ASYMPTOTIC PROPERTIES OF HIGHER ORDER CAYLEY TRANSFORMS

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Abstract

In this short paper we generalize some previous results on attitude representations using higher order Cayley transforms. We show that the kinematic parameters generated by these higher order Cayley transforms have a a very simple limit, i.e., the well-known Euler vector.

Introduction

In a recent paper[1] we introduced the concept of Higher Order Cayley Transforms (HOCT) as a means of generating three-dimensional parameterizations of the rotation group $SO(3)$. Since $SO(3)$ is the configuration space of the rotational motion of a freely rotating body, these Cayley transforms can be used to generate new kinematic descriptions of the attitude motion. It is a classical result that the well-known Cayley-Rodrigues parameters (CRP's) can be generated by a first order Cayley transform[2,3]. In Ref. 4 and Ref. 1 we showed that a second order Cayley transform can be used to generate the (not so well-known) Modified Rodrigues parameters (MRP's)[5–11]. The advantages of the MRP's over the CRP's are discussed, for example, in Refs. 8,10,11 and they essentially stem from the fact that the MRP's are well-defined for all eigen-rotations $-2\pi < \phi < 2\pi$, whereas the CRP's can be used for describing eigen-rotations only in the interval $-\pi < \phi < \pi$. Thus, the MRP's have twice the domain of validity of the CRP's (measured in terms of the eigenaxis rotation angle ϕ). Higher order Cayley transforms can be used to increase the domain of ϕ even further thus, essentially, increasing the region of validity of the corresponding kinematic parameters.

In this note we investigate the asymptotic behavior of these higher order Cayley transforms and we show that, as the order of the transformation tends to infinity, the corresponding kinematic parameters tend to the so-called "Euler vector". This *may* provide a proof to the conjecture that the Euler vector provides the "best" three-dimensional attitude description in the sense that it has the smallest kinematic singularity measure*. In the process of showing this result, we provide an alternative – more straightforward – proof of the main result in Ref. 1, which reveals the connection between matrix transformations of the form in Eq. (3) and the corresponding parameters given in Eq. (13). Finally, we discuss the implications of these results to attitude kinematics problems.

Higher Order Cayley Transforms

As usual, let $so(n)$ denote the space of $n \times n$ skew symmetric matrices† and let $SO(n)$ denote the space of all $n \times n$ proper orthogonal matrices‡. The standard Cayley transform parameterizes a proper orthogonal matrix $C \in SO(n)$ as a function of a skew-symmetric matrix $Q \in so(n)$ via

$$C = (I-Q)(I+Q)^{-1} = (I+Q)^{-1}(I-Q) \qquad (1)$$

The Cayley transform is invertible and its inverse is the transformation itself

$$Q = (I-C)(I+C)^{-1} = (I+C)^{-1}(I-C) \qquad (2)$$

More information on the classical Cayley transform for the 3-dimensional case and its use in the description of the attitude motion can be found in Ref. 2.

In Ref. 1, drawing on some insightful comments by Halmos[13], we interpreted Eq. (1) as a "conformal mapping" in the space of matrices. This allowed a generalization of Eq. (1) to higher order and the introduction of 'Higher Order Cayley Parameters" (HOCP's) via the series of "Higher Order Cayley Transforms" (HOCT) defined by

$$C = (I-Q)^m(I+Q)^{-m} = (I+Q)^{-m}(I-Q)^m \qquad (3)$$

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*The kinematic singularity measure of a 3-dimensional parameterization of $SO(3)$ is defined in Ref. 12 as the ratio of all possible configurations over the non-singular configurations.

†That is, $so(n) = \{A \in \mathbb{R}^{n\times n} : A = -A^T\}$.

‡That is, $SO(n) = \{A \in \mathbb{R}^{n\times n} : AA^T = I, \ \det(A) = +1\}$.

for $m = 1,2,3,\ldots$, where $Q \in so(n)$. Note that the mapping in Eq. (3) is not one-to-one and there are multiple solutions for the inverse of Eq. (3). The inverse mapping of Eq. (3) and its domain of validity have been analyzed in Ref. 1 with great detail.

CRP's, MRP's and HOCP's

The following discussion will concentrate on the three-dimensional case ($n = 3$) which is of particular interest in attitude kinematics. The main objective of this paper is to investigate the limit of Eq. (3) as $m \to \infty$. In the process, we present an alternative – more direct and rigorous – proof of the results in Ref. 1.

Let $[\,\cdot\,] : \mathbb{R}^3 \to so(3)$ denote the standard isomorphism between the 3-dimensional skew symmetric matrices and the three-dimensional vectors defined by

$$[\mathbf{x}] = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ -x_1 & x_2 & 0 \end{bmatrix} \qquad \mathbf{x} \in \mathbb{R}^3 \tag{4}$$

It is well known that Euler's theorem parameterizes $SO(3)$ via the *Euler vector*

$$\mathbf{v} = \phi\,\hat{\mathbf{e}} \tag{5}$$

where ϕ is the principal angle and $\hat{\mathbf{e}}$ is the principal (eigenaxis) *unit* vector. Specifically, given any $C \in SO(3)$, there exists a vector $\mathbf{v}$ (equivalently, an angle ϕ and a unit vector $\hat{\mathbf{e}}$) such that

$$C = e^{[\mathbf{v}]} = e^{\phi[\hat{\mathbf{e}}]} \tag{6}$$

This equation will be central to the subsequent developments.

In order to proceed, recall that given an analytic function $F(z)$ inside a disk of radius r in the complex plane, and a square matrix A, one can unambiguously define the *matrix-valued function* $F(A)$ from the Taylor series of $F(z)$[14,15]. That is, if $F(z) = \sum_{i=0}^{\infty} \alpha_i z^i$, ($|z| \leq r$) then $F(A) = \sum_{i=0}^{\infty} \alpha_i A^i$. The series for $F(A)$ converges (and thus the matrix $F(A)$ is well defined) if the eigenvalues of A lie inside the disk of radius r in the complex plane. It is this fundamental result from linear algebra that ensures that "intuitive" expressions like $\sin(A)$, $\cos(A)$, $\exp(A)$, $\tanh(A)$ etc., actually make sense.

The results of this note are based on the following trigonometric identity which can be easily verified by the interested reader

$$\left(\frac{1 + \tanh\left(\dfrac{x}{2m}\right)}{1 - \tanh\left(\dfrac{x}{2m}\right)} \right)^m = e^x, \qquad x \in \mathbb{R} \tag{7}$$

for $m = 1,2,3,\ldots$. We emphasize the fact that the previous identity holds for *all* $m = 1,2,3,\ldots$.

Let now the skew-symmetric matrix $[\mathbf{v}]$. According to the previous discussion, it is clear that Eq. (7) implies that

$$\left(I + \tanh\left[\frac{\mathbf{v}}{2m}\right] \right)^m \left(I - \tanh\left[\frac{\mathbf{v}}{2m}\right] \right)^{-m} = e^{[\mathbf{v}]} \tag{8}$$

for $m = 1,2,3,\ldots$, where I denotes the identity matrix. In the Appendix it is shown that

$$\tanh[\mathbf{v}] = \tanh[\phi\,\hat{\mathbf{e}}] = [\tan(\phi)\,\hat{\mathbf{e}}] \tag{9}$$

The last equation, along with Eq. (8), yield

$$\left(I + \left[\hat{\mathbf{e}} \tan\left(\frac{\phi}{2m}\right) \right] \right)^m \left(I - \left[\hat{\mathbf{e}} \tan\left(\frac{\phi}{2m}\right) \right] \right)^{-m} = e^{[\phi\,\hat{\mathbf{e}}]} = e^{\phi[\hat{\mathbf{e}}]} \qquad m = 1,2,3,\ldots \tag{10}$$

For $m = 1$ we get the classical Cayley transform which states that $C = (I - R)(I + R)^{-1}$ where $R = -[\rho]$ is the skew-symmetric matrix defined in terms of the Cayley-Rodrigues parameters[§]

$$\rho = \hat{\mathbf{e}} \tan\frac{\phi}{2} \tag{11}$$

For $m = 2$ we get the second order Cayley transform[4,1] which states that $C = (I+S)^2(I-S)^{-2}$ where $S = -[\sigma]$ is the skew-symmetric matrix defined in terms of the Modified Rodrigues parameters

$$\sigma = \hat{\mathbf{e}} \tan\frac{\phi}{4} \tag{12}$$

Higher order parameters can be generated by taking increasingly larger values for m (see Ref. 1 for details).

Equation (10) makes it now clear how higher order Cayley transforms can be used to generate higher-order Cayley parameters defined by

$$\mathbf{p}_m = \hat{\mathbf{e}} \tan\left(\frac{\phi}{2m}\right) \qquad m = 3,4,\ldots \tag{13}$$

Moreover, it should be obvious why HOCT's increase the domain of validity and the linearity of the corresponding kinematic parameters (with respect to the principal angle ϕ).

[§]The minus sign in the definition of R is inconsequential and stems from our (arbitrary) convention for the isomorphism $\mathbf{x} \leftrightarrow [\mathbf{x}]$ which leads to $\mathbf{x} \times \mathbf{y} = -[\mathbf{x}]\,\mathbf{y}$. This, in turn, implies that the kinematic equations are given by $\dot{C} = [\omega]C$ where C is the rotation matrix from inertial to body frame and ω is the angular velocity in body coordinates.

Asymptotic Limit of the HOCP's

Simple observation of Eq. (13) begs the natural question: "What is the limit of these parameters as $m \to \infty$?" In this section we answer this question. To this end, recall the well-known formula from elementary calculus

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m = e \tag{14}$$

or, more generally,

$$\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = e^x \qquad x\in \mathbb{R} \tag{15}$$

which leads to the very useful formula

$$\lim_{m\to\infty}\left(\frac{1+\dfrac{x}{2m}}{1-\dfrac{x}{2m}}\right)^m = e^x \tag{16}$$

The last equation implies that

$$\lim_{m\to\infty}\left(I+[\frac{\mathbf{v}}{2m}]\right)^m\left(I-[\frac{\mathbf{v}}{2m}]\right)^{-m} = \lim_{m\to\infty}\left(I+[\hat{\mathbf{e}}\frac{\phi}{2m}]\right)^m\left(I-[\hat{\mathbf{e}}\frac{\phi}{2m}]\right)^{-m} = e^{[\mathbf{v}]} \tag{17}$$

Using now the fact that

$$\lim_{m\to\infty}\tan\left(\frac{\phi}{2m}\right) = \frac{\phi}{2m} \tag{18}$$

and Eq. (10), one immediately obtains that the asymptotic Cayley transform as $m \to \infty$ generates the (better, a scaled version of) Euler vector $\mathbf{v}$.

Remark: One could *formally* obtain the same result directly from Eqs. (13) and (18). However, strictly speaking, this would be incorrect, since passing the limit operation inside the parenthesis in Eq. (10) is not justified *a priori*. The derivation of the result via Eq. (17) circumvents this difficulty.

Kinematics

The kinematic equation for the Euler vector $\mathbf{v}$ has been analyzed extensively in the literature[16,17,9]. A more recent investigation of the differential equation satisfied by the Euler vector, along with applications in control problems appears in Ref. 18. According to Refs. 17 and 16,

$$\dot{\mathbf{v}} = G(\mathbf{v})\,\omega \tag{19}$$

where

$$G(\mathbf{v}) = k_1(\phi)\,I - \frac{1}{2}[\mathbf{v}] + k_2(\phi)\mathbf{v}\mathbf{v}^T \tag{20}$$

and

$$k_1(\phi) = \frac{\phi}{2}\cot\left(\frac{\phi}{2}\right), \qquad k_2(\phi) = \frac{1-k_1(\phi)}{\phi^2} \tag{21}$$

We next show that the kinematic equation (19) has no finite escape times. In other words, for every time t_c such that $\phi(t_c) = 2k\pi$, the limits $\lim_{t\to t_c^+}\mathbf{v}(t) = \mathbf{v}(t_c^+)$ and $\lim_{t\to t_c^-}\mathbf{v}(t) = \mathbf{v}(t_c^-)$ exist and are bounded. The apparent singularity at $\phi = 2k\pi$, $k = 0,1,2,\ldots$ is thus removable and one can re-define the differential equation at the instances when $\phi = 2k\pi$ for $k = 1,2,3,\ldots$. This is in accordance with similar results in Ref. 1 for the CRP's and MRP's[¶]. Before proceeding, let us analyze the case when $\phi = 0$. Note that

$$\lim_{\phi\to 0} k_1(\phi) = \lim_{\phi\to 0}\frac{\phi}{2}\cot\left(\frac{\phi}{2}\right) = 1 \tag{22}$$

and

$$\begin{aligned}\lim_{\phi\to 0} k_2(\phi)\mathbf{v}\mathbf{v}^T &= \frac{1-k_1(\phi)}{\phi^2}\,\phi^2\,\hat{\mathbf{e}}\hat{\mathbf{e}}^T \\ &= \lim_{\phi\to 0}(1-k_1(\phi)) = 0\end{aligned} \tag{23}$$

Thus, $\lim_{\phi\to 0} G(\mathbf{v}) = I$ and for $\phi = 0$, one can substitute Eq. (19) with

$$\dot{\mathbf{v}} = \omega \tag{24}$$

For the case when $\phi = 2k\pi$, $k = 1,2,3,\ldots$, note that Eq. (19) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 &= \mathbf{v}^T\dot{\mathbf{v}} \\ &= \mathbf{v}^T G(\mathbf{v})\,\omega \\ &= k_1(\phi)\,\mathbf{v}^T\omega + k_2(\phi)\phi^2\mathbf{v}^T\omega \\ &= \mathbf{v}^T\omega \le \|\mathbf{v}\|\,\|\omega\|\end{aligned} \tag{25}$$

Here $\|\cdot\|$ denotes, as usual, the Euclidean norm in $\mathbb{R}^3$, i.e., $\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v}$ and the last relationship is the Cauchy-Schwarz inequality. Since ω is assumed to be bounded, Eq. (25) implies that the derivative of the magnitude of $\mathbf{v}$ is bounded for all $t \ge 0$ and, consequently, Eq. (19) has no finite escape times. Nevertheless, discontinuities in $\mathbf{v}$ may still occur at $\phi = 2k\pi$, $k = 1,2,3,\ldots$ since $\mathbf{v}(t_c^-) \ne \mathbf{v}(t_c^+)$ although $\|\mathbf{v}(t_c^-)\| = \|\mathbf{v}(t_c^+)\| = 2k\pi$.

To this end, let $t = t_c$ denote the time at which ϕ becomes equal to $2k\pi$ for $k = 1,2,3,\ldots$. To investigate more closely the behavior of Eq. (19) at $t = t_c$, we use the definition of the Euler vector, $\mathbf{v} = \hat{\mathbf{e}}\phi$ to obtain from Eq. (25) the well-known formula[16,18]

$$\dot{\phi} = \hat{\mathbf{e}}^T\omega \tag{26}$$

[¶]The argument used in Ref. 1 to prove no finite escape times at $\phi = 2\pi$ for the HOCP's is incorrect. The results remain valid, however, if one follows the approach used in this paper.

In case $\hat{\mathbf{e}}^T\omega \neq 0$ at $t = t_c$ then $\dot{\phi}(t_c) \neq 0$ and ϕ will cross transversally the singularity at $t = t_c$. As before, without loss of generality, one may substitute Eq. (19) with Eq. (24) at $t = t_c$.

In case $\hat{\mathbf{e}}^T\omega = 0$ at $t = t_c$, a discontinuity of $\mathbf{v}$ may occur due to change in the direction of $\hat{\mathbf{e}}$. The differential equation for $\hat{\mathbf{e}}$ can be easily computed as follows[16]

$$\dot{\hat{\mathbf{e}}} = \frac{1}{2}\left(\cot\left(\frac{\phi}{2}\right)I - [\hat{\mathbf{e}}]\right)\omega_\perp \tag{27}$$

where $\omega_\perp = (I - \hat{\mathbf{e}}\hat{\mathbf{e}}^T)\omega$ is the component of ω perpendicular to $\hat{\mathbf{e}}$. The component of ω along $\hat{\mathbf{e}}$ is clearly $\omega_\| = \hat{\mathbf{e}}\hat{\mathbf{e}}^T\omega$. If $\hat{\mathbf{e}}^T\omega = 0$ at $t = t_c$ the component of ω along the Euler vector is zero and $\omega = \omega_\perp$. In this case, the magnitude of $\mathbf{v}$ remains constant. It appears that the motion is trapped at a state where $\phi = 2k\pi$. Notice, however, that for $\phi = 2k\pi$, $k = 0, 1, 2, \ldots$, the rotation matrix C in Eq. (6) is independent of $\hat{\mathbf{e}}$. This can be easily seen from the equivalent expression for C[9] (Euler's formula)

$$C(\phi, \hat{\mathbf{e}}) = \cos\phi I + (1 - \cos\phi)\hat{\mathbf{e}}\hat{\mathbf{e}}^T + \sin\phi[\hat{\mathbf{e}}] \tag{28}$$

In this case one can use any suitable unit vector for $\hat{\mathbf{e}}$. Based on the additional fact that $\phi = 2k\pi$ for $k \neq 0$ and $\phi = 0$, correspond to the same physical orientation, one can choose $\hat{\mathbf{e}} = 0$ and still use Eq. (24) in this case. This observation also justifies Eq. (24) when $\hat{\mathbf{e}}^T\omega \neq 0$. It essentially ensures that one can choose $\hat{\mathbf{e}}$ along ω in this case.

Based on the previous discussion, any integration routine that handles discontinuities and/or stiff differential equations can be used to (reliably) integrate Eq. (19) over finite or *infinite* time intervals. On the contrary, CRP's MRP's and HOCP's have finite escape times, where the solution itself blows up. Besides, this is evident from the definition of these parameters in Eq. (13).

Remark: In practice, one may want to keep the magnitude of $\mathbf{v}$ between $-\pi \leq \phi \leq \pi$ since this domain is enough to describe any physical orientation. This can be achieved using a similar approach as the one described above, where now jump discontinuities in the direction of $\mathbf{v}$ need to be handled whenever $\phi = \pm\pi$[18].

Numerical Examples

In this section we demonstrate via numerical simulations the behavior of the kinematics at the singularities for the HOCP's as well as for the Euler vector. Figure 1 shows time simulations for three different cases of Cayley parameters. The case $m = 1$ (denoted by ρ) corresponds to the CRP's, the case $m = 2$ (denoted on the plot by σ) corresponding to the MRP's, and the case $m = 4$ (denoted by τ) corresponds to the parameters introduced in Ref. 1. The Euler vector is denoted by $\mathbf{v}$. The results in Fig. 1 represent an eigenaxis spin-up about the first body-axis with a linearly increasing angular velocity $\omega = (1 + t, 0, 0)$ rad/sec.

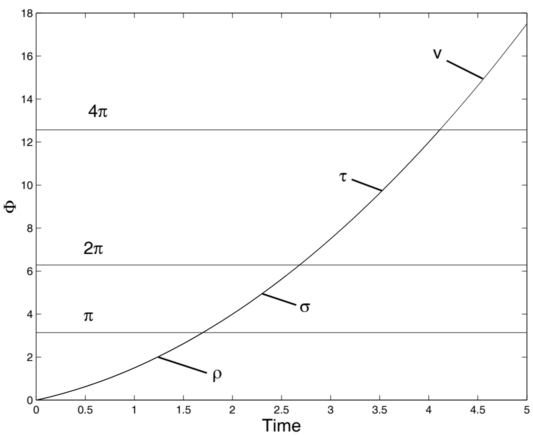

Figure 1: Comparison of several Cayley parameters.

On this plot we indicate the different instances of singularities during integration. The horizontal lines indicate where the integration of the corresponding kinematic equations quit because of the finite escape times of the corresponding parameters. In particular, propagation of ρ, σ and τ can be continued upto $\pi, 2\pi$ and 4π, respectively. No finite escape times occur for the Euler vector $\mathbf{v}$ which, in this case, is integrated upto approximately $\phi = 6\pi$.

These numerical results confirm the theoretical predictions. The numerical simulations were performed using the general purpose routine `ode45` of MATLAB.

Conclusions

In this short note we complete our previous work[1] by answering the question of the limiting behavior of the HOCT. The answer is surprisingly simple, since it is shown that, as the order of the HOCT tends to infinity, the corresponding kinematic parameters tend to the well-known Euler vector. This may provide an answer to the conjecture in Ref. 12 that the Euler vector is indeed the 3-dimensional kinematic parameterization with the smallest kinematic singularity measure.

Appendix

The series expansion for $\tanh(x)$ is given by[19]

$$\tanh(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n}{(2n)!} x^{2n-1}$$

$$= x - \frac{x^3}{3} + \frac{2}{15}x^5 - \cdots \quad |x| < \frac{\pi}{2} \qquad (A.1)$$

where B_n denote the Bernoulli numbers. One can therefore calculate $\tanh[\mathbf{v}]$ from

$$\tanh[\mathbf{v}] = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n}{(2n)!}[\mathbf{v}]^{2n-1} \qquad (A.2)$$

Direct calculation shows that $[\hat{\mathbf{e}}]^2 = \hat{\mathbf{e}}\hat{\mathbf{e}}^T - I$ and thus $[\hat{\mathbf{e}}]^3 = -[\hat{\mathbf{e}}]$. In general,

$$[\hat{\mathbf{e}}]^{2n-1} = (-1)^{n-1}[\hat{\mathbf{e}}] \qquad n = 1, 2, 3, \ldots \qquad (A.3)$$

Use of Eq. (A.3) in Eq. (A.2) yields

$$\begin{aligned} \tanh[\phi\hat{\mathbf{e}}] &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_n}{(2n)!}\phi^{2n-1}[\hat{\mathbf{e}}]^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}\phi^{2n-1}[\hat{\mathbf{e}}] \\ &= \tan(\phi)[\hat{\mathbf{e}}] = [\tan(\phi)\,\hat{\mathbf{e}}] \qquad (A.4) \end{aligned}$$

where we have made use of the series expansion of $\tan(x)$[19]

$$\tan(x) = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}x^{2n-1} \qquad (A.5)$$

Thus, we have shown that

$$\tanh[\mathbf{v}] = \tanh[\phi\hat{\mathbf{e}}] = [\tan(\phi)\,\hat{\mathbf{e}}] \qquad (A.6)$$

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