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TRACKING RIGID BODY MOTION USING THRUSTERS AND MOMENTUM WHEELS*

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Abstract

We develop tracking control laws for a rigid spacecraft using both thrusters and momentum wheels. The model studied comprises a rigid body with external thrusters and with N rigid axisymmetric wheels controlled by axial torques. The thruster torques and the axial motor torques are the controls used to track given attitude motions. Specifically, the thruster torques are used to implement the coarse tracking, and the momentum wheels are used to provide the fine control.

Introduction

In this paper, we develop a linear feedback controller that asymptotically stabilizes the motion of an N-rotor gyrostat. The controller is a logical extension to the controllers developed in Ref. 5.

Using a body-fixed reference frame, the rotational equations of motion for a rigid body with internal momentum wheels may be expressed as

$$\dot{\mathbf{h}} = \mathbf{h}^{\times}\mathbf{J}^{-1}(\mathbf{h}-\mathbf{A}\mathbf{h}_a)+\mathbf{g}_e \tag{1}$$
$$\dot{\mathbf{h}}_a = \mathbf{g}_a \tag{2}$$

where $\mathbf{h}$ is the system angular momentum, $\mathbf{h}_a$ is the $N\times 1$ matrix of the axial angular momenta of the rotors, $\mathbf{g}_e$ is the 3×1 matrix of external torques, $\mathbf{g}_a$ is the $N\times 1$ matrix of the internal axial torques applied by the platform to the rotors, $\mathbf{A}$ is the $3\times N$ matrix containing the axial vectors of the N rotors, and $\mathbf{J}$ is an inertia-like matrix defined as

$$\mathbf{J} = \mathbf{I}-\mathbf{A}\mathbf{I}_s\mathbf{A}^T \tag{3}$$

Here $\mathbf{I}$ is the angular momentum of the system, including the rotors, whereas $\mathbf{I}_s = \mathrm{diag}\{I_{s1},\cdots,I_{sN}\}$ is an $N\times N$ matrix with the axial moments of inertia of the rotors on the diagonal. The matrix $\mathbf{J}$ may be interpreted as the inertia matrix of an equivalent system where all the rotors have zero axial moment of inertia.

The angular velocity of the body frame may be written as

$$\boldsymbol{\omega} = \mathbf{J}^{-1}(\mathbf{h}-\mathbf{A}\mathbf{h}_a) \tag{4}$$

Observe that $\boldsymbol{\omega}$ may also be written as ∇H, where the ∇ is with respect to $\mathbf{h}$, and H is a Hamiltonian function. This formulation is especially useful for identifying relative equilibrium motions and for characterizing their stability.

The axial angular momenta of the rotors can be written in terms of the body angular velocity, $\boldsymbol{\omega}$, and the rotors' axial angular velocities relative to the body, $\boldsymbol{\omega}_s$:

$$\mathbf{h}_a = \mathbf{I}_s\mathbf{A}^T\boldsymbol{\omega}+\mathbf{I}_s\boldsymbol{\omega}_s \tag{5}$$

Note that $\boldsymbol{\omega}_s$ is an $N\times 1$ matrix, and that these relative angular velocities are those that would be measured by tachometers fixed to the platform.

The equations describing the kinematics of a reference frame may be given in several different forms. Here we choose to use the so-called "modified Rodrigues parameters," which are defined in terms of the Euler principal vector, $\mathbf{e}$, and angle, Φ, by

$$\boldsymbol{\sigma} = \mathbf{e}\tan(\Phi/4) \tag{6}$$

The kinematic differential equations are

$$\dot{\boldsymbol{\sigma}} = \mathbf{G}(\boldsymbol{\sigma})\boldsymbol{\omega} \tag{7}$$

where

$$\mathbf{G}(\boldsymbol{\sigma}) = \frac{1}{2}\left(\mathbf{1}+\boldsymbol{\sigma}^{\times}+\boldsymbol{\sigma}\boldsymbol{\sigma}^T-\frac{1+\boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\mathbf{1}\right) \tag{8}$$

Ignoring the external torques, $\mathbf{g}_e$, we consider the problem of stabilizing the origin using only internal torques. Note that using $\mathbf{g}_a$ as the controls is different from applying equivalent external torques, since the interaction of the rotors with the platform includes both axial and transverse (constraint) torques, and only the axial torques are being controlled. It would also be interesting to consider the effects of a constant, nonzero $\mathbf{h}_a$ on the applicability of previously developed external torque control laws, or to consider a combination of internal and external torques.

Following Tsiotras, we consider the candidate Liapunov function

$$V = \frac{1}{2}(\mathbf{h} - \mathbf{A}\mathbf{h}_a)^T \mathbf{K}(\mathbf{h} - \mathbf{A}\mathbf{h}_a) + k\ln\left(1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}\right) \quad (9)$$

where $\mathbf{K} = \mathbf{K}^T > 0$, and $k > 0$. Then

$$\dot{V} = \left(\dot{\mathbf{h}} - \mathbf{A}\dot{\mathbf{h}}_a\right)^T \mathbf{K}(\mathbf{h} - \mathbf{A}\mathbf{h}_a) + k\frac{\boldsymbol{\sigma}^T\mathbf{G}(\boldsymbol{\sigma})}{1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}}\boldsymbol{\omega} \quad (10)$$

Choosing $\mathbf{K} = \mathbf{J}^{-1}$, and using Eqs. (1 and 4), one can show that

$$\dot{V} = \left[-\left(\mathbf{A}\dot{\mathbf{h}}_a\right)^T + \frac{k}{4}\boldsymbol{\sigma}^T\right]\boldsymbol{\omega} \quad (11)$$

where we have made use of the identity

$$\boldsymbol{\sigma}^T\mathbf{G}(\boldsymbol{\sigma})\boldsymbol{\omega} = \frac{1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{4}\boldsymbol{\sigma}^T\boldsymbol{\omega} \quad (12)$$

Thus if we choose internal torques $\mathbf{g}_a = \dot{\mathbf{h}}_a$ so that

$$\mathbf{A}\mathbf{g}_a = k_1\boldsymbol{\omega} + k_2\boldsymbol{\sigma} \quad (13)$$

then

$$\dot{V} = -k_2\omega^2 \quad (14)$$

Since $\boldsymbol{\omega} = \mathbf{0}$ with $\boldsymbol{\sigma} \neq \mathbf{0}$ would mean the control torques would be nonzero, the fact that $\dot{V} = 0$ when $\boldsymbol{\omega} = \mathbf{0}$ causes no difficulty. The torque due to $\boldsymbol{\sigma} \neq \mathbf{0}$ would change $\mathbf{h}_a$, in turn channging $\mathbf{h}$ and $\boldsymbol{\omega}$. Thus V is a Liapunov function and the control stabilizes the origin.

One difficulty is that the controls are actually the internal torques $\mathbf{g}_a$, whereas Eq. (13 is a (possibly) non-square linear system for $\mathbf{g}_a$. If $N < 3$, then the system is overdetermined, and no solution may exist. If $N = 3$ and the rotors are not coplanar ($\mathbf{A}$ is nonsingular), then a unique solution exists. If $N > 3$ and the rotors are not coplanar ($\mathbf{A}$ has rank 3), then an infinite number of solutions exist. One interesting possibility is to use the torques in the nullspace of $\mathbf{A}$ for energy storage.

Results

In the two figures, we show the angular velocities and modified Rodriguez parameters for an example maneuver. It is evident that all six states are being driven to zero.

System Model

In this section, we develop a combined control scheme to track rigid spacecraft attitude motions using *both* thrusters and momentum wheels. The thrusters may act as the feedforward portion of the controller while the momentum wheels implement the feedback portion of the controller. Alternatively, in case the thrusters can generate continuous control profiles, one may choose to implement the feedforward plus the nonlinear feedback portion of the control law through the thrusters. In this case, the controller for the momentum wheels implements a linear feedback control law in terms of the angular velocity and attitude errors. Both of these implementation schemes globally asymptotically stabilize the tracking error.

We consider a rigid spacecraft $\mathcal{P}$ with N rigid and symmetric, balanced momentum wheels $\mathcal{W}_i$, $i = 1, \cdots, N$ and three thrusters mounted along the principal axes of the body frame. (The wheel axes are allowed to have an arbitrary orientation with respect to the body.) Let $\mathbf{N}$ denote the inertial frame, and $\mathbf{B}$ denote the body frame with the origin at the center of mass of the system $\mathcal{P} + \sum_{i=1}^{i=N}\mathcal{W}_i$. The desired trajectory to be tracked is one generated by a "virtual" spacecraft with the same inertia properties or total angular momentum as the rigid spacecraft. Let $\mathbf{R}$ denote the reference frame which is fixed at the center of mass of this virtual spacecraft.

The purpose of the controller is to make the body frame $\mathbf{B}$ asymptotically track the reference frame $\mathbf{R}$. In addition, in the absence of any disturbances and for the same initial conditions, the tracking controller should keep $\mathbf{B}$ and $\mathbf{R}$ aligned at all times.

Dynamics

Let $\mathbf{I}$ be the moment of inertia of the system, including the wheels and thrusters, and let $I_{si}, i = 1, 2, \cdots, N$ denote the axial moments of inertia of each momentum wheel. Defining the matrix $\mathbf{I}_s = \mathrm{diag}\{\mathrm{I}_{s1}, \cdots, \mathrm{I}_{sN}\}$, we have the dynamics of the system described by the following equations[1]

$$\dot{\mathbf{h}}_B = \mathbf{h}_B^\times\mathbf{J}^{-1}(\mathbf{h}_B - \mathbf{A}\mathbf{h}_a) + \mathbf{g}_e \quad (15a)$$

$$\dot{\mathbf{h}}_a = \mathbf{g}_a \quad (15b)$$

where $\mathbf{h}_B$ is the system angular momentum vector in $\mathbf{B}$ frame given by

$$\mathbf{h}_B = \mathbf{I}\boldsymbol{\omega}_B + \mathbf{A}\mathbf{I}_s\boldsymbol{\omega}_s \quad (16)$$

$\mathbf{h}_a$ is the $N \times 1$ matrix of the *axial* angular momenta of the wheels, $\mathbf{g}_e$ is the 3×1 matrix of external torques applied by the thrusters, $\mathbf{g}_a$ is the $N \times 1$ matrix of the internal axial torques applied by the platform to the momentum wheels, $\mathbf{A}$ is the $3 \times N$ matrix containing the axial unit vectors of the N momentum wheels, and $\mathbf{J}$ is an inertia-like matrix defined as

$$\mathbf{J} = \mathbf{I} - \mathbf{A}\mathbf{I}_s\mathbf{A}^T \tag{17}$$

From Eqs. (16) and (17) the angular velocity of the body frame can be written as

$$\boldsymbol{\omega}_B = \mathbf{J}^{-1}(\mathbf{h}_B - \mathbf{A}\mathbf{h}_a) \tag{18}$$

and the *axial* angular momenta of the momentum wheels can be written as

$$\mathbf{h}_a = \mathbf{I}_s\mathbf{A}^T\boldsymbol{\omega}_B + \mathbf{I}_s\boldsymbol{\omega}_s \tag{19}$$

where $\boldsymbol{\omega}_s = (\omega_{s1}, \omega_{s2}, \cdots, \omega_{sN})^T$ is an $N \times 1$ vector denoting the axial angular velocities of the momentum wheels with respect to the body.

Kinematics

Here we choose the "Modified Rodrigues Parameters" (MRP's) to describe the kinematics of the attitude motion which are defined as

$$\boldsymbol{\sigma} = \hat{\mathbf{e}} \tan(\Phi/4) \tag{20}$$

where $\hat{\mathbf{e}}$ is the unit vector along the Euler principal axis, and Φ is the Euler principal rotation angle.[2] The differential equations of the kinematics in terms of the MRP's are

$$\dot{\boldsymbol{\sigma}} = \mathbf{G}(\boldsymbol{\sigma})\boldsymbol{\omega} \tag{21}$$

where

$$\mathbf{G}(\boldsymbol{\sigma}) = \frac{1}{2}\left(\mathbf{1} + \boldsymbol{\sigma}^\times + \boldsymbol{\sigma}\boldsymbol{\sigma}^T - \frac{1 + \boldsymbol{\sigma}^T\boldsymbol{\sigma}}{2}\mathbf{1}\right) \tag{22}$$

and $\mathbf{1}$ is the 3×3 identity matrix. Therefore, the kinematics of the body frame can then be written as

$$\dot{\boldsymbol{\sigma}}_B = \mathbf{G}(\boldsymbol{\sigma}_B)\boldsymbol{\omega}_B \tag{23}$$

Suppose a reference motion is designed and, at the design stage, only the thrusters provide control torques, while the momentum wheels are non-rotating (i.e., $\omega_{si} = 0, i = 1, 2, \cdots, N$). In this case, from Eq. (16) $\mathbf{h}_B = \mathbf{I}\boldsymbol{\omega}_B$. With the reference frame denoted by $\mathbf{R}$, the reference dynamics is assumed to be

$$\dot{\mathbf{h}}_R = \mathbf{h}_R^\times\mathbf{I}^{-1}\mathbf{h}_R + \mathbf{g}_R \tag{24}$$

where $\mathbf{h}_R = \mathbf{I}\boldsymbol{\omega}_R$ and $\boldsymbol{\omega}_R$ is the angular velocity of the virtual body in the $\mathbf{R}$ frame. Note that if the wheels are non-rotating and $\mathbf{g}_e = \mathbf{g}_R$ then Eqs. (15) and (24) are identical. Thus, Eq. (24) describes the dynamics of the attitude motion of a "virtual" spacecraft with the same inertia properties as the real spacecraft. This virtual spacecraft will be used to generate the desired (nominal or optimal) trajectory to be tracked. Hence, $\mathbf{g}_R$ in Eq. (24) is the desired nominal control torque which, if acted upon the *real* spacecraft (assuming stationary wheels), would generate the desired trajectory.

The kinematics of the $\mathbf{R}$ frame is given by

$$\dot{\boldsymbol{\sigma}}_R = \mathbf{G}(\boldsymbol{\sigma}_R)\boldsymbol{\omega}_R \tag{25}$$

where $\boldsymbol{\sigma}_R$ denote the MRP's of the $\mathbf{R}$ frame with respect to the inertial frame $\mathbf{N}$.

Let us now define the tracking error of the angular velocity expressed in $\mathbf{B}$ frame as

$$\delta\boldsymbol{\omega} = \boldsymbol{\omega}_B - \mathbf{C}_R^B(\delta\boldsymbol{\sigma})\,\boldsymbol{\omega}_R \tag{26}$$

with $\mathbf{C}_R^B(\delta\boldsymbol{\sigma})$ the rotation matrix from the reference frame $\mathbf{R}$ to the body frame $\mathbf{B}$, and $\delta\boldsymbol{\sigma}$ the kinematics error between the frames $\mathbf{B}$ and $\mathbf{R}$ defined by

$$\mathbf{C}_R^B(\delta\boldsymbol{\sigma}) = \mathbf{C}_N^B(\boldsymbol{\sigma}_B)[\mathbf{C}_N^R(\boldsymbol{\sigma}_R)]^T \tag{27}$$

From Eqs. (21) and (26), the differential equation for the error kinematics takes the form

$$\delta\dot{\boldsymbol{\sigma}} = \mathbf{G}(\delta\boldsymbol{\sigma})\delta\boldsymbol{\omega} \tag{28}$$

From Eq. (24), we have

$$\dot{\boldsymbol{\omega}}_R = \mathbf{I}^{-1}\mathbf{h}_R^\times\mathbf{I}^{-1}\mathbf{h}_R + \mathbf{I}^{-1}\mathbf{g}_R \tag{29}$$

thus

$$\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R = \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{h}_R^\times\mathbf{I}^{-1}\mathbf{h}_R + \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{g}_R \tag{30}$$

According to the definition of the tracking error of the angular velocity in Eq. (26), we define the following tracking error of the angular momentum expressed in $\mathbf{B}$ frame

$$\begin{aligned} \delta\mathbf{h} &= \mathbf{h}_B - \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\boldsymbol{\omega}_R \\ &= \mathbf{I}\boldsymbol{\omega}_B + \mathbf{A}\mathbf{I}_s\boldsymbol{\omega}_s - \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\boldsymbol{\omega}_R \\ &= \mathbf{J}(\boldsymbol{\omega}_B - \mathbf{C}_R^B(\delta\boldsymbol{\sigma})\boldsymbol{\omega}_R) + \mathbf{A}(\mathbf{I}_s\boldsymbol{\omega}_s + \mathbf{I}_s A^T\boldsymbol{\omega}_B) \end{aligned} \tag{31}$$

and using Eq. (19) we have finally that

$$\delta\mathbf{h} = \mathbf{J}\delta\boldsymbol{\omega} + \mathbf{A}\mathbf{h}_a \tag{32}$$

From Eq. (32) we have the error dynamics as

$$\begin{aligned} \delta\dot{\mathbf{h}} &= \dot{\mathbf{h}}_B - \mathbf{J}\frac{d\mathbf{C}_R^B(\delta\boldsymbol{\sigma})}{dt}\boldsymbol{\omega}_R - \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R \\ &= \dot{\mathbf{h}}_B - \mathbf{J}\boldsymbol{\omega}_B^\times\delta\boldsymbol{\omega} - \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R \\ &= \mathbf{h}_B^\times\mathbf{J}^{-1}(\mathbf{h}_B - \mathbf{A}\mathbf{h}_a) + \mathbf{g}_e - \mathbf{J}\boldsymbol{\omega}_B^\times\delta\boldsymbol{\omega} - \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R \end{aligned} \tag{33}$$

where $\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R$ is given in Eq. (30). Here we have used the fact that

$$\frac{d\mathbf{C}_R^B(\delta\boldsymbol{\sigma})}{dt}\boldsymbol{\omega}_R = \boldsymbol{\omega}_B^\times \delta\boldsymbol{\omega} \tag{34}$$

We give a brief proof of this fact in the Appendix.

Tracking Controllers

Consider the following Lyapunov function candidate

$$\begin{aligned} V &= \frac{1}{2}\delta\boldsymbol{\omega}^T\mathbf{K}\delta\boldsymbol{\omega} + 2k_2\ln(1+\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}) \\ &= \frac{1}{2}(\delta\mathbf{h}-\mathbf{A}\mathbf{h}_a)^T\mathbf{K}(\delta\mathbf{h}-\mathbf{A}\mathbf{h}_a) + 2k_2\ln(1+\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}) \end{aligned} \tag{35}$$

where $\mathbf{K}=\mathbf{K}^T>0$, and $k_2>0$. This function is positive definite and radially unbounded[3] in terms of the tracking errors $\delta\boldsymbol{\omega}$ and $\delta\boldsymbol{\sigma}$. Calculation of the derivative of V along the error dynamics and kinematics, Eqs. (33) and (28), yields

$$\begin{aligned} \dot{V} &= (\delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a)^T\mathbf{K}(\delta\mathbf{h}-\mathbf{A}\mathbf{h}_a) + 4k_2\frac{\delta\boldsymbol{\sigma}^T\mathbf{G}(\delta\boldsymbol{\sigma})}{1+\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\sigma}}\delta\boldsymbol{\omega} \\ &= (\delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a)^T\mathbf{K}(\mathbf{h}_B-\mathbf{A}\mathbf{h}_a-\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\boldsymbol{\omega}_R) + k_2\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\omega} \\ &= (\delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a)^T\mathbf{K}(\mathbf{J}\boldsymbol{\omega}_B-\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\boldsymbol{\omega}_R) + k_2\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\omega} \\ &= (\delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a)^T\mathbf{K}\mathbf{J}\delta\boldsymbol{\omega} + k_2\delta\boldsymbol{\sigma}^T\delta\boldsymbol{\omega} \end{aligned} \tag{36}$$

By choosing $\mathbf{K}=\mathbf{J}^{-1}$, we get

$$\begin{aligned} \dot{V} &= \delta\boldsymbol{\omega}^T(\delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a+k_2\delta\boldsymbol{\sigma}) \\ &= \delta\boldsymbol{\omega}^T\big[\mathbf{h}_B^\times\mathbf{J}^{-1}(\mathbf{h}_B-\mathbf{A}\mathbf{h}_a)+\mathbf{g}_e-\mathbf{J}\boldsymbol{\omega}_B^\times\delta\boldsymbol{\omega} \\ &\quad -\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{h}_R^\times\mathbf{I}^{-1}\mathbf{h}_R-\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{g}_R-\mathbf{A}\mathbf{g}_a+k_2\delta\boldsymbol{\sigma}\big] \end{aligned} \tag{37}$$

Controller I

In case the thrusters are of the on-off type they may not be able to implement a continuously varying control profile (unless a PWPM scheme is used) and thus, it will be assumed that $\mathbf{g}_R$ is a bang-bang command. In this case we choose the thrusters to perform the designed nominal control $\mathbf{g}_R$, i.e.,

$$\mathbf{g}_e = \mathbf{g}_R \tag{38}$$

and the momentum wheels are used to correct for the tracking errors. From Eq. (37), letting the feedback control law for the momentum wheels satisfy

$$\begin{aligned} \mathbf{A}\mathbf{g}_a &= \mathbf{h}_B^\times\mathbf{J}^{-1}(\mathbf{h}_B-\mathbf{A}\mathbf{h}_a)+\mathbf{g}_R-\mathbf{J}\boldsymbol{\omega}_B^\times\delta\boldsymbol{\omega} \\ &\quad -\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{h}_R^\times\mathbf{I}^{-1}\mathbf{h}_R-\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{I}^{-1}\mathbf{g}_R+k_1\delta\boldsymbol{\omega}+k_2\delta\boldsymbol{\sigma} \end{aligned} \tag{39}$$

yields

$$\dot{V} = -k_1\delta\boldsymbol{\omega}^T\delta\boldsymbol{\omega} \le 0 \tag{40}$$

where $k_1>0$. This implies that the tracking trajectories are bounded and furthermore,

$$\lim_{t\to\infty}\delta\boldsymbol{\omega}(t) = 0 \tag{41}$$

From Eqs. (32) and (33), we have that

$$\begin{aligned} \mathbf{J}\delta\dot{\boldsymbol{\omega}} &= \delta\dot{\mathbf{h}}-\mathbf{A}\dot{\mathbf{h}}_a \\ &= -k_1\delta\boldsymbol{\omega}-k_2\delta\boldsymbol{\sigma} \end{aligned} \tag{42}$$

Thus, and because of Eq. (41), we have

$$\lim_{t\to\infty}\delta\boldsymbol{\sigma}(t) = 0 \tag{43}$$

From LaSalle's Theorem,[3] the tracking error dynamics and kinematics with the feedback control law (39) are globally asymptotically stable.

In the absence of any initial condition errors, i.e., $\delta\boldsymbol{\omega}(0)=\delta\boldsymbol{\sigma}(0)=0$ it is easy to show that the control law in Eqs. (38) and (39) ensures perfect tracking, i.e., $\boldsymbol{\omega}_B(t)=\boldsymbol{\omega}_R(t)$ and $\boldsymbol{\sigma}_B(t)=\boldsymbol{\sigma}_R(t)$ for all $t\ge 0$.

Controller II

Equations (38) and (39) show that Controller I is such that if initially $\boldsymbol{\omega}_B(0)=\boldsymbol{\omega}_R(0)$ and $\boldsymbol{\sigma}_B(0)=\boldsymbol{\sigma}_R(0)$ and, in addition, $\boldsymbol{\omega}_s(0)=0$, then $\mathbf{h}_B(t)=\mathbf{h}_R(t)$ for all $t\ge 0$. In particular, the last equality implies that $\boldsymbol{\omega}_s(t)=0$ for all $t\ge 0$ and the momentum wheels remain stationary with respect to the platform.

An alternative control implementation is to choose the control law so that, in the absence of initial condition errors and if the total axial momentum is initially zero, $\mathbf{h}_a(0)=0$, it remains zero during the maneuver, i.e., $\mathbf{h}_a(t)=0$ for all $t\ge 0$. In this case, $\mathbf{h}_B=\mathbf{J}\boldsymbol{\omega}_B$ and we can therefore assume that the reference dynamics is given by

$$\dot{\mathbf{h}}_R = \mathbf{h}_R^\times\mathbf{J}^{-1}\mathbf{h}_R+\mathbf{g}_R \tag{44}$$

where $\mathbf{h}_R=\mathbf{J}\boldsymbol{\omega}_R$. Choosing the same Lyapunov function as before and the thruster control law as

$$\mathbf{g}_e = \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{J}^{-1}\mathbf{g}_R \tag{45}$$

and the momentum wheel control law

$$\begin{aligned} \mathbf{A}\mathbf{g}_a &= \mathbf{h}_B^\times\mathbf{J}^{-1}(\mathbf{h}_B-\mathbf{A}\mathbf{h}_a)-\mathbf{J}\boldsymbol{\omega}_B^\times\delta\boldsymbol{\omega} \\ &\quad -\mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\mathbf{J}^{-1}\mathbf{h}_R^\times\mathbf{J}^{-1}\mathbf{h}_R+k_1\delta\boldsymbol{\omega}+k_2\delta\boldsymbol{\sigma} \end{aligned} \tag{46}$$

one obtains again that

$$\dot{V} = -k_1\delta\boldsymbol{\omega}^T\delta\boldsymbol{\omega} \le 0 \tag{47}$$

where $k_1>0$. Using similar arguments as before, one can show that this control law achieves global asymptotic stability for the tracking error dynamics.

Note that in the absence of any initial condition errors, $\delta\boldsymbol{\sigma}(0) = \delta\boldsymbol{\omega}(0) = 0$, and if $\mathbf{h}_a(0) = 0$, the control law in Eqs. (45) and (46) guarantees that $\delta\boldsymbol{\sigma}(t) = \delta\boldsymbol{\omega}(t) = 0$ for all $t \geq 0$ and the control law becomes

$$\mathbf{g}_e = \mathbf{g}_R \tag{48}$$

and

$$\mathbf{A}\mathbf{g}_a = \mathbf{h}_B^\times \mathbf{J}^{-1}\mathbf{h}_B - \mathbf{h}_R^\times \mathbf{J}^{-1}\mathbf{h}_R = (\mathbf{A}\mathbf{h}_a)^\times \boldsymbol{\omega}_R \tag{49}$$

The last equation, along with Eq. (15b) implies that if $\mathbf{h}_a(0) = 0$, then $\mathbf{g}_a = 0$ and hence, $\mathbf{h}_a(t) = 0$, for all $t \geq 0$.

Controller III

Another alternative way to implement the control law is to enforce a linear feedback control law for the wheels,

$$\mathbf{A}\mathbf{g}_a = k_1\delta\boldsymbol{\omega} + k_2\delta\boldsymbol{\sigma} \tag{50}$$

Then one needs to choose the thruster control law as

$$\mathbf{g}_e = -\mathbf{h}_B^\times \mathbf{J}^{-1}(\mathbf{h}_B - \mathbf{A}\mathbf{h}_a) + \mathbf{J}\boldsymbol{\omega}_B^\times \delta\boldsymbol{\omega} + \mathbf{J}\mathbf{C}_R^B(\delta\boldsymbol{\sigma})\dot{\boldsymbol{\omega}}_R \tag{51}$$

where $\dot{\boldsymbol{\omega}}_R$ is given either from Eq. (24) or from Eq. (44). In the first case, we have that $\mathbf{h}_R = \mathbf{I}\boldsymbol{\omega}_R$ whereas in the second case we have that $\mathbf{h}_R = \mathbf{J}\boldsymbol{\omega}_R$.

The Lyapunov function in Eq. (35) can be used to show that this control law renders the error system $(\delta\boldsymbol{\omega}, \delta\boldsymbol{\sigma})$ globally asymptotically stable.

Numerical Example

To show the effectiveness of the previous control laws, we apply them to track a trajectory of a minimum-time rest-to-rest maneuver. Three momentum wheels are used to provide the feedback control. They are aligned with the principal axes and their axial moments of inertia are given by $\mathbf{I}_s = \text{diag}\{0.01, 0.01, 0.01\}\text{kgm}^2$. The spacecraft moment of inertia matrix is

$$\mathbf{J} = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 150 & 0 \\ 0 & 0 & 175 \end{bmatrix} \text{kgm}^2$$

The nominal control $\mathbf{g}_R$, which is known to be bang-bang, is designed to drive the spacecraft from an initial attitude $\boldsymbol{\sigma}_R(0) = (0.1, 0.2, 0.3)$ which can be represented in 3-2-1 Euler angles as $(42.5°, 20.2°, 77.7°)$, to a position aligned with the inertial frame, i.e., $\boldsymbol{\sigma}(t_f) = (0,0,0)$. It is assumed that the actual initial attitude of the body frame is $\boldsymbol{\sigma}_B(0) = (0.11, 0.15, 0.28)$, which can be expressed in 3-2-1 Euler angles as $(37°, 13.3°, 69°)$.

For the sake of brevity we present the results only for the Controllers I and III. The results are shown in Figs. (1)-(6). In both cases, the gains were chosen as $k_1 = 54$ and $k_2 = 47$.

Figure 1 shows the time history of $\delta\boldsymbol{\omega}$, and Fig. 2 shows the time history of $\delta\boldsymbol{\sigma}$. Note that the time histories of the tracking error are the same for both controllers. Figures 3 and 4 show the time history of the controls when the thrusters perform the nominal control $\mathbf{g}_R$ and the control law for the momentum wheels is given by Eq. (39). Figures 5 and 6 show the time history of the control inputs when the thrusters perform the control law in Eq. (51) and the momentum wheels perform the linear feedback control law in Eq. (50), with reference input generated by Eq. (24).

Appendix

We now give a brief proof of the fact that

$$\frac{d\mathbf{C}_R^B}{dt}\boldsymbol{\omega}_R = \boldsymbol{\omega}_B^\times \delta\boldsymbol{\omega}$$

It is well known in the analytical dynamics[4] that

$$\dot{\mathbf{C}}_N^B = -\boldsymbol{\omega}_B^\times \mathbf{C}_N^B, \text{ and } \dot{\mathbf{C}}_N^R = -\boldsymbol{\omega}_R^\times \mathbf{C}_N^R$$

so

$$\begin{aligned} \frac{d\mathbf{C}_R^B}{dt} &= \frac{d[\mathbf{C}_N^B \mathbf{C}_R^N]}{dt} \\ &= \dot{\mathbf{C}}_N^B \mathbf{C}_R^N + \mathbf{C}_N^B \dot{\mathbf{C}}_R^N \\ &= -\boldsymbol{\omega}_B^\times \mathbf{C}_N^B \mathbf{C}_R^N + \mathbf{C}_N^B[-\boldsymbol{\omega}_R^\times \mathbf{C}_N^R]^T \\ &= -\boldsymbol{\omega}_B^\times \mathbf{C}_N^B \mathbf{C}_R^N + \mathbf{C}_N^B[-\boldsymbol{\omega}_R^\times \mathbf{C}_B^R \mathbf{C}_N^B]^T \\ &= -\boldsymbol{\omega}_B^\times \mathbf{C}_N^B \mathbf{C}_R^N + \mathbf{C}_N^B[\mathbf{C}_N^B]^T \mathbf{C}_R^B \boldsymbol{\omega}_R^\times \\ &= -\boldsymbol{\omega}_B^\times \mathbf{C}_R^B + \mathbf{C}_R^B \boldsymbol{\omega}_R^\times \end{aligned}$$

Thus

$$\begin{aligned} \frac{d\mathbf{C}_R^B}{dt}\boldsymbol{\omega}_R &= -\boldsymbol{\omega}_B^\times \mathbf{C}_R^B \boldsymbol{\omega}_R \\ &= -\boldsymbol{\omega}_B^\times [\boldsymbol{\omega}_B - \delta\boldsymbol{\omega}] \\ &= \boldsymbol{\omega}_B^\times \delta\boldsymbol{\omega} \end{aligned}$$

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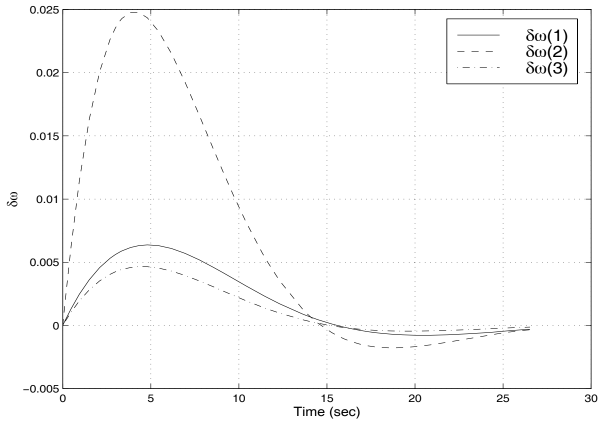

Figure 1: The time history of $\delta\boldsymbol{\omega}$.

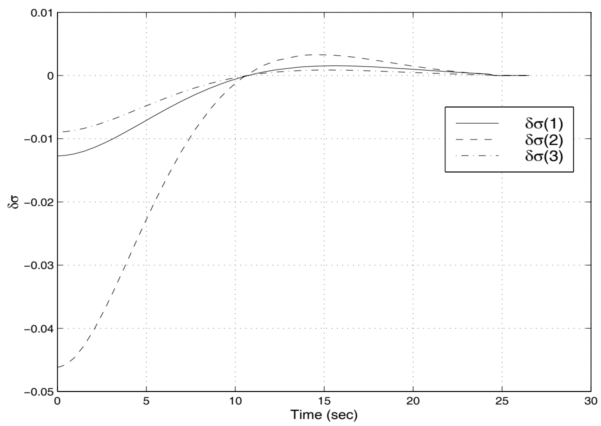

Figure 2: The time history of $\delta\boldsymbol{\sigma}$.

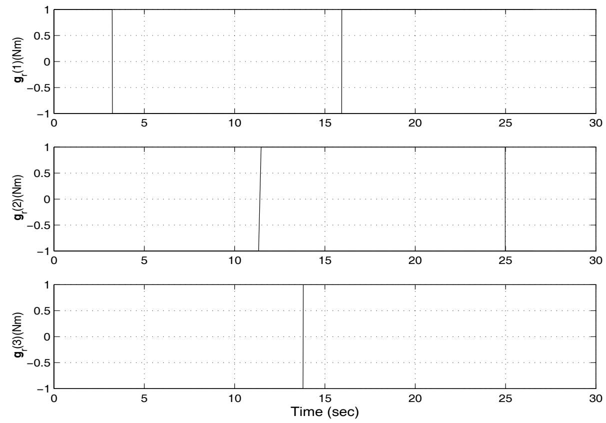

Figure 3: The nominal control performed by the thrusters

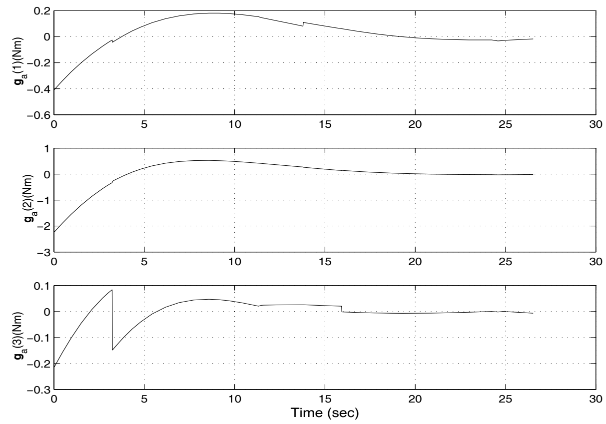

Figure 4: The feedback nonlinear control law in Eq. (39) for the momentum wheel

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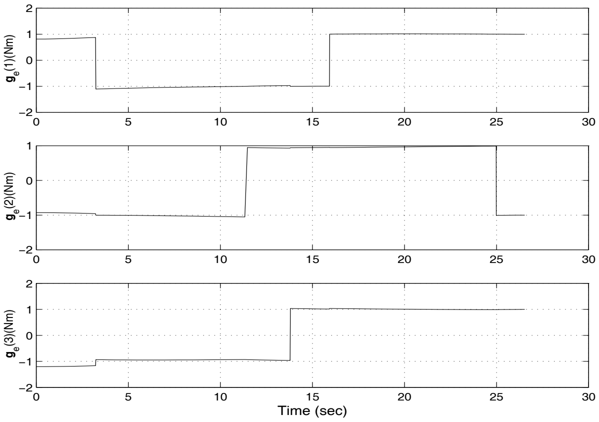

Figure 5: The control law in Eq. (51) performed by the thrusters

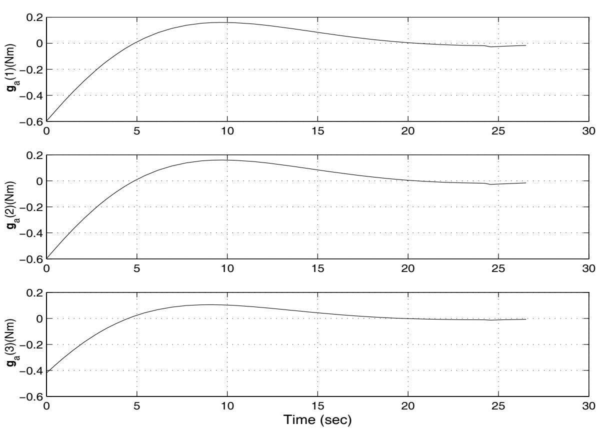

Figure 6: The feedback linear control law in Eq. (50) for the momentum wheel