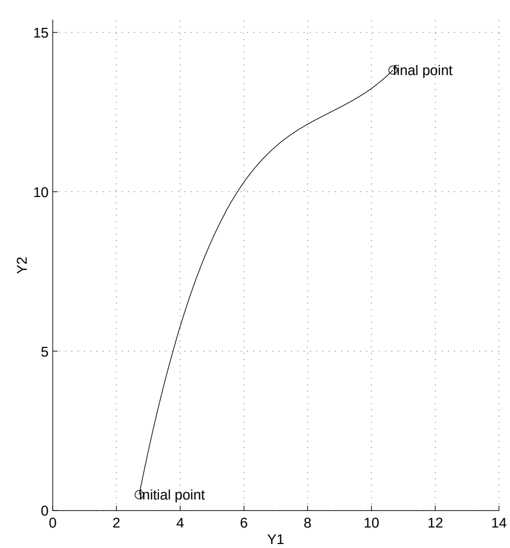
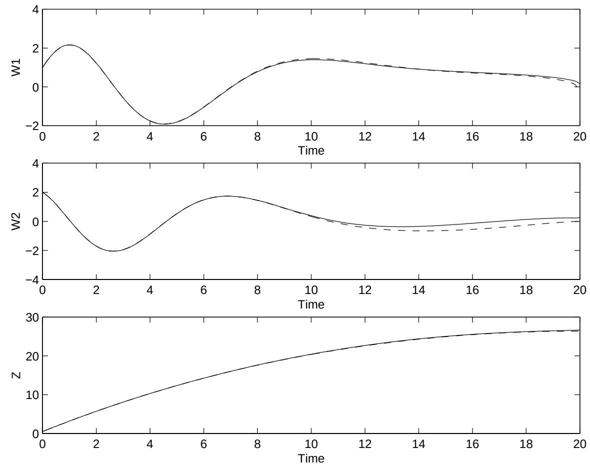
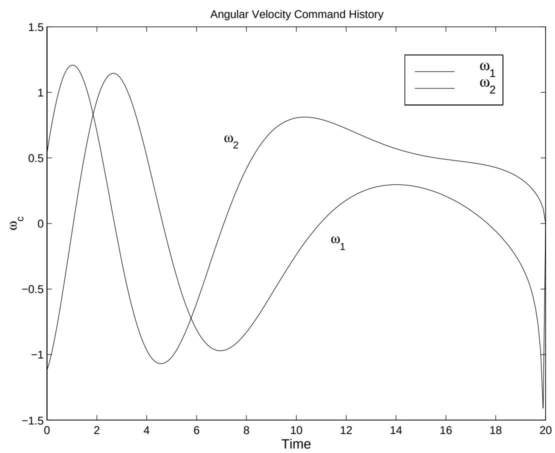


Fig. 3 Corresponding trajectories in the flat output space.

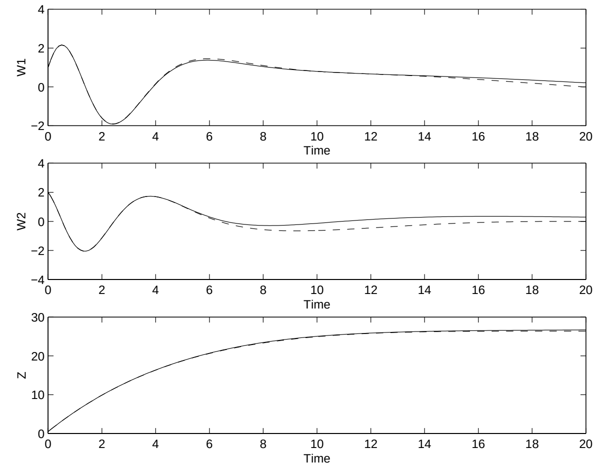


a) Trajectories generated using Eq. (25).

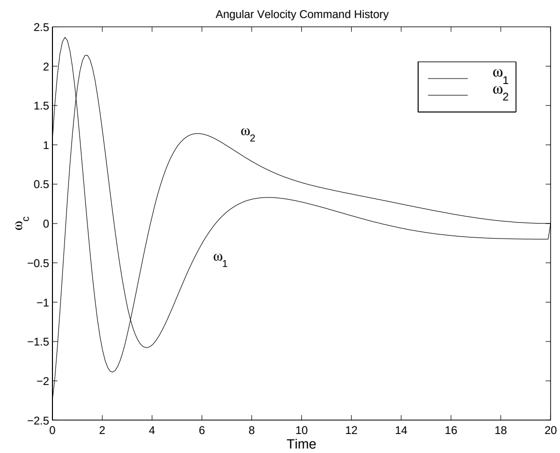


b) Corresponding angular velocity history.

Fig. 4 Feasible trajectory generation for $|w(t_f)| = 0$.



a) Trajectories generated using Eq. (33).



b) Corresponding angular velocity history.

Fig. 5 Feasible trajectory generation for $|w(t_f)| = 0$.

$(w_1(t_f), w_2(t_f), z(t_f)) = (0,0,0)$. The trajectories using the linear parameterization are shown in Fig. 6(a) and the corresponding control inputs are shown in Fig. 6(b). The dashed lines in Fig. 6(a) correspond to the trajectories as given directly by the flat outputs, and the solid lines correspond to the trajectories as given by the integrating the system of differential equations using the angular velocity history in Fig. 6(b). Notice that although the trajectory generated by the flat output approach can be generated very closely by the dynamical system, the angular velocity history requires large values at both the initial and final points.

By using the cubic parameterization of $y_1(t)$ in Eq. (34) we get the results in Fig. 7. The trajectories are essentially the same with the previous ones, but the angular velocity history is much better behaved, especially at the initial and final time.

Robustness to Small Asymmetries

The results presented thus far are valid assuming that $\omega_3 \equiv 0$. This can only occur when the initial condition $\omega_3(0)$ is zero and the body is completely axisymmetric about the $\hat{b}_3$ axis. This is a restrictive and rather unrealistic assumption. If the initial condition $\omega_3(0) \neq 0$ or if the body is "almost" axi-symmetric, then a small residual value ω_3 always acts on the system. The differential equations in this case can be written as

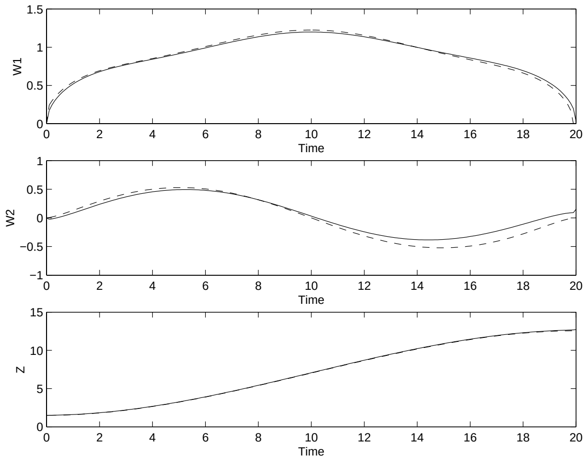
$$\dot{\omega}_3 = e\,\omega_1\omega_2 \tag{35a}$$

$$\dot{w}_1 = \omega_3 w_2 + \frac{1}{2}(1 + w_1^2 - w_2^2)\,\omega_1 + w_1 w_2 \omega_2 \tag{35b}$$

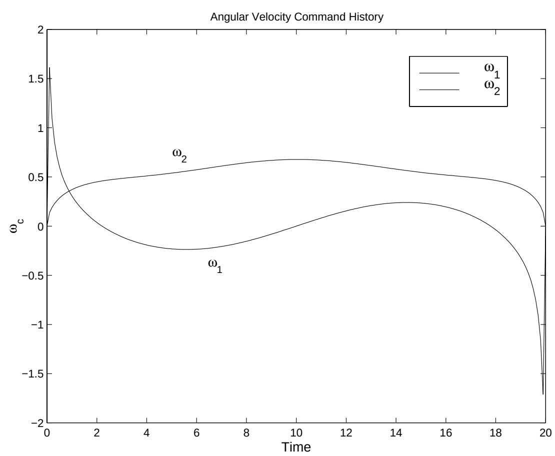
$$\dot{w}_2 = -\omega_3 w_1 + \frac{1}{2}(1 - w_1^2 + w_2^2)\,\omega_2 + w_1 w_2 \omega_1 \tag{35c}$$

$$\dot{z} = \omega_3 + w_1\omega_2 - w_2\omega_1 \tag{35d}$$

where $e << 1$ is a constant which depends on the principle moments of inertia of the body, and captures the effect of the asymmetry about the $\hat{b}_3$ axis. If $e = 0$ the

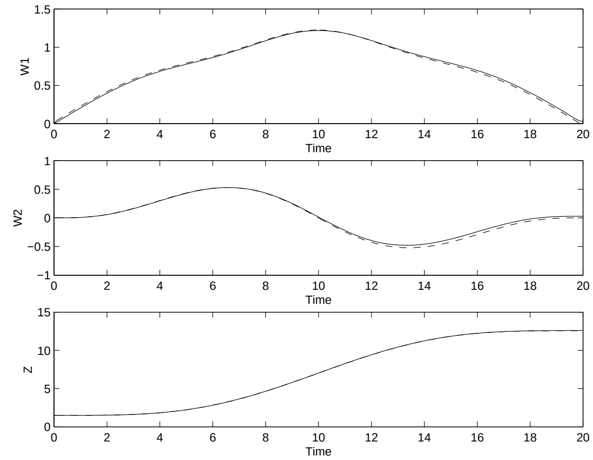


a) Trajectories generated using Eq. (25).

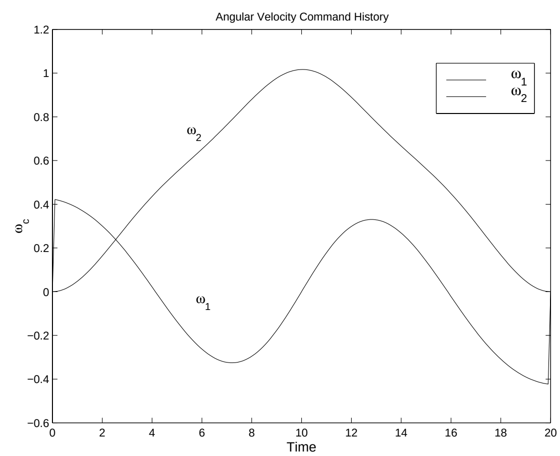


b) Corresponding angular velocity history.

Fig. 6 Feasible trajectory generation for $|w(0)| = |w(t_f)| = 0$.



a) Trajectories generated using Eq. (34).



b) Corresponding angular velocity history.

Fig. 7 Feasible trajectory generation for $|w(0)| = |w(t_f)| = 0$.

body is axi-symmetric about the $\hat{b}_3$ axis.

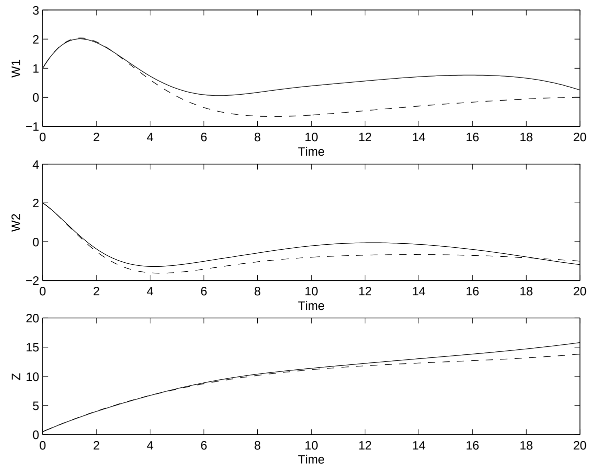
Whether the system in Eq. (35) is flat is an open problem. Here we will investigate via numerical simulations the effect of small values of ω_3 on the trajectory generation approach developed in the previous sections. To this end, let $e = 0.1$ and $\omega_3(0) = 0$. The initial and final conditions for w and z are given as $(w_1(0), w_2(0), z(0)) = (1, 2, 0.5)$ and $(w_1(t_f), w_2(t_f), z(t_f)) = (0, -1, 1.25)$. The results of the simulations are shown in Fig. 8(a). One should compare these results with the corresponding ones for the axi-symmetric case in Fig. 2(a). The dashed lines denote the nominal (ideal) trajectory that should be followed, and the solid lines indicate the actual trajectory. Figure 8(b) shows the variation of ω_3 with time.

In the second case we let $e = 0$ and $\omega_3(0) = 0.1$. In this case, although the body is axi-symmetric, the component ω_3 is nonzero (constant) because of nonzero initial conditions. The results of the simulations are shown in Fig. 9.

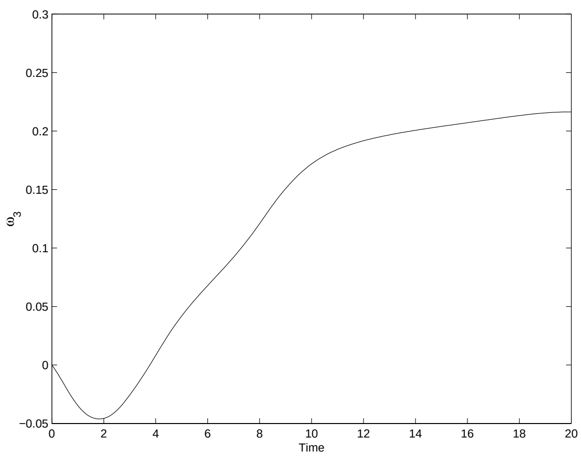
The third case combines the effects of non-symmetry and non-zero initial conditions in ω_3. We let $e = 0.1$ and $\omega_3(0) = 0.1$. The results are shown in Fig. 10(a). Figure 10(b) shows the corresponding variation of ω_3 with time.

These simulations show that small asymmetries or small non-zero initial conditions do not have a catastrophic effect on the final maneuver orientation. Nevertheless, the errors are non-negligible for accurate precision targeting and pointing maneuvers. It is therefore imperative to develop similar algorithms for the general (non axi-symmetric) underactuated rigid body case. In light of the absence of a characterization of the flat outputs (if they exist) this case seems to be a very challenging one, and it is left to future investigation.

Nevertheless, an *ad hoc* procedure that seems to be working well for small asymmetries is to correct the values of $\dot{w}_1, \dot{w}_2, \dot{z}$ in Eq. (17) by subtracting the cor-



a) Trajectories in the (w,z) space.



b) Time history for ω_3.

Fig. 8 Feasible trajectory generation for nearly axi-symmetric case ($e = 0.1$) and $\omega_3(0) = 0$.

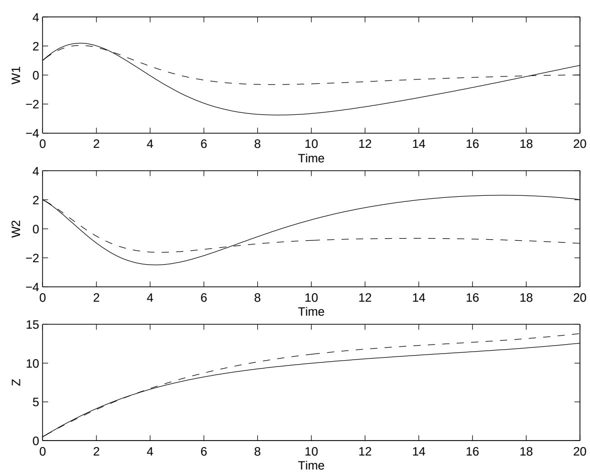
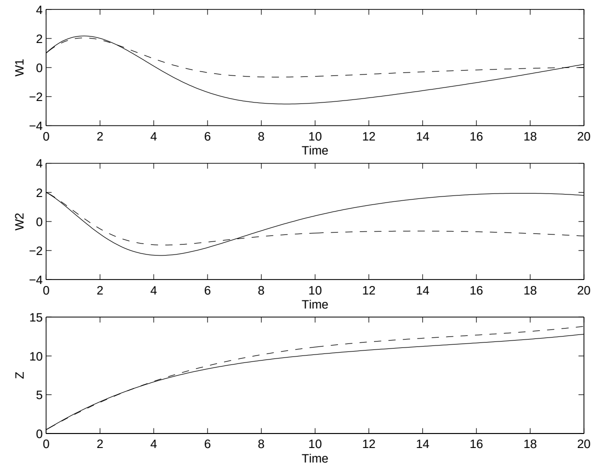
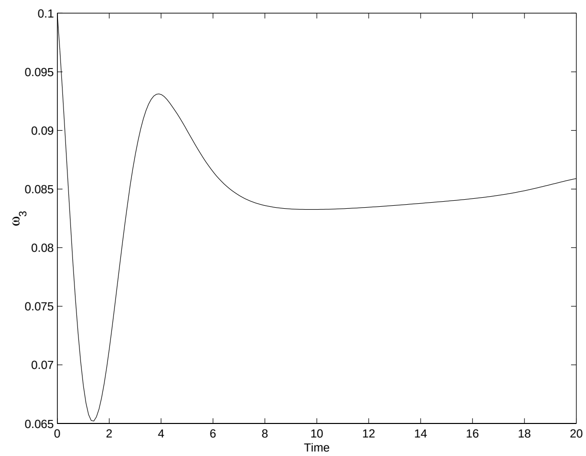


Fig. 9 Feasible trajectory generation for axi-symmetric case ($e = 0$) and nonzero initial conditions in ω_3 ($\omega_3(0) = 0.1$).



a) Trajectories in the (w,z) space.



b) Time history for ω_3.

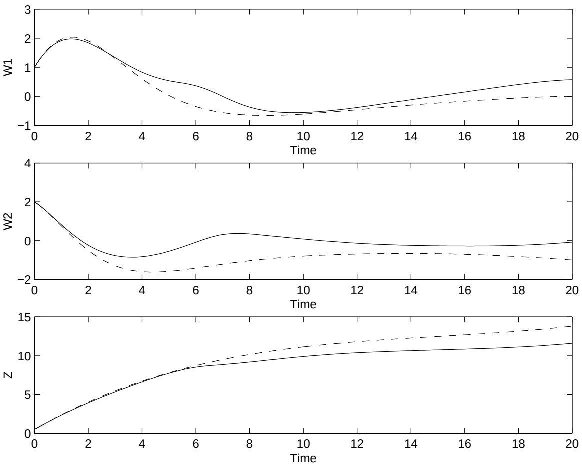
Fig. 10 Feasible trajectory generation for nearly axi-symmetric body ($e = 0.1$) and nonzero initial conditions in ω_3 ($\omega_3(0) = 0.1$).

responding terms due to ω_3 from the lhs of Eq. (35b)-(35d). Figures 11(a) and 11(b) show the simulations using this correction for $e = 0.2$ and $\omega_3(0) = 0$.

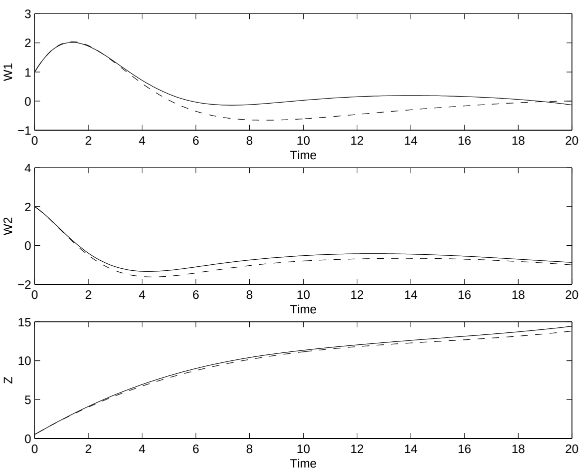
Conclusions

In this paper we provide an algorithm for solving the problem of feasible trajectory generation for an underactuated rigid spacecraft. The spacecraft is underactuated in the sense that there is no control authority along one of its principal axis. An example of this situation is the case of an axi-symmetric rigid spacecraft with a thruster failure along the symmetry axis.

We derive the necessary and sufficient conditions that must be satisfied for a state trajectory to be feasible. We show that the system is differentially flat by deriving the corresponding flat outputs. The feasible trajectory design is then performed in the flat output space. We finally propose a simple methodology for designing trajectories in the flat output space that are everywhere non-singular. Singularities man-



a) Trajectories in the (w,z) space without correction.



b) Trajectories in the (w,z) space with correction.

Fig. 11 Feasible trajectory generation for nearly axi-symmetric body ($e = 0.2$) and $\omega_3(0) = 0$. Original and *ad hoc* correction results.

ifest themselves as points where the derivative of the angular velocity history goes to infinity. This can occur at the initial and final points of the trajectory. The proposed approach avoids any singularities by simple time-reparameterization of one of the flat outputs (the "independent" flat output.) Singularity avoidance is very important because any angular velocity history must be implemented through appropriate torque histories. It is therefore imperative to ensure that any angular velocity used to generate the feasible trajectories is bounded and has a bounded derivative.

The feasible trajectories generated by the proposed approach can be used as reference trajectories for tracking problems. The whole approach can be readily automated and can thus be used for autonomous, on-line trajectory generation and tracking.

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Appendix

Consider the curve in the $y_1 - y_2$ plane given by

$$y_2 = a_0 + a_1 y_1 + a_2 y_1^2 + a_3 y_1^2 \qquad \text{(A.1)}$$

where a_0, a_1, a_2, a_3 are given by the solution to the following linear set of equations

$$\begin{bmatrix} 1 & y_{10} & y_{10}^2 & y_{10}^3 \\ 1 & y_{1f} & y_{1f}^2 & y_{1f}^3 \\ 0 & 1 & 2y_{10} & 3y_{10}^2 \\ 0 & 1 & 2y_{1f} & 3y_{1f}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_4 \end{bmatrix} = \begin{bmatrix} y_{20} \\ y_{2f} \\ w_0^2 \\ w_f^2 \end{bmatrix}, \qquad \text{(A.2)}$$

and where $y_{10}, y_{1f}, y_{20}, w_0^2$ and w_f^2 are positive constants, with $y_{1f} > y_{10}$.

We will show that, under these conditions, there always exist a y_{2f} large enough (positive), such that

$$\frac{\mathrm{d}y_2}{\mathrm{d}y_1} \geq 0 \qquad \text{(A.3)}$$

To this end, calculation of the previous derivative gives

$$y_2' = a_1 + 2a_2 y_1 + 3a_3 y_1^2 \qquad \text{(A.4)}$$

Clearly,

$$\min y_2' = y_2(y_1^*) = a_1 - \frac{a_2^2}{3a_3} \qquad \text{(A.5)}$$

where y_1^* as in Eq. (29). Since by assumption, $y_2'(y_{10}) > 0$ and $y_2'(y_{1f}) > 0$, then Eq. (A.3) holds if and only if $a_1 - a_2/3a_3 \geq 0$.

A tedious but straightforward calculation shows that a_1, a_2 and a_3 are linear functions of y_{2f} given by

$$a_1 = -6\frac{y_{10}y_{1f}}{(y_{1f} - y_{10})^3} y_{2f} + c_1 \qquad \text{(A.6a)}$$

$$a_2 = 3\frac{y_{10} + y_{1f}}{(y_{1f} - y_{10})^3} y_{2f} + c_2 \qquad \text{(A.6b)}$$

$$a_3 = -2\frac{1}{(y_{1f} - y_{10})^3} y_{2f} + c_3 \qquad \text{(A.6c)}$$

where c_1, c_2 and c_3 are constants independent of y_{2f}.

Substituting Eqs. (A.6) in the rhs of Eq. (A.5) and for large enough y_{2f}, one obtains that

$$a_1 - \frac{a_2^2}{3a_3} \approx \frac{3}{2(y_{1f} - y_{10})} y_{2f} + c \qquad \text{(A.7)}$$

where c is a constant independent of y_{2f}.

Since by assumption $y_{1f} - y_{10} > 0$, the last equation shows that for y_{2f} large enough we have that $y_2'(y_1) \geq 0$ and the proof is complete.