Astrodynamics Specialists Conference
San Diego, CA, July 29-31, 1996

COMMENTS ON A NEW PARAMETERIZATION OF THE ATTITUDE KINEMATICS

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Abstract

Recently, a new formulation has been introduced for the description of attitude kinematics, which is based on two perpendicular rotations. The new parameterization bridges the gap between the Euler-angle (three rotations) and Euler-Rodrigues (one rotation) parameterizations and sheds new light on attitude kinematics. In this paper we present a slightly different derivation (again based on stereographic projection of a column of the rotation matrix) with a different choice of variables. We show the relation of the new parameterization to established formulations and cite examples in which the new description presents special advantages in deriving analytic solutions and in designing control laws.

1 Introduction

In 1995, a new parameterization of the attitude kinematics was reported.[1] This new formulation, which is based on two orthogonal rotations, results in a set of kinematic equations which contain quadratic nonlinearities (in the form of the Riccati equation). Thus, the new kinematic equations are "less" nonlinear than those associated with the three-rotation Euler angles, which have trigonometric nonlinearities, and "more" nonlinear than those of the one-rotation Euler-Rodrigues (quaternion) parameterization which are linear. This parameterization appears to be a new result in the literature, at least as far as the authors know. (See for example, the excellent recent survey paper by Shuster.[2])

The motivation for constructing such a formulation issued from the search for closed-form analytic solutions of the *self-excited rigid body*, which Grammel[3] and Leimanis[4] define as a body free to rotate about a point fixed in the body and space, when it is acted upon by a torque vector arising from internal reactions which do not appreciably change the mass or mass distribution. Many authors[3–22] have contributed to the pursuit of such analytic solutions of the self-excited rigid body and closely related spacecraft attitude dynamics problems.

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Euler angles are the variables of choice in most of these analytical investigations, in spite of their notorious nonlinearities. This is because, in many applications, the spacecraft does not make large angular excursions from its initial orientation in inertial space. Thus, small angles are assumed, and the resulting kinematic equations are linear. On the other hand, the linear kinematic equations of the Euler-Rodrigues parameters, have not been quite so popular in this pursuit due to their time-varying nature. There are a few examples, however. Analytic solutions have been constructed for the special case of a torque-free rotating body.[11] Kane[9] has obtained approximate solutions for an axisymmetric rigid body subject to body-fixed transverse torques of constant magnitude, by employing an averaging technique. Similar approximate solutions are reported by Kane and Levinson.[18]

A first step in developing the new parameterization was provided by Tsiotras and Longuski[23] in which an old, but relatively unknown method due to Darboux[24] is used to formulate the attitude problem as the solution of a single but complex-valued Riccati equation. An important characteristic of this equation is that when the quadratic terms are dropped, it reverts to the linearized form of the Euler angle kinematics. Thus, the quadratic terms contain the correction term for the large angle theory, a fact which is exploited by Longuski and Tsiotras.[25] It appears that all analytic theories based on the small angle assumption may be extended to cover large angular excursions if the quadratic terms can be integrated.

The final step was taken by Tsiotras and Longuski[1] where they introduced the third parameter, consisting of an initial rotation about a body axis. It is interesting to note that this third parameter first appeared in Tsiotras, Corless and Longuski[26] with regard to control laws for an axisymmetric spacecraft. In that paper it is conjectured that the new variable could be used as an alternative new description of the kinematics of the attitude motion. But the full import of the new variable and its physical interpretation were not completely recognized.

In this paper we derive the new parameterization in a slightly different fashion from that of Tsiotras and Longuski.[1] We also make a different choice of variables in the stereographic projection which is more convenient to remember. The kinematic equations appear in a form which may be slightly more appealing than the ones reported in Ref. 1. We hope that the derivation which follows will make these equations more accessible and more widely available to scientists and engineers.

2 The w Parameter

Consider a point (a, b, c) located on a unit sphere. Let this point be represented by a stereographic projection (represented by a line through the point and the south pole of the sphere) onto the complex plane where each complex number is associated with the ordered pairs (w_1, w_2). For convenience we choose the real axis to be aligned with x_1 and the imaginary axis with x_2. From Fig. 1 it is clear that the complex number, w, is given by

$$w = w_1 + i\,w_2 = \frac{a + i\,b}{1 + c} \tag{1}$$

In previous work,[1,23,25,26] the slightly less convenient relation $\tilde{w} = (b - i\,a)/(1 + c)$ is used. Notice that w and $\tilde{w}$ are related by $w = i\,\tilde{w}$.

We want the point (a, b, c) to somehow represent the final orientation of a rigid body frame $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$. Let us assume that the original orientation of the $\hat{\mathbf{b}}$ frame is coincident with a set of orthogonal unit vectors $(\hat{i}'_1, \hat{i}'_2, \hat{i}'_3)$ and that the rotation is about an axis restricted to the $\hat{i}'_1$, $\hat{i}'_2$ plane. (The reason for the primes will be clear later when we discuss the third parameter of the new parameterization).

Now let us consider a point (0, 0, 1) in the $\hat{i}'$ frame which represents the initial orientation of one of the body axes (say the z axis). The effect of the rotation is to transform the original coordinates of the $\hat{\mathbf{b}}$ frame (0, 0, 1) to the new coordinates (a, b, c).

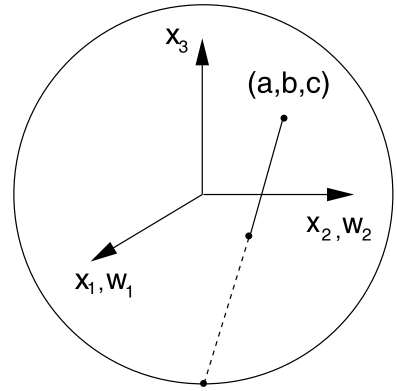

Fig. 1 Stereographic projection.

This transformation can be written as

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = R_2(w) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2}$$

where the notation for the rotation matrix, $R_2(w)$, will become apparent in the discussion of the third parameter. It is obvious from Eq.(2) that the third column of $R_2(w)$ must be $(a, b, c)^T$. Another way to express the meaning of Eq. (2) is by the vector equation

$$\hat{i}'_3 = a\,\hat{b}_1 + b\,\hat{b}_2 + c\,\hat{b}_3 \tag{3}$$

That is, after rotation, the unit vector $\hat{i}'_3$ has coordinates (a, b, c) in the $\hat{\mathbf{b}}$ frame.

By symmetry, we can express the $\hat{b}_3$ unit vector in the $\hat{\mathbf{i}}'$ frame as

$$\hat{b}_3 = -a\,\hat{i}'_1 - b\,\hat{i}'_2 + c\,\hat{i}'_3 \tag{4}$$

Figure 2 presents a sketch of the orientation of $\hat{b}_3$ in the $\hat{\mathbf{i}}'$ frame.

Also shown is the unit vector, $\hat{u}$, which shows the direction of the rotation axis through the angle $\cos^{-1} c$. It is readily computed by

$$\begin{aligned} \hat{u} &= \frac{\hat{i}'_3 \times \hat{b}_3}{\|\hat{i}'_3 \times \hat{b}_3\|} = \frac{b\,\hat{i}'_1 - a\,\hat{i}'_2}{\sqrt{a^2 + b^2}} \\ &= \frac{b\,\hat{b}'_1 - a\,\hat{b}_2}{\sqrt{a^2 + b^2}} \end{aligned} \tag{5}$$

Let the angular velocity of the $\hat{\mathbf{b}}$ frame with respect to the $\hat{\mathbf{i}}'$ frame be represented by

$${}^{i'}\vec{\omega}^{b} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3 \tag{6}$$

We note that for general motion the $\hat{\mathbf{b}}$ frame is free to rotate about any axis (and so the single rotation

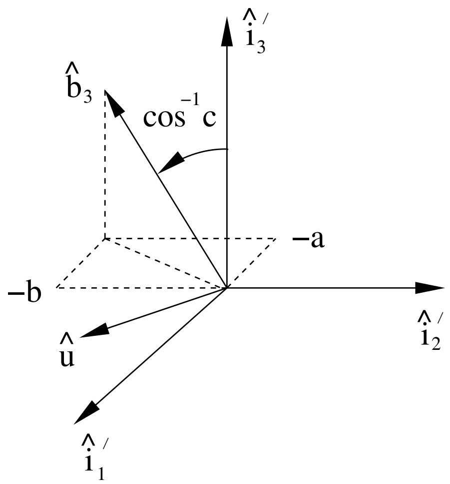

Fig. 2 Orientation of body frame.

along $\hat{u}$ cannot be adequate for the complete parameterization). To find the time rate of change of $\hat{i}_3'$ with respect to the $\hat{\mathbf{b}}$ frame, we write

$$\begin{aligned} \frac{{}^b d\hat{i}_3'}{dt} &= \dot{a}\,\hat{b}_1 + \dot{b}\,\hat{b}_2 + \dot{c}\,\hat{b}_3 \\ &= {}^b\vec{\omega}^{i'} \times \hat{i}_3' \end{aligned} \tag{7}$$

Substituting Eqs. (3) and (6) into Eq. (7) and noticing that ${}^b\vec{\omega}^{i'} = -{}^{i'}\vec{\omega}^b$ provides the system of differential equations

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = S({}^{i'}\vec{\omega}^b) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{8}$$

where

$$S({}^{i'}\vec{\omega}^b) \equiv \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \tag{9}$$

and where it is important to remember that ${}^{i'}\vec{\omega}^b$ is given in terms of $\hat{\mathbf{b}}$ coordinates, i.e. as in Eq. (6). We note that some authors define[2] a matrix $[{}^{i'}\vec{\omega}^b \times] = -S({}^{i'}\vec{\omega}^b)$. Using Euler's formula,[3,27,28] we can easily now compute the rotation matrix which corresponds to a rotation by an angle $\cos^{-1} c$ about the unit vector $\hat{u}$:

$$R_2(w) = I + \sin(\cos^{-1} c) S(\hat{u}) + [1 - \cos(\cos^{-1} c)] S^2(\hat{u}) \tag{10}$$

where $S(\cdot)$ is defined by Eq. (9) and $\hat{u}$ is given by Eq. (5). Carrying out the algebra and noting that $\sin(\cos^{-1} c) = \sqrt{1 - c^2}$, we obtain

$$R_2(w) = \begin{bmatrix} 1 - \frac{a^2}{1+c} & -\frac{ab}{1+c} & a \\ -\frac{ab}{1+c} & 1 - \frac{b^2}{1+c} & b \\ -a & -b & c \end{bmatrix} \tag{11}$$

Eq. (11) is equivalent to Eq. (18) of Ref. 1. As expected from Eqs. (2) and (3), the third column of $R_2(w)$ is $(a, b, c)^T$. Also, Eq. (11) verifies the correctness of Eq. (4).

3 The Kinematic Equation for w

We can obtain the corresponding kinematic equation for the complex parameter, w, by differentiating Eq. (1)

$$\dot{w} = \frac{\dot{a} + i\,\dot{b} - w\dot{c}}{1 + c} \tag{12}$$

Substituting for $\dot{a}$, $\dot{b}$, $\dot{c}$ from Eqs. (8) and (9) into Eq. (12) provides (after some algebra)

$$\dot{w} = -i\,(\omega_3 w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2) \tag{13}$$

where we have defined

$$\omega = \omega_1 + i\,\omega_2 \tag{14}$$

and where we have made use of the inverse relations based on Eq. (1)

$$a = \frac{w + \bar{w}}{1 + |w|^2} = \frac{2w_1}{1 + w_1^2 + w_2^2} \tag{15a}$$

$$b = \frac{i(\bar{w} - w)}{1 + |w|^2} = \frac{2w_2}{1 + w_1^2 + w_2^2} \tag{15b}$$

$$c = \frac{1 - |w|^2}{1 + |w|^2} = \frac{1 - w_1^2 - w_2^2}{1 + w_1^2 + w_2^2} \tag{15c}$$

where $|w|^2 = w\bar{w}$, denotes magnitude of a complex number.

4 The Third Parameter, z

We need a third parameter to complete the set (we note that the complex variable, w, counts as two parameters). It seems natural, at first, to perform a second rotation — about the $\hat{b}_3$ axis. But this will result in the appearance of the new variable on the right hand side of all three kinematic equations — destroying its "ignorable" character. (Here we use the term ignorable rather loosely to describe a variable that does not appear explicitly in the differential

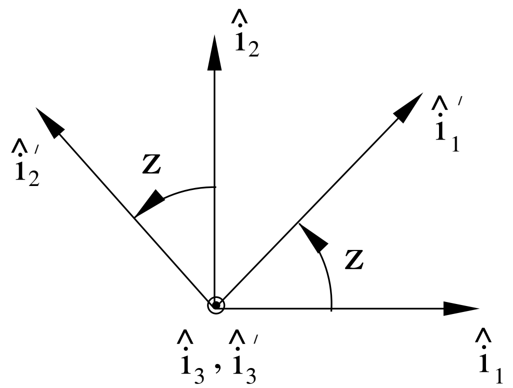

Fig. 3 Rotation through the angle z.

equations.) It turns out that it is best to perform a rotation about the body z axis first and complete the parameterization by a second rotation about $\hat{u}$.

Let $(\hat{i}_1,\ \hat{i}_2,\ \hat{i}_3)$ represent a set of orthogonal unit vectors, fixed in inertial space. Let $(\hat{i}_1',\ \hat{i}_2',\ \hat{i}_3')$ represent the orientation of the rigid body frame after a rotation through an angle, z, about the $\hat{i}_3$ axis. Then the associated transformation equation is

$$\begin{bmatrix} \hat{i}_1' \\ \hat{i}_2' \\ \hat{i}_3' \end{bmatrix} = R_1(z) \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix} \tag{16}$$

where

$$R_1(z) = \begin{bmatrix} \cos z & \sin z & 0 \\ -\sin z & \cos z & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

Combining the first rotation implied by Eq. (16) with the second rotation implied by Eq. (11) we deduce the relation

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = R(z, w) \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \hat{i}_3 \end{bmatrix} \tag{18}$$

where we define the transformation matrix

$$R(z, w) = R_2(w) R_1(z) \tag{19}$$

Here we note a crucial fact: the third column of $R(z, w)$ is $(a, b, c)^T$, i.e., identical with the third column of $R_1(z)$. This means that a point fixed in the $\hat{\mathbf{i}}$ frame at $(0, 0, 1)$ transforms to the point (a, b, c) in the $\hat{\mathbf{b}}$ frame after the two successive rotations. That is

$$\hat{i}_3 = a\,\hat{b}_1 + b\,\hat{b}_2 + c\,\hat{b}_3 \tag{20}$$

Tantamount in this argument, we have

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{bmatrix} = S({}^{i}\vec{\omega}^{b}) \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{21}$$

where we have ${}^{i}\vec{\omega}^{b}$ in lieu of ${}^{i'}\vec{\omega}^{b}$. Thus, our kinematic equation for w has the identical form of Eq. (13). The only difference is that the angular velocity components are now interpreted to be those of ${}^{i}\vec{\omega}^{b}$, namely

$${}^{i}\vec{\omega}^{b} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3 \tag{22}$$

5 The Kinematic Equation for z

We next proceed to derive the kinematic equation for z. First, let us write $R_2(w)$ in terms of w_1 and w_2. Substituting Eqs. (15) into Eq. (11) we obtain

$$R_2(w) = \frac{1}{1 + w_1^2 + w_2^2} \times \begin{bmatrix} 1 - w_1^2 + w_2^2 & -2w_1 w_2 & 2w_1 \\ -2w_1 w_2 & 1 + w_1^2 - w_2^2 & 2w_2 \\ -2w_1 & -2w_2 & 1 - w_1^2 - w_2^2 \end{bmatrix} \tag{23}$$

Now, substituting Eqs. (23) and (17) into Eq. (19), we find $R(z, w)$

$$R(z, w) = \frac{1}{1 + w_1^2 + w_2^2} \times \begin{bmatrix} [(1 - w_1^2 + w_2^2)cz + 2w_1 w_2 sz] & [(1 - w_1^2 + w_2^2)sz - 2w_1 w_2 cz] & 2w_1 \\ [-2w_1 w_2 cz - (1 + w_1^2 - w_2^2)sz] & [-2w_1 w_2 sz + (1 + w_1^2 - w_2^2)cz] & 2w_2 \\ -2w_1 cz + 2w_2 sz & -2w_1 sz - 2w_2 cz & [1 - w_1^2 - w_2^2] \end{bmatrix} \tag{24}$$

where sz and cz denote $\sin z$ and $\cos z$, respectively. According to Kane et al.,[29] $R(z, w)$ obeys the following equation

$$\dot{R}(z, w) = S({}^{i}\vec{\omega}^{b}) R(z, w) \tag{25}$$

which we recognize as a generalization of Eq. (21). To find the differential equation for z, we make use of Eq. (25) in the scalar form

$$tr[\dot{R}(z, w)] = tr[S({}^{i}\vec{\omega}^{b}) R(z, w)] \tag{26}$$

where $tr(\cdot)$ denotes the trace of the matrix. Taking the trace of $\dot{R}(z, w)$ we obtain

$$tr[\dot{R}(z, w)] = \frac{-2\dot{z}sz}{1 + w_1^2 + w_2^2} - \frac{4(1 + cz)(w_1 \dot{w}_1 + w_2 \dot{w}_2)}{(1 + w_1^2 + w_2^2)^2} \tag{27}$$

We obtain $\dot{w}_1$ and $\dot{w}_2$ from the real and imaginary parts of Eq. (13) as

$$\dot{w}_1 = \omega_3 w_2 + \omega_1 w_1 w_2 - \frac{\omega_2}{2}(1 + w_1^2 - w_2^2) \tag{28a}$$

$$\dot{w}_2 = -\omega_3 w_1 - \omega_2 w_1 w_2 + \frac{\omega_1}{2}(1 - w_1^2 + w_2^2) \tag{28b}$$

Substituting Eqs. (28) into Eq. (27), we find

$$tr[\dot{R}(z,w)] = \frac{-2\dot{z}sz - 2(cz+1)(\omega_1 w_2 - \omega_2 w_1)}{1 + w_1^2 + w_2^2} \tag{29}$$

Using the definition of $S(^i\vec{\omega}^b)$ from Eq. (9) and Eq. (24) we compute the trace

$$tr[S(^i\vec{\omega}^b)R(w,z)] = \frac{-2sz(\omega_3 + \omega_1 w_1 + \omega_2 w_2) - 2(cz+1)(\omega_1 w_2 - \omega_2 w_1)}{1 + w_1^2 + w_2^2} \tag{30}$$

Equating Eqs. (27) and (30), in accordance with Eq. (25), we obtain the kinematic equation for z

$$\dot{z} = \omega_3 + \omega_1 w_1 + \omega_2 w_2 \tag{31}$$

Rewriting Eq. (31) in terms of the complex variables ω and w and restating Eq. (13), we finally obtain

$$\dot{z} = \omega_3 + \frac{1}{2}(\omega \bar{w} + \bar{\omega} w) \tag{32a}$$

$$\dot{w} = -i\,(\omega_3 w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2) \tag{32b}$$

Equations (32) describe the kinematic equations in terms of the new parameterization (z, w).

(Here we note that this final formulation is slightly different form that of Ref. 1, where instead of defining $w = (a+ib)/(1+c)$, we used $w = (b-ia)/(1+c)$, resulting in

$$\dot{z} = \omega_3 + \frac{i}{2}(\bar{\omega} w - \omega \bar{w}) \tag{33a}$$

$$\dot{w} = -i\,\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2 \tag{33b}$$

The form of Eqs. (32) is perhaps a bit more appealing since the first equation is real and doesn't display i explicitly, while the second equation is complex and has a common factor of i on the right-hand side. But perhaps the best argument in favor of $w = (a+ib)/(1+c)$ is that it is easier to remember!)

6 Other Formulations

Equation (1) is only one of the possible definitions of the parameter w. Other combinations will provide different kinematic equations for w and z. For example, we have seen that by defining $w = (b - i\,a)/(1+c)$, the corresponding kinematic equations are given by Eqs. (33). Table 1 summarizes some of the possible choices for w from the stereographic projection and the corresponding kinematic equations.

According to the specific application at hand, one may choose the most convenient form for the attitude kinematics from this table.

Table 1 Stereographic coordinate w and corresponding kinematics.

w	Kinematics	ω
$\frac{a+ib}{1+c}$	$\dot{w} = -i\,(\omega_3 w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2)$ $\dot{z} = \omega_3 + \frac{1}{2}(\omega \bar{w} + \bar{\omega} w)$	$\omega_1 + i\,\omega_2$
$\frac{b-ia}{1+c}$	$\dot{w} = -i\,\omega_3 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2$ $\dot{z} = \omega_3 + \frac{i}{2}(\bar{\omega} w - \omega \bar{w})$	$\omega_1 + i\,\omega_2$
$\frac{b+ic}{1+a}$	$\dot{w} = -i\,(\omega_1 w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2)$ $\dot{z} = \omega_1 + \frac{1}{2}(\omega \bar{w} + \bar{\omega} w)$	$\omega_2 + i\,\omega_3$
$\frac{c-ib}{1+a}$	$\dot{w} = -i\,\omega_1 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2$ $\dot{z} = \omega_1 + \frac{i}{2}(\bar{\omega} w - \omega \bar{w})$	$\omega_2 + i\,\omega_3$
$\frac{c+ia}{1+b}$	$\dot{w} = -i\,(\omega_2 w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2)$ $\dot{z} = \omega_2 + \frac{1}{2}(\omega \bar{w} + \bar{\omega} w)$	$\omega_3 + i\,\omega_1$
$\frac{a-ic}{1+b}$	$\dot{w} = -i\,\omega_2 w + \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2$ $\dot{z} = \omega_2 + \frac{i}{2}(\bar{\omega} w - \omega \bar{w})$	$\omega_3 + i\,\omega_1$

7 Relation to Other Parameterizations

We now present the connection of the (z, w) parameters with some of the other standard kinematic parameters.

7.1 Eulerian Angles

Consider a Type 1: 3-2-1 Euler angle sequence[30] (ϕ_z, ϕ_y, ϕ_x), in which the rigid body frame is rotated successively by angles ϕ_z, ϕ_y and ϕ_x about the z, y and x body axes, respectively. Then the rotation matrix, $R_{321}(\phi_z, \phi_y, \phi_x)$, corresponding to $R(z, w)$ is given by

$$R_{321}(\phi_z,\phi_y,\phi_x) = \begin{bmatrix} c_z c_y & s_z c_y & -s_y \\ -s_z c_x + c_z s_y s_x & c_z c_x + s_z s_y s_x & c_y s_x \\ s_z s_x + c_z s_y c_x & -c_z s_x + s_z s_y c_x & c_y c_x \end{bmatrix} \tag{34}$$

where s and c denote sine and cosine and subscripts x, y and z denote ϕ_x, ϕ_y and ϕ_z, respectively.

To find the connection to the w parameter we recall that the third column of any rotation matrix can be set equal to $(a, b, c)^T$. Thus

$$w = \frac{a+ib}{1+c} = \frac{-\sin\phi_y + i\cos\phi_y \sin\phi_x}{1 + \cos\phi_y \cos\phi_x} \tag{35}$$

To find the relation to the z parameter, we take the trace of $R(z, w)$

$$\begin{aligned} tr[R(z,w)] &= \frac{1 - w_1^2 - w_2^2 + 2\cos z}{1 + w_1^2 + w_2^2} \\ &= c + (1+c)\cos z \end{aligned} \tag{36}$$

Taking the trace of $R_{321}(\phi_z, \phi_y, \phi_x)$

$$tr[R_{321}(\phi_z, \phi_y, \phi_x)] = c_z c_y + c_z c_x + s_z s_y s_x + c_y c_x \tag{37}$$

and equating Eq. (37) to Eq. (36) we have (noting that $c = \cos\phi_y \cos\phi_x$)

$$\cos z = \frac{c\phi_z c\phi_y + c\phi_z c\phi_x + s\phi_z s\phi_y s\phi_x}{1 + c\phi_y c\phi_x} \tag{38}$$

The kinematic equations corresponding to this Euler-angle sequence are[31]

$$\dot{\phi}_z = (\omega_2 \sin\phi_x + \omega_3 \cos\phi_x)\sec\phi_y \tag{39a}$$

$$\dot{\phi}_y = \omega_2 \cos\phi_x - \omega_3 \sin\phi_x \tag{39b}$$

$$\dot{\phi}_x = \omega_1 + (\omega_2 \sin\phi_x + \omega_3 \cos\phi_x)\tan\phi_y \tag{39c}$$

which are "highly" nonlinear, as mentioned above. We also note that the first rotation angle, ϕ_z, is the "ignorable" variable because it does not appear explicitly in these equations.

Equations (39) can be linearized by assuming that ϕ_x and ϕ_y are small angles. If we also assume that the term $\omega_2\phi_x$ is small compared to ω_3 (as is usually the case for spin-stabilized bodies), then we obtain the following linear system

$$\dot{\phi}_z = \omega_3 \tag{40a}$$

$$\dot{\phi}_y = \omega_2 - \omega_3\phi_x \tag{40b}$$

$$\dot{\phi}_x = \omega_1 + \omega_3\phi_y \tag{40c}$$

By defining

$$\phi = \phi_x + i\phi_y \tag{41}$$

the last two equations of Eqs. (40) become

$$\dot{\phi} = -i\omega_3\phi + \omega \tag{42}$$

By comparing Eq. (42) with the linearized equivalent of Eq. (32b), namely

$$\dot{w} = -i\omega_3 w + i\frac{\omega}{2} \tag{43}$$

we see that

$$w \approx i\,\frac{\phi}{2} \tag{44}$$

This is confirmed by applying the small angle assumption to Eq. (35). The most important conclusion for the development of analytic solutions is that small angle theories correspond directly to small w theories and that any improvement obtained in integrating the quadratic term in Eq. (32b) corresponds to a large angle theory.[25]

After finding a solution for w, we can obtain the solution for z by quadrature through the integration of Eq. (32a), which corresponds to Eq. (40a). This quadrature integral is always available because the variable associated with the first rotation (z and ϕ_z) always decouples from the kinematic equations (i.e., is "ignorable").

7.2 Euler-Rodrigues Parameters

The Euler-Rodrigues parameters are defined by

$$q_o = \cos(\Phi/2), \qquad q_i = \hat{e}_i \sin(\Phi/2) \tag{45}$$

where Φ is the principal angle of rotation and the $\hat{e}_i$ are the components of the principal unit vector. The associated rotation matrix is

$$R(q_0, q_1, q_2, q_3) = \begin{bmatrix} [q_0^2 + q_1^2 - q_2^2 - q_3^2] & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & [q_0^2 - q_1^2 + q_2^2 - q_3^2] & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & [q_0^2 - q_1^2 - q_2^2 + q_3^2] \end{bmatrix} \tag{46}$$

Setting the third column of Eq. (46) equal to $(a, b, c)^T$, we obtain the relation to the w parameter

$$w = \frac{a + ib}{1 + c} = \frac{2(q_1q_3 - q_0q_2) + 2i(q_2q_3 + q_0q_1)}{1 + q_0^2 - q_1^2 - q_2^2 + q_3^2} \tag{47}$$

Since

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{48}$$

Eq. (47) simplifies to

$$w = \frac{q_1q_3 - q_0q_2 + i\,(q_2q_3 + q_0q_1)}{q_0^2 + q_3^2} \tag{49}$$

Setting the trace of $R(q_0, q_1, q_2, q_3)$ equal to the trace of $R(z, w)$ in Eq. (36) we obtain an equation for the z relation

$$\cos z = \frac{q_0^2 - q_3^2}{q_0^2 + q_3^2} \tag{50}$$

The kinematic equations for the Euler-Rodrigues parameters consist of the linear system

$$\begin{bmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{51}$$

In spite of the much touted linearity of these equations, they are in fact quite difficult to apply in analytic developments because of the time-varying ω's. The chief defect in these equations, in this regard, is that there are no physical assumptions (analogous to using the small angle assumption with the Euler angle formulation) which can ease the analytic integration. This is the reason that most researchers pursuing analytic solutions prefer to work with Eulerian angles.

Of course one very important strength of the linear system is that it allows for fast and accurate numerical integration, in contrast to the Eulerian angle kinematics, Eqs. (39), which require time-consuming calculations of trigonometric functions. For this reason Eqs. (51) tend to be preferred for onboard numerical integration.[30] But we hasten to add that the kinematic equations for the new parameterization, Eqs. (32), also provide a very efficient formulation for onboard integration, compared to the Eulerian angles, since they avoid the computation of trigonometric functions.

7.3 Principal Angle and Axis

The (w, z) parameterization is realized by two successive rotations at angles z and $\theta = \cos^{-1} c$ about the axes $\hat{\imath}_3$ and $\hat{u}$, respectively (see Figs. 2 and 3). The angle θ can also be expressed in terms of w through the relationship

$$\cos\theta = \frac{1 - |w|^2}{1 + |w|^2} \tag{52}$$

Recalling that the trace of any rotation matrix is equal to $1+2\cos\Phi$, and using Eq. (37) we have immediately the following equation for the principal angle Φ in terms of w and z

$$\cos\theta + (1 + \cos\theta)\cos z = 1 + 2\cos\Phi \tag{53}$$

where $\cos\theta$ is given in Eq. (52). By adding 1 to both sides of the previous equation and using the trigonometric identity $\cos\gamma = 2\cos^2\frac{\gamma}{2} - 1$, Eq. (53) reduces to the simple formula

$$\cos^2\frac{\Phi}{2} = \cos^2\frac{z}{2}\,\cos^2\frac{\theta}{2} \tag{54}$$

or that

$$\cos\frac{\Phi}{2} = \cos\frac{z}{2}\,\cos\frac{\theta}{2} \tag{55}$$

where the angles are $-\pi \le \Phi \le \pi$, $-\pi \le z \le \pi$, and $0 \le \theta \le \pi$. This equation indicates that the half-angles $\Phi/2$, $z/2$ and $\theta/2$ are related through a right spherical triangle as in Fig. 4.

The derivation of the equation for the unit vector along the principal axis, $\hat{e}$ is somewhat more complicated and it is deduced as follows. Recall first that $\hat{e}$ is the eigenvector of the rotation matrix which corresponds to the eigenvalue +1. After some extensive calculations, it can be shown that the eigenvector of the matrix $R(z, w)$ in Eq. (24) corresponding to the +1 eigenvalue is given by

$$v = \begin{bmatrix} w_1 cz - w_2 sz - w_1 \\ w_2 cz + w_1 sz - w_2 \\ cz - 1 \end{bmatrix} \tag{56}$$

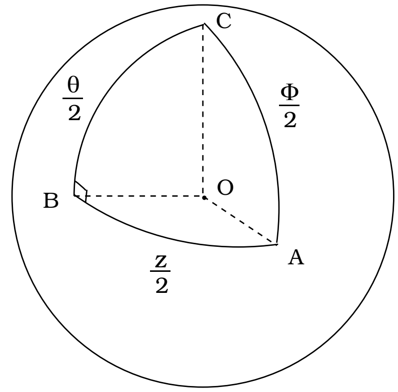

Fig. 4 Right spherical triangle.

The unit vector $\hat{e}$ is then given by

$$\hat{e} = \frac{v}{\|v\|} \tag{57}$$

where

$$\|v\| = (2|w|^2 + 1 - \cos z)^{\frac{1}{2}}(1 - \cos z)^{\frac{1}{2}} \tag{58}$$

is the magnitude of the vector v.

8 Control Applications

In this section we discuss some of the potential advantages of the proposed (w, z) parameterization in attitude control problems. In particular, the advantages of the (w, z) parameters become more apparent when control of only one of the body axes is required, as for axisymmetric bodies. In such cases, the w parameter can be used to describe the deviation of the axis from the desired position. There is no need to keep track of the time history of the parameter z if only the alignment of the specific body axis with the inertial axis is desirable. Because of the ignorable character of z, one can then work only with equation Eq. (32b), completely discarding any reference to the z coordinate. Besides, this is the main reason we chose z in a way such that it does not enter into the right-hand side of the kinematic equations.

In order to concretely demonstrate these ideas, let the case of an axisymmetric spacecraft where it is desirable to stabilize its symmetry (e.g., the $\hat{b}_3$) axis along the inertial $\hat{\imath}_3$ axis. From Eq. (20) this implies that we need $a = b = 0$ and $c = 1$. Consulting Table 1 and choosing the first row of this table we see that the previous requirement is equivalent to $w = 0$. Moreover, the following system completely describes the relevant dynamics

$$\dot{\omega} \;=\; -i\,\alpha\omega_{30}\omega + u \tag{59a}$$

$$\dot{w} = -i\,(\omega_{30} w - \frac{\omega}{2} + \frac{\bar{\omega}}{2} w^2) \qquad (59b)$$

where $\alpha = (I_2 - I_3)/I_1$, $u = u_1 + i\,u_2$ and $\omega_{30} = \omega_3(0)$. The objective here is to choose a feedback control law $u = u(\omega, w)$ such that the closed-loop system has the origin $\omega = w = 0$ as a stable equilibrium point.

It can be shown[32] that the *linear* control

$$u = -k_1 \omega - i\,k_2 w \qquad (60)$$

where $k_1 > 0, k_2 > 0$, globally asymptotically stabilizes the system in Eqs. (59). Notice that the only measurements required for feedback for this control are ω_1 and ω_2 and the kinematic parameters w_1 and w_2. Moreover, only actuation along the two principal axes perpendicular to the symmetry axis is necessary to achieve the stabilization objective. We also mention that the control law in Eq. (60) can be used to stabilize nonsymmetric bodies. The only difference with the axisymmetric case is that ω_3 is no longer constant, i.e., fixed at its initial value ω_{30}, but is an *a priori* unknown function of time. Due to the structure of the equation, however, the actual value of ω_3 has no effect on either the magnitude of ω or w.

Other control laws for the system in Eqs. (59), based on the theory of cascade systems, are also given in Ref. 32. Additional applications of this system in control applications, as well as its passivity and optimality properties, have been reported in Refs. 26 and 33.

9 Conclusions

The new parameterization provides some interesting insights into the description of attitude kinematics. It fits neatly between the two best known parameterizations in the literature, namely the Eulerian angles and the Euler-Rodrigues parameters. It is presented here as a different formulation — not necessarily as a better one — for its usefulness is highly dependent on the application at hand.

Acknowledgment: The authors wish to thank Professor Malcolm Shuster and Professor John Junkins for helpful discussions. This research has been supported by the National Science Foundation under Grant No. MSS-9114388.

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