Relative Pose Stabilization using Backstepping Control with Dual Quaternions

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In this paper we propose a backstepping controller to stabilize the relative pose of a follower spacecraft with respect to a leader spacecraft. The relative dynamic equations of motion are derived in dual quaternion algebra, and used to constructively design a controller from an existing rotational-only control law. Asymptotic stability of the time-invariant closed loop system is proven through the use of Lyapunov stability theory in conjunction with the definition of two backstepping variables associated to the error pose and the error dual velocity. Numerical results show that the controller possesses desirable characteristics for proximity operations, such as the ability to maintain small transient relative distances and relative velocities, while still commanding small forces and torques.

Nomenclature

t	=	time (s)
$\mathbb{H}$	=	space of quaternions
$\mathbb{H}_d$	=	space of dual quaternions
$r^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	position quaternion from point Y to point X, in Z frame coordinates (m)
$v^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	linear velocity quaternion of point X with respect to frame Y, in Z frame coordinates (m/s)
$\omega^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	angular velocity quaternion of frame X relative to frame Y, in Z frame coordinates (rad/s)
$q_{\mathrm{X/Y}}$	=	quaternion describing attitude change from frame Y to frame X
$\mathsf{1}$	=	$(1, \bar{0})$
$\mathsf{0}$	=	$(0, \bar{0})$
ϵ	=	dual unit
$\boldsymbol{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}}$	=	dual velocity of frame X relative to frame Y, in Z frame coordinates
$\boldsymbol{q}_{\mathrm{X/Y}}$	=	dual quaternion describing pose change from frame Y to frame X
$\mathbf{1}$	=	$\mathsf{1} + \epsilon \mathsf{0}$
$\mathbf{0}$	=	$\mathsf{0} + \epsilon \mathsf{0}$
m_{s}	=	mass of satellite s (kg)
$\bar{I}_{\mathrm{s}}$	=	inertia matrix about the center of mass of body s, in s-frame coordinates ($kg.m^2$)
M_{s}	=	dual inertia matrix about the center of mass of body s, in s-frame coordinates
τ^{s}	=	net torque quaternion in s-frame coordinates ($N.m$)
f^{s}	=	net force quaternion in s-frame coordinates (N)
$\boldsymbol{f}^{\mathrm{s}}_{\mathrm{s}}$	=	net dual force about the center of mass of body s, in s-frame coordinates

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$\boldsymbol{f}^{\mathrm{s}}_{c,\mathrm{s}}$ = dual control force about the center of mass of body s, in s-frame coordinates
$\boldsymbol{f}^{\mathrm{s}}_{d,\mathrm{s}}$ = dual force due to disturbances about the center of mass of body s, in s-frame coordinates
$\boldsymbol{f}^{\mathrm{s}}_{g,\mathrm{s}}$ = dual force due to gravity about the center of mass of body s, in s-frame coordinates
$\boldsymbol{z}_1, \boldsymbol{z}_2$ = backstepping variables
$\boldsymbol{\alpha}$ = stabilizing function
$\boldsymbol{G}, \boldsymbol{H}$ = functions $\mathbb{H}_d \to \mathbb{R}^{8\times 8}$
K_1, K_2 = gain matrices in $\mathbb{R}^8$

I. Introduction

In the past, much of spacecraft control literature has focused on performing attitude (only) reference tracking through the use of a wide range of techniques and attitude parameterizations[1–5]. With the advent of space missions (both commercial and military), spacecraft proximity operations have become increasingly common, and they remain among the most critical phases for space-related activities. Ranging from on-orbit servicing, asteroid sample return, or just rendezvous and docking, these maneuvers pose a challenging technological problem that requires addressing the natural coupling between both the spacecraft's attitude as well as its position.

Originally, the modeling of rigid body motion to address proximity operations was decoupled into the corresponding attitude and position (pose) subproblems [6, 7]. This tends to be the simpler approach, as it takes advantage of conventional techniques. The drawback is usually efficiency and accuracy. New techniques that treat attitude and position on the same footing increase numerical efficiency and accuracy. The benefits have been especially dominant in the field of estimation, where combined representations of pose have led to significant improvements in the estimation of position [8–11], and also in the area of control design, where new interpretations of historically effective control laws for attitude problems, can be naturally re-interpreted to solve the equivalent pose-problem [12].

In this paper, and following on the footsteps of the previous work of [12, 13] we use dual quaternions to describe the complete relative pose of two spacecraft. No artificial separation of the translational and rotational motions is imposed and the natural motion between the two spacecraft is captured exactly. Dual quaternions are an extension of quaternions in the context of dual algebra. While unit quaternions carry information about the relative attitude between two frames, dual quaternions contain relative pose information between two frames, and as such, they allow for the extension of well-known results in the area of attitude control into the realm of pose control. Dual quaternions have been shown to have better computational efficiency and lower memory requirements than other conventional methods for kinematic modeling [14–17]. Since computational power is often limited in space-related tasks, this makes quaternions and dual quaternions more appropriate than, say, working directly with the more natural spaces $SO(3)$ (for attitude) or $SE(3)$ (for pose).

Performing on-orbit proximity operations is of interest due to a wide variety of reasons ranging from typical rendezvous and docking (R&D) maneuvers, commonly used in the context of operation of the ISS, to military and strategic inspection of an orbiting satellite. One of these applications is being able to statically (relative pose constant) inspect a known, co-operative target, regardless of what its state is. In Ref. [18] the problem of stabilization of the relative attitude of the spacecraft is approached. This method uses a backstepping control algorithm based on quaternions to stabilize the desired angular velocity and the attitude relative to the leader spacecraft. This can be useful in the context of a pointing based requirement, but it lacks relevance when the relative position of the leader with respect to the follower is important. A similar control design strategy is used by the authors in Refs. [19, 20], except with different control actuation schemes, or attitude perturbation models. While stabilizing the attitude of a follower relative to the leader is a necessary aspect for on-orbit inspection, it is not sufficient, particularly if these are separated by distances on the orbital scale, and their positions are drifting by orbital perturbations. Hence, the control of relative position will be addressed in this paper, by extension of the attitude result given in [18] to dual quaternions.

In this paper, we will show pose stabilization of a follower relative to a leader spacecraft, allowing us to *park* the follower satellite in a desired pose regardless of the state of the leader. The usefulness of this approach lies in the fact that it would allow us to observe a face of a satellite from a fixed relative pose, regardless of position changes or attitude changes of the leader.

The paper is structured as follows; Section II contains mathematical preliminaries, including an overview of quaternion and dual quaternion algebra, and its use to model kinematics and dynamics. The main result of the paper is derived in constructive fashion in Section III. In Section IV, the proposed controller is compared through simulation to an existing controller. Some conclusions are then listed in Section V.

II. Mathematical Preliminaries

This section will introduce the basic concepts of quaternions, dual quaternions, and their use in representing kinematics and dynamics of rigid bodies. For an exhaustive description the reader is referred to Refs. [12, 13, 21, 22], from which the notation has been adopted.

II.A. Quaternions

Quaternions are a mathematical tool commonly used to represent rotations in three-dimensional space. Quaternions define an associative, non-commutative algebra, defined as $\mathbb{H} := \{q = q_0 + q_1 i + q_2 j + q_3 k : i^2 = j^2 = k^2 = ijk = -1,\ q_i \in \mathbb{R}\}$. In practice, quaternions are referred to by their scalar and vectors parts as $q = (q_0, \bar{q})$, where $q_0 \in \mathbb{R}$ and $\bar{q} = [q_1, q_2, q_3]^{\mathrm{T}} \in \mathbb{R}^3$. The properties of quaternion algebra are summarized in Table 1. Previous literature has defined quaternion multiplication as the multiplication between a 4×4 matrix and a vector in $\mathbb{R}^4$.

Table 1. Quaternion Operations

Operation	Definition
Addition	$a + b = (a_0 + b_0, \bar{a} + \bar{b})$
Multiplication by a scalar	$\lambda a = (\lambda a_0, \lambda a)$
Multiplication	$ab = (a_0 b_0 - \bar{a} \cdot \bar{b}, a_0 \bar{b} + b_0 \bar{a} + \bar{a} \times \bar{b})$
Conjugate	$a^* = (a_0, -\bar{a})$
Dot product	$a \cdot b = (a_0 b_0 + \bar{a} \cdot \bar{b}, 0_{3\times 1})$
Cross product	$a \times b = (0, a_0 \bar{b} + b_0 \bar{a} + \bar{a} \times \bar{b})$
Norm	$\|a\| = \sqrt{a \cdot a}$

Since any rotation can be described by three parameters, the unit norm constraint is imposed on quaternions for attitude representation. *Unit* quaternions are closed under multiplication, but not under addition. A quaternion describing the orientation of frame X with respect to frame Y, $q_{\mathrm{X/Y}}$, will satisfy $q^*_{\mathrm{X/Y}} q_{\mathrm{X/Y}} = q_{\mathrm{X/Y}} q^*_{\mathrm{X/Y}} = \mathtt{1}$, where $\mathtt{1} = (1, 0_{3\times 1})$. This quaternion can be constructed as $q_{\mathrm{X/Y}} = (\cos(\phi/2), \bar{n} \sin(\theta/2))$, where $\bar{n}$ and θ are the *unit* Euler axis, and Euler angle of the rotation respectively. It is worth emphasizing that $q^*_{\mathrm{Y/X}} = q_{\mathrm{X/Y}}$, and that $q_{\mathrm{X/Y}}$ and $-q_{\mathrm{X/Y}}$ represent the same rotation. Furthermore, given quaternions $q_{\mathrm{Y/X}}$ and $q_{\mathrm{Z/Y}}$, the quaternion describing the rotation from X to Z is given by $q_{\mathrm{Z/X}} = q_{\mathrm{Y/X}} q_{\mathrm{Z/Y}}$.

Three-dimensional vectors can be interpreted as quaternions. That is, given $\bar{s}^{\mathrm{X}} \in \mathbb{R}^3$, the coordinates of a vector expressed in frame X, its quaternion representation is given by $s^{\mathrm{X}} = (0, \bar{s}^{\mathrm{X}}) \in \mathbb{H}^v$, where $\mathbb{H}^v$ is the set of *vector* quaternions defined as $\mathbb{H}^v \triangleq \{(q_0, \bar{q}) \in \mathbb{H} :\ q_0 = 0\}$ (see Ref. 12 for further information). The change of reference frame on a vector quaternion is achieved by the adjoint operation, and is given by $s^{\mathrm{Y}} = q^*_{\mathrm{Y/X}} s^{\mathrm{X}} q_{\mathrm{Y/X}}$.

In general, the attitude kinematics evolve as

$$\dot{q}_{\mathrm{X/Y}} = \tfrac{1}{2} q_{\mathrm{X/Y}} \omega^{\mathrm{X}}_{\mathrm{X/Y}} = \tfrac{1}{2} \omega^{\mathrm{Y}}_{\mathrm{X/Y}} q_{\mathrm{X/Y}}, \tag{1}$$

where $\omega^{\mathrm{Z}}_{\mathrm{X/Y}} \triangleq (0, \bar{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}}) \in \mathbb{H}^v$ and $\bar{\omega}^{\mathrm{Z}}_{\mathrm{X/Y}} \in \mathbb{R}^3$ is the angular velocity of frame X with respect to frame Y expressed in Z-frame coordinates.

II.B. Dual Quaternions

Dual quaternions are an extension of quaternions that arise in the study of dual numbers. A dual number can be described by $\boldsymbol{x} = x_r + \epsilon x_d$ for $x_r, x_d \in \mathbb{R}$, where ϵ is such that $\epsilon \neq 0, \epsilon^2 = 0$. A dual quaternion is a quaternion whose entries are dual numbers. This is analogous to defining the space of dual quaternions as $\mathbb{H}_d = \{\boldsymbol{q} = q_r + \epsilon q_d : q_r, q_d \in \mathbb{H}\}$. The nilpotent term ϵ commutes with the quaternion basis elements i, j, k, allowing us to define the basic properties listed in Table 2. Reference 23 also conveniently defines a multiplication between matrices and dual quaternions that resembles the well-known matrix-vector multiplication by simply representing the dual quaternion coefficients as a vector in $\mathbb{R}^8$. Additionally, the following

Table 2. Dual Quaternion Operations

Operation	Definition
Addition	$\boldsymbol{a} + \boldsymbol{b} = (a_r + b_r) + \epsilon(a_d + b_d)$
Multiplication by a scalar	$\lambda \boldsymbol{a} = (\lambda a_r) + \epsilon(\lambda a_d)$
Multiplication	$\boldsymbol{ab} = (a_r b_r) + \epsilon(a_d b_r + a_r b_d)$
Conjugate	$\boldsymbol{a}^* = (a_r^*) + \epsilon(a_d^*)$
Dot product	$\boldsymbol{a} \cdot \boldsymbol{b} = (a_r \cdot b_r) + \epsilon(a_d \cdot b_r + a_r \cdot b_d)$
Cross product	$\boldsymbol{a} \times \boldsymbol{b} = (a_r \times b_r) + \epsilon(a_d \times b_r + a_r \times b_d)$
Circle product	$\boldsymbol{a} \circ \boldsymbol{b} = (a_r \cdot b_r + a_d \cdot b_d) + \epsilon \mathsf{0}$
Swap	$\boldsymbol{a}^{\mathsf{s}} = a_d + \epsilon a_r$
Norm	$\|a\| = \sqrt{a \circ a}$
Vector part	$\mathrm{vec}\big(\boldsymbol{a}\big) = (0, \overline{a_r}) + \epsilon(0, \overline{a_d})$

property of the circle product for dual quaternions will be used (Lemma 39 of Ref. [12])

$$\boldsymbol{b} \circ (M \star \boldsymbol{a}) = \boldsymbol{a} \circ (M^{\mathrm{T}} \star \boldsymbol{b}). \tag{2}$$

Since rigid body motion has six degrees of freedom, a dual quaternion needs two constraints to parameterize it. The dual quaternion describing the relative pose of frame B relative to I is given by $\boldsymbol{q}_{\mathrm{B/I}} = q_{\mathrm{B/I},r} + \epsilon q_{\mathrm{B/I},d} = q_{\mathrm{B/I}} + \epsilon \frac{1}{2} q_{\mathrm{B/I}} r^{\mathrm{B}}_{\mathrm{B/I}}$, where $r^{\mathrm{B}}_{\mathrm{B/I}}$ is the position quaternion describing the location of the origin of frame B relative to that of frame I, expressed in B-frame coordinates. It can be easily observed that $q_{\mathrm{B/I},r} \cdot q_{\mathrm{B/I},r} = \mathsf{1}$ and $q_{\mathrm{B/I},r} \cdot q_{\mathrm{B/I},d} = \mathsf{0}$, where $\mathsf{0} = (0, \bar{0})$, providing the two necessary constraints. Thus, we say that a dual quaternion representing a pose transformation is a *unit* dual quaternion, since it satisfies $\boldsymbol{q} \cdot \boldsymbol{q} = \boldsymbol{q}^* \boldsymbol{q} = \mathbf{1}$, where $\mathbf{1} = \mathsf{1} + \epsilon \mathsf{0}$. For completeness purposes, let us also define $\mathbf{0} = \mathsf{0} + \epsilon \mathsf{0}$.

Furthermore, similar to quaternion relationships, the frame transformations laid out in Table 3 can be easily verified.

Table 3. Unit Dual Quaternion Operations

Composition of rotations	$\boldsymbol{q}_{\mathrm{Z/X}} = \boldsymbol{q}_{\mathrm{Y/X}} \boldsymbol{q}_{\mathrm{Z/Y}}$
Inverse, Conjugate	$\boldsymbol{q}^*_{\mathrm{Y/X}} = \boldsymbol{q}_{\mathrm{X/Y}}$

Analogous to the set of vector quaternions $\mathbb{H}^v$, we can define the set of vector dual quaternions as $\mathbb{H}^v_d \triangleq \{\boldsymbol{q} = q_r + \epsilon q_d : \; q_r, q_d \in \mathbb{H}^v\}$, to which the dual velocity belongs. The dual velocity is the generalization of velocity in dual algebra, and it contains a linear and an angular velocity term. The dual velocity is defined as

$$\boldsymbol{\omega}^{\mathrm{X}}_{\mathrm{Y/Z}} = \boldsymbol{q}^*_{\mathrm{X/Y}} \boldsymbol{\omega}^{\mathrm{Y}}_{\mathrm{Y/Z}} \boldsymbol{q}_{\mathrm{X/Y}} = \omega^{\mathrm{X}}_{\mathrm{Y/Z}} + \epsilon(v^{\mathrm{X}}_{\mathrm{Y/Z}} + \omega^{\mathrm{X}}_{\mathrm{Y/Z}} \times r^{\mathrm{X}}_{\mathrm{X/Y}}), \tag{3}$$

allowing us to define the dual quaternion kinematics as

$$\dot{\boldsymbol{q}}_{\mathrm{X/Y}} = \tfrac{1}{2} \boldsymbol{q}_{\mathrm{X/Y}} \boldsymbol{\omega}^{\mathrm{X}}_{\mathrm{X/Y}} = \tfrac{1}{2} \boldsymbol{\omega}^{\mathrm{Y}}_{\mathrm{X/Y}} \boldsymbol{q}_{\mathrm{X/Y}}. \tag{4}$$

The form of the kinematics expressed in dual quaternions resembles quaternion kinematics, differing only in the underlying algebra. This has particular advantages when extending control laws from rotational(-only) to combined rotational and translational results.

II.C. Frames

Four reference frames will be used throughout the paper. These are shown in Figure 1 and described as:

- BL (leader body reference frame): origin at the center of mass of the leader spacecraft, and the axes are fixed on the leader spacecraft.
- BF (follower body reference frame): origin at the center of mass of the follower spacecraft, and the axes are fixed on the follower spacecraft.
- D (desired reference frame): this is the reference frame for the follower spacecraft.
- I (inertial frame): this frame coincides with the Earth-Centered Inertial (ECI) frame.

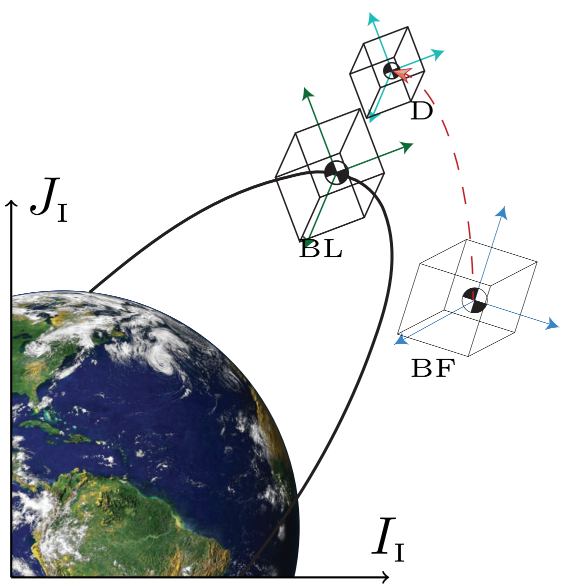

Figure 1. Frames used in the solution of this problem.

II.D. Dynamics

In Ref. [23], Filipe and Tsiotras represented the inertial *pose* dynamics in a manner that closely resembles the attitude(-only) dynamic equations of motion through the introduction of the swap operator and the redefinition of the matrix-dual quaternion multiplication. For both, the leader and the follower, identified as subscripts $_{\mathrm{BL}}$ and $_{\mathrm{BF}}$ respectively, the equation is given by:

$$M_{\mathrm{BL}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BL}}_{\mathrm{BL/I}})^{\mathsf{s}} = \boldsymbol{f}^{\mathrm{BL}} - (\boldsymbol{\omega}^{\mathrm{BL}}_{\mathrm{BL/I}}) \times (M_{\mathrm{BL}} \star (\boldsymbol{\omega}^{\mathrm{BL}}_{\mathrm{BL/I}})^{\mathsf{s}}), \tag{5}$$

$$M_{\mathrm{BF}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{BF/I}})^{\mathsf{s}} = \boldsymbol{f}^{\mathrm{BF}} - (\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/I}}) \times (M_{\mathrm{BF}} \star (\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/I}})^{\mathsf{s}}), \tag{6}$$

where $\boldsymbol{f}^{\mathrm{BL}}_{\mathrm{BL}} = f^{\mathrm{BL}} + \epsilon\tau^{\mathrm{BL}}$, and $\boldsymbol{f}^{\mathrm{BF}}_{\mathrm{BF}} = f^{\mathrm{BF}} + \epsilon\tau^{\mathrm{BF}}$ are the net external *dual forces* about the center of mass expressed in the reference frame of the corresponding body, and M_{BL}, $M_{\mathrm{BF}} \in \mathbb{R}^{8\times 8}$ are the *dual inertia matrices* defined as

$$M_{\mathrm{BL}} = \begin{bmatrix} 1 & 0_{1\times3} & 0 & 0_{1\times3} \\ 0_{3\times1} & m_{\mathrm{BL}} I_{3\times3} & 0_{3\times1} & 0_{3\times3} \\ 0 & 0_{1\times3} & 1 & 0_{1\times3} \\ 0_{3\times1} & 0_{3\times3} & 0_{3\times1} & \bar{I}_{\mathrm{BL}} \end{bmatrix}, \text{ and } M_{\mathrm{BF}} = \begin{bmatrix} 1 & 0_{1\times3} & 0 & 0_{1\times3} \\ 0_{3\times1} & m_{\mathrm{BF}} I_{3\times3} & 0_{3\times1} & 0_{3\times3} \\ 0 & 0_{1\times3} & 1 & 0_{1\times3} \\ 0_{3\times1} & 0_{3\times3} & 0_{3\times1} & \bar{I}_{\mathrm{BF}} \end{bmatrix},$$

where $\bar{I}_{\mathrm{BL}}$, $\bar{I}_{\mathrm{BF}} \in \mathbb{R}^{3\times3}$ are the mass moment of inertia of the corresponding body about the center of mass, and m_{BL}, m_{BF} is the corresponding mass of the bodies.

The dual force for each body can be decomposed into

$$\boldsymbol{f}^{\rm BF}_{\rm BF} = \boldsymbol{f}^{\rm BF}_{c,\rm BF} + \boldsymbol{f}^{\rm BF}_{d,\rm BF} + \boldsymbol{f}^{\rm BF}_{g,\rm BF}, \tag{7}$$

$$\boldsymbol{f}^{\rm BL}_{\rm BL} = \boldsymbol{f}^{\rm BL}_{c,\rm BL} + \boldsymbol{f}^{\rm BL}_{d,\rm BL} + \boldsymbol{f}^{\rm BL}_{g,\rm BL}, \tag{8}$$

where $\boldsymbol{f}^{\rm s}_{c,\rm s}$ is the control dual force exerted by the spacecraft, $\boldsymbol{f}^{\rm s}_{d,\rm s}$ is the dual disturbance force, and $\boldsymbol{f}^{\rm s}_{g,\rm s}$ is the gravitational force on body s (s = BF or s = BL), expressed in s-frame coordinates. In particular, it is assumed that $\boldsymbol{f}^{\rm s}_{g,\rm s}$ is given by

$$\begin{aligned}
\boldsymbol{f}^{\rm s}_{g,\rm s} &= m_{\rm s} a^{\rm s}_{g,\rm s} + \epsilon \mathsf{0}, \\
a^{\rm s}_{g,\rm s} &= (0, \bar{a}^{\rm s}_{g,\rm s}), \\
\bar{a}^{\rm s}_{g,\rm s} &= -\frac{\mu_E}{\|\bar{r}^{\rm s}_{\rm s/I}\|^3}\bar{r}^{\rm s}_{\rm s/I}.
\end{aligned}$$

II.E. Relative Kinematics and Dynamics

Given that it is our objective to stabilize the angular velocity of the follower, relative to that of the leader, we need to determine the equations that describe their relative motion. The relative pose of frame BF relative to frame BL is given by

$$\boldsymbol{q}_{\rm BF/BL} = \boldsymbol{q}^*_{\rm BL/I}\boldsymbol{q}_{\rm BF/I}. \tag{9}$$

It can be easily derived from its definition, and Equation (4), that this dual quaternion evolves as

$$\dot{\boldsymbol{q}}_{\rm BF/BL} = \tfrac{1}{2}\boldsymbol{q}_{\rm BF/BL}\boldsymbol{\omega}^{\rm BF}_{\rm BF/BL}. \tag{10}$$

This allows us to define the error angular velocity as

$$\boldsymbol{\omega}^{\rm BF}_{\rm BF/BL} = \boldsymbol{\omega}^{\rm BF}_{\rm BF/I} - \boldsymbol{\omega}^{\rm BF}_{\rm BL/I} = \boldsymbol{\omega}^{\rm BF}_{\rm BF/I} - \boldsymbol{q}^*_{\rm BF/BL}\boldsymbol{\omega}^{\rm BL}_{\rm BL/I}\boldsymbol{q}_{\rm BF/BL}. \tag{11}$$

Differentiating both sides of Equation (11), applying the swap operator and multiplying by $M_{\rm BF}$ leads to

$$\begin{aligned}
M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/BL})^{\mathsf{s}} = {}& M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/I})^{\mathsf{s}} - M_{\rm BF} \star (\dot{\boldsymbol{q}}^*_{\rm BF/BL}\boldsymbol{\omega}^{\rm BL}_{\rm BL/I}\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} \\
& - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}\dot{\boldsymbol{\omega}}^{\rm BL}_{\rm BL/I}\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}\boldsymbol{\omega}^{\rm BL}_{\rm BL/I}\dot{\boldsymbol{q}}_{\rm BF/BL})^{\mathsf{s}}.
\end{aligned}$$

Evaluating the error kinematics given by Equation (10) and simplifying,

$$\begin{aligned}
M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/BL})^{\mathsf{s}} &= M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/I})^{\mathsf{s}} - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}(\dot{\boldsymbol{\omega}}^{\rm BL}_{\rm BL/I} - \boldsymbol{\omega}^{\rm BL}_{\rm BF/BL} \times \boldsymbol{\omega}^{\rm BL}_{\rm BL/I})\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} \\
&= M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/I})^{\mathsf{s}} + M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}(\boldsymbol{\omega}^{\rm BL}_{\rm BF/BL} \times \boldsymbol{\omega}^{\rm BL}_{\rm BL/I})\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}\dot{\boldsymbol{\omega}}^{\rm BL}_{\rm BL/I}\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}}.
\end{aligned}$$

Using the dynamics given by Equations (5) and (6), we get

$$\begin{aligned}
M_{\rm BF} \star (\dot{\boldsymbol{\omega}}^{\rm BF}_{\rm BF/BL})^{\mathsf{s}} = {}& \boldsymbol{f}^{\rm BF} - (\boldsymbol{\omega}^{\rm BF}_{\rm BF/I}) \times (M_{\rm BF} \star (\boldsymbol{\omega}^{\rm BF}_{\rm BF/I})^{\mathsf{s}}) + M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}(\boldsymbol{\omega}^{\rm BL}_{\rm BF/BL} \times \boldsymbol{\omega}^{\rm BL}_{\rm BL/I})\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} \\
& - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}((M_{\rm BL})^{-1} \star (\boldsymbol{f}^{\rm BL} - (\boldsymbol{\omega}^{\rm BL}_{\rm BL/I}) \times (M_{\rm BL} \star (\boldsymbol{\omega}^{\rm BL}_{\rm BL/I})^{\mathsf{s}})))^{\mathsf{s}}\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} \\
= {}& \boldsymbol{f}^{\rm BF} + \boldsymbol{C}(\boldsymbol{f}^{\rm BL}),
\end{aligned} \tag{12}$$

where $\boldsymbol{C}(\boldsymbol{f}^{\rm BL})$ is defined for simplicity as

$$\begin{aligned}
\boldsymbol{C}(\boldsymbol{f}^{\rm BL}) \triangleq {}& -(\boldsymbol{\omega}^{\rm BF}_{\rm BF/I}) \times (M_{\rm BF} \star (\boldsymbol{\omega}^{\rm BF}_{\rm BF/I})^{\mathsf{s}}) + M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}(\boldsymbol{\omega}^{\rm BL}_{\rm BF/BL} \times \boldsymbol{\omega}^{\rm BL}_{\rm BL/I})\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}} \\
& - M_{\rm BF} \star (\boldsymbol{q}^*_{\rm BF/BL}((M_{\rm BL})^{-1} \star (\boldsymbol{f}^{\rm BL} - (\boldsymbol{\omega}^{\rm BL}_{\rm BL/I}) \times (M_{\rm BL} \star (\boldsymbol{\omega}^{\rm BL}_{\rm BL/I})^{\mathsf{s}})))^{\mathsf{s}}\boldsymbol{q}_{\rm BF/BL})^{\mathsf{s}}.
\end{aligned} \tag{13}$$

Notice that in the case where the leader actively cancels out disturbances ($\boldsymbol{f}^{\rm BL}_{c,\rm BL} = -\boldsymbol{f}^{\rm BL}_{d,\rm BL}$), then $\boldsymbol{C}(\boldsymbol{f}^{\rm BL}) = \boldsymbol{C}(\boldsymbol{f}^{\rm BL}_{g,\rm BL})$.

Finally, the desired reference for the follower will be given relative to the leader's body frame, BL. A general time-varying desired pose is assumed, which will then be specialized for the main result of this paper to a constant reference. The desired pose of the follower will be prescribed by $\boldsymbol{q}_{\rm D/BL}$. The dual velocity of frame D is also prescribed relative to frame BL by $\boldsymbol{\omega}^{\rm BL}_{\rm D/BL}$. Therefore, the error quaternion from the desired frame to the body of the follower is given by

$$\boldsymbol{q}_{\rm BF/D} = \boldsymbol{q}^*_{\rm D/BL}\boldsymbol{q}_{\rm BF/BL}, \tag{14}$$

and it evolves with the following equation

$$\dot{\boldsymbol{q}}_{\rm BF/D} = \tfrac{1}{2}\boldsymbol{q}_{\rm BF/D}\boldsymbol{\omega}^{\rm BF}_{\rm BF/D} = \tfrac{1}{2}\boldsymbol{q}_{\rm BF/D}(\boldsymbol{\omega}^{\rm BF}_{\rm BF/BL} - \boldsymbol{\omega}^{\rm BF}_{\rm D/BL}). \tag{15}$$

III. Backstepping Control

Our control objective is to asymptotically stabilize the unit dual quaternion $\boldsymbol{q}_{\mathrm{BF/D}}$ and the dual velocity $\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}}$ to $\mathbf{1}$ and $\mathbf{0}$ respectively. This will mean that the frame of the follower spacecraft, BF will be aligned with the desired reference frame, D. The proof for the stability of our controller is constructive and detailed next. Define $\boldsymbol{z}_1 \in \mathbb{H}_d$ as

$$\boldsymbol{z}_1 \triangleq (\mathbf{1} - \boldsymbol{q}_{\mathrm{BF/D}})^*. \tag{16}$$

Furthermore, define $\boldsymbol{\alpha} \in \mathbb{H}_d$, a stabilizing function to be designed, and $\boldsymbol{z}_2 \in \mathbb{H}_d$ such that

$$\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}} \triangleq -(\boldsymbol{\alpha} + (\boldsymbol{z}_2)^{\mathsf{s}}). \tag{17}$$

Then,

$$\dot{\boldsymbol{z}}_1 = -\dot{\boldsymbol{q}}^*_{\mathrm{BF/D}} = \tfrac{1}{2}\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}}\boldsymbol{q}^*_{\mathrm{BF/D}} = \tfrac{1}{2}\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}})\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}},$$

where the operator $\boldsymbol{G} : \mathbb{H}_d \to \mathbb{R}^{8\times 8}$ is defined such that

$$\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}) \star \boldsymbol{\omega} \triangleq \boldsymbol{\omega}\boldsymbol{q}^*. \tag{18}$$

Substituting $\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}}$ by Equation (17), yields

$$\dot{\boldsymbol{z}}_1 = -\tfrac{1}{2}\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star (\boldsymbol{\alpha} + (\boldsymbol{z}_2)^{\mathsf{s}}). \tag{19}$$

Let us now define the Lypanuov function candidate $\boldsymbol{V}_1(\boldsymbol{z}_1)$ by

$$\boldsymbol{V}_1(\boldsymbol{z}_1) = \tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{z}_1. \tag{20}$$

Notice that $\boldsymbol{V}_1(0_{8\times 1}) = 0$, and $\boldsymbol{V}_1(\boldsymbol{z}_1) > 0$ for $\boldsymbol{z}_1 \neq 0_{8\times 1}$. Taking the time derivative of $\boldsymbol{V}_1(\boldsymbol{z}_1)$ along the flow of the system yields

$$\begin{aligned}
\dot{\boldsymbol{V}}_1(\boldsymbol{z}_1) &= \boldsymbol{z}_1 \circ \dot{\boldsymbol{z}}_1 \\
&= -\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star (\boldsymbol{\alpha} + (\boldsymbol{z}_2)^{\mathsf{s}}) \\
&= -\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star \boldsymbol{\alpha} - \tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}}.
\end{aligned}$$

Choosing $\boldsymbol{\alpha}$ to be

$$\boldsymbol{\alpha} \triangleq K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\mathrm{BF/D}}) \star \boldsymbol{z}_1, \tag{21}$$

where $K_1 \in \mathbb{R}^{8\times 8}$ is a positive definite matrix, yields

$$\dot{\boldsymbol{V}}_1(\boldsymbol{z}_1) = -\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\mathrm{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}}$$

and $\dot{\boldsymbol{z}}_1$ in Eq. (19) becomes

$$\dot{\boldsymbol{z}}_1 = -\tfrac{1}{2}\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\mathrm{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{G}^{\mathrm{T}}(\boldsymbol{q}_{\mathrm{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}}. \tag{22}$$

Now, consider the time derivative of $\boldsymbol{z}_2$ from Eq. (17). Then,

$$\begin{aligned}
(\dot{\boldsymbol{z}}_2)^{\mathsf{s}} &= -\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{BF/D}} - \dot{\boldsymbol{\alpha}} \\
(\dot{\boldsymbol{z}}_2)^{\mathsf{s}} &= \dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{D/BL}} - \dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{BF/BL}} - \dot{\boldsymbol{\alpha}} \\
M_{\mathrm{BF}} \star \dot{\boldsymbol{z}}_2 &= M_{\mathrm{BF}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{D/BL}} - \dot{\boldsymbol{\alpha}})^{\mathsf{s}} - M_{\mathrm{BF}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{BF/BL}})^{\mathsf{s}},
\end{aligned}$$

and substituting the error dynamics given by Eq. (12) yields

$$M_{\mathrm{BF}} \star \dot{\boldsymbol{z}}_2 = M_{\mathrm{BF}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{D/BL}} - \dot{\boldsymbol{\alpha}})^{\mathsf{s}} - \boldsymbol{f}^{\mathrm{BF}} - \boldsymbol{C}(\boldsymbol{f}^{\mathrm{BL}}).$$

Assuming that the leader cancels non-gravitational perturbations, i.e. $\boldsymbol{f}^{\mathrm{BL}}_{c,\mathrm{BL}} = -\boldsymbol{f}^{\mathrm{BL}}_{d,\mathrm{BL}}$, yields

$$M_{\mathrm{BF}} \star \dot{\boldsymbol{z}}_2 = M_{\mathrm{BF}} \star (\dot{\boldsymbol{\omega}}^{\mathrm{BF}}_{\mathrm{D/BL}} - \dot{\boldsymbol{\alpha}})^{\mathsf{s}} - \boldsymbol{f}^{\mathrm{BF}}_{c,\mathrm{BF}} - \boldsymbol{f}^{\mathrm{BF}}_{d,\mathrm{BF}} - \boldsymbol{f}^{\mathrm{BF}}_{g,\mathrm{BF}} - \boldsymbol{C}(\boldsymbol{f}^{\mathrm{BL}}_{g,\mathrm{BL}}).$$

Define the control of the follower spacecraft to be

$$\boldsymbol{f}_{c,\text{BF}}^{\text{BF}} = M_{\text{BF}} \star (\dot{\boldsymbol{\omega}}_{\text{D/BL}}^{\text{BF}} - \dot{\boldsymbol{\alpha}})^{\mathsf{s}} - \boldsymbol{f}_{d,\text{BF}}^{\text{BF}} - \boldsymbol{f}_{g,\text{BF}}^{\text{BF}} - \boldsymbol{C}(\boldsymbol{f}_{g,\text{BL}}^{\text{BL}}) + K_2 \star \boldsymbol{z}_2 - \boldsymbol{H}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1, \tag{23}$$

where $K_2 \in \mathbb{R}^{8\times 8}$ is positive definite, and $\boldsymbol{H}(\cdot)$ is yet to be defined. Then, the dynamics for $\boldsymbol{z}_2$ become

$$M_{\text{BF}} \star \dot{\boldsymbol{z}}_2 = -K_2 \star \boldsymbol{z}_2 + H(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1. \tag{24}$$

Let us now define the Lypanuov function candidate $\boldsymbol{V}_2(\boldsymbol{z}_1, \boldsymbol{z}_2)$ by

$$\boldsymbol{V}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = \boldsymbol{V}_1(\boldsymbol{z}_1) + \tfrac{1}{4}\boldsymbol{z}_2 \circ M_{\text{BF}} \star \boldsymbol{z}_2 = \tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{z}_1 + \tfrac{1}{4}\boldsymbol{z}_2 \circ M_{\text{BF}} \star \boldsymbol{z}_2. \tag{25}$$

Notice that $\boldsymbol{V}_2(0_{8\times 1}, 0_{8\times 1}) = 0$, and $\boldsymbol{V}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) > 0$ for $\boldsymbol{z}_1 \neq 0_{8\times 1}$ and $\boldsymbol{z}_2 \neq 0_{8\times 1}$. Taking the time derivative of $\boldsymbol{V}_2(\boldsymbol{z}_1, \boldsymbol{z}_2)$ yields

$$\dot{\boldsymbol{V}}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = \boldsymbol{z}_1 \circ \dot{\boldsymbol{z}}_1 + \tfrac{1}{2}\boldsymbol{z}_2 \circ M_{\text{BF}} \star \dot{\boldsymbol{z}}_2. \tag{26}$$

Substituting Eqs. (22)-24) yields

$$\begin{aligned}\dot{\boldsymbol{V}}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = &-\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}} \\ &+ \tfrac{1}{2}\boldsymbol{z}_2 \circ (-K_2 \star \boldsymbol{z}_2 + H(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1).\end{aligned}$$

Re-arranging and using Eq. (2) yields

$$\begin{aligned}\dot{\boldsymbol{V}}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = &-\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{z}_2 \circ K_2 \star \boldsymbol{z}_2 \\ &-\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}} + \tfrac{1}{2}\boldsymbol{z}_1 \circ H(\boldsymbol{q}_{\text{BF/D}})^T \star \boldsymbol{z}_2.\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{V}}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = &-\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{z}_2 \circ K_2 \star \boldsymbol{z}_2 \\ &+\tfrac{1}{2}\boldsymbol{z}_1 \circ (H^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_2 - \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}}).\end{aligned}$$

Defining $\boldsymbol{H}(\boldsymbol{q})$ such that

$$H^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_2 \triangleq \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{z}_2)^{\mathsf{s}}, \tag{27}$$

yields

$$\dot{\boldsymbol{V}}_2(\boldsymbol{z}_1, \boldsymbol{z}_2) = -\tfrac{1}{2}\boldsymbol{z}_1 \circ \boldsymbol{G}^{\text{T}}(\boldsymbol{q}_{\text{BF/D}}) \star K_1 \star \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 - \tfrac{1}{2}\boldsymbol{z}_2 \circ K_2 \star \boldsymbol{z}_2 \leq 0. \tag{28}$$

Since assuming $\boldsymbol{\omega}_{\text{D/BL}} = \boldsymbol{0}$ makes the overall system time-invariant, Lyapunov's direct theorem ensures that $\boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are asymptotically stable. To proceed with the stability analysis of the closed loop system, we need the following Lemma.

Lemma 1. $\boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 = \boldsymbol{0}$ *if and only if* $\boldsymbol{q}_{\text{BF/D}} = \pm\boldsymbol{1}$

Proof.

($\Leftarrow$) Let $\boldsymbol{q}_{\text{BF/D}} = \boldsymbol{1}$, then $\boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 = \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{1} - \boldsymbol{q}_{\text{BF/D}})^* = \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{1} - \boldsymbol{1})^* = \boldsymbol{0}$. Let $\boldsymbol{q}_{\text{BF/D}} = -\boldsymbol{1}$. Then, $\boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 = 2 \cdot \text{diag}(0, -I_3, 0, -I_3) \star \boldsymbol{1} = \boldsymbol{0}$.
($\Rightarrow$) Let us denote $\boldsymbol{q}_{\text{BF/D}} = q_{\text{BF/D},r} + q_{\text{BF/D},d} = q_{\text{BF/D}} + \epsilon\frac{1}{2}q_{\text{BF/D}}r_{\text{BF/D}}^{\text{BF}} = (q_1, [q_2, q_3, q_4]^{\text{T}}) + \epsilon(q_5, [q_6, q_7, q_8]^{\text{T}})$. Then, by hypothesis

$$\boldsymbol{0} = \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star \boldsymbol{z}_1 = \boldsymbol{G}(\boldsymbol{q}_{\text{BF/D}}) \star (\boldsymbol{1} - \boldsymbol{q}_{\text{BF/D}})^* = \left(0, \begin{bmatrix} q_2 \\ q_3 \\ q_4 \end{bmatrix}\right) + \epsilon\left(0, \begin{bmatrix} q_1q_6 - q_2q_5 - q_3q_8 + q_4q_7 \\ q_1q_7 - q_3q_5 + q_2q_8 - q_4q_6 \\ q_1q_8 - q_2q_7 + q_3q_6 - q_4q_5 \end{bmatrix}\right). \tag{29}$$

Equating the real parts of the dual quaternions implies that $[q_2, q_3, q_4]^{\text{T}} = 0_{3\times 1}$. Since a quaternion has unit norm, this means that $q_1 = \pm 1$. Therefore, $q_{\text{BF/D},r} = q_{\text{BF/D}} = \pm\mathtt{1}$. The dual part of the dual quaternion then becomes $q_{\text{BF/D},d} = \frac{1}{2}q_{\text{BF/D}}r_{\text{BF/D}}^{\text{BF}} = \frac{1}{2}(\pm\mathtt{1})r_{\text{BF/D}}^{\text{BF}} = \pm\frac{1}{2}r_{\text{BF/D}}^{\text{BF}}$, which implies that $q_5 = 0$. Substituting $q_2 = q_3 = q_4 = q_5 = 0$ and $q_1 = \pm 1$ in Equation (29) yields $\boldsymbol{0} = (0, [0, 0, 0]^{\text{T}}) + \epsilon(0, \pm[q_6, q_7, q_8]^{\text{T}})$, which implies that $[q_6, q_7, q_8]^{\text{T}} = 0_{3\times 1}$. It is worth emphasizing that this condition implies that $r_{\text{BF/D}}^{\text{BF}} = \mathtt{0}$, and $q_{\text{BF/D}} = \mathtt{1}$, which implies that the pose of the follower is the same as the pose of the desired reference frame. ∎

Since $\boldsymbol{G}(\boldsymbol{q}_{\mathrm{BF/D}}) \star \boldsymbol{z}_1 \to \mathbf{0}$ as $t \to \infty$, one immediately obtains that $\boldsymbol{\alpha} \to \mathbf{0}$ as $t \to \infty$ from Equation (21), and from Lemma 1 we deduce that $\boldsymbol{q}_{\mathrm{BF/D}} \to \pm\mathbf{1}$ as $t \to \infty$. Additionally, since $\boldsymbol{z}_2 \to \mathbf{0}$ as $t \to \infty$, using Equation (17) we obtain that $\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}} \to \mathbf{0}$ as $t \to \infty$. Finally, since $\boldsymbol{q}_{\mathrm{BF/D}} = -\mathbf{1}$ is an unstable equilibrium, we have proven (almost) global asymptotic stability of the system to $\boldsymbol{q}_{\mathrm{BF/D}} = \mathbf{1}$ and $\boldsymbol{\omega}^{\mathrm{BF}}_{\mathrm{BF/D}} = \mathbf{0}$, which fulfills the control objective.

Remark 1. While the controller is based on stabilizing to a fixed desired pose, this pose is affixed relative to the frame of the leader. This implies that if the leader, and thus the desired frame, are tumbling, the control law will still be able to track the path traced by the desired frame to ensure the follower converges to this moving frame.

Remark 2. Dual quaternions inherit the unwinding phenomenon from quaternions. Therefore, even though $\boldsymbol{q}_{\mathrm{BF/D}} = -\mathbf{1}$ is an equilibrium point, it is unstable. Thus, the spacecraft may undergo large rotations even when it is near its desired equilibrium. The problem is well known and has been documented as a topological obstruction in the space $\mathcal{S}^3$, the 3-sphere in $\mathbb{R}^4$. For approaches on how to mitigate these effects, the reader is referred to Refs. 24–27.

IV. Numerical Results

The control law was implemented in simulation. The leader is in an almost-circular orbit whose parameters are listed in Table 4. The inertial position and velocity of the follower are perturbed by $\delta\bar{r} = [-0.027, -0.013, 0.022]^{\mathrm{T}}$ km and $\delta\bar{v} = [-0.0077, 0.0021, 0.0063]^{\mathrm{T}}$ km/s, relative to the initial conditions of the leader.

Table 4. Orbital elements for leader spacecraft.

Parameter	Value
Semi-major axis (a)	6775 km
Eccentricity (e)	0.00038111
Inclination (i)	51.6417°
Longitude of the Ascending Node (Ω)	299.0882°
Argument of perigee (w)	137.9621°
True anomaly (ν)	0°

The initial inertial attitude of the leader and the follower are given by $q_{\mathrm{BL/I}}(0) = \left(\sqrt{2}/2, [0, 0, \sqrt{2}/2]^{\mathrm{T}}\right)$ and $q_{\mathrm{BF/I}}(0) = \left(\sqrt{2}/2, [0, \sqrt{2}/2, 0]^{\mathrm{T}}\right)$ respectively. The inertial angular velocities are given by $\bar{\omega}^{\mathrm{BL}}_{\mathrm{BL/I}}(0) = [2\pi/180, -0.5\pi/180, 3\pi/180]^{\mathrm{T}}$ rad/s and $\bar{\omega}^{\mathrm{BF}}_{\mathrm{BF/I}}(0) = [-10\pi/180, -7\pi/180, -\pi/180]^{\mathrm{T}}$ rad/s. The mass of the leader is $m_{\mathrm{BL}} = 91$ kg, while the follower has a mass of $m_{\mathrm{BF}} = 5$ kg. Their inertias are respectively

$$\bar{I}_{\mathrm{BL}} = \begin{bmatrix} 100 & 5 & 2 \\ 5 & 100 & 5 \\ 2 & 5 & 100 \end{bmatrix} \quad \text{and} \quad \bar{I}_{\mathrm{BF}} = \begin{bmatrix} 0.5 & 0.2 & -0.1 \\ 0.2 & 1.0 & 0.1 \\ -0.1 & 0.1 & 0.3 \end{bmatrix} \tag{30}$$

The controller gains were selected as $K_1 = 0.1I_8$ and $K_2 = 4I_8$. Furthermore, the desired pose of the follower relative to the leader was prescribed to be $q_{\mathrm{D/BL}} = -\mathtt{1}$ and $\bar{r}^{\mathrm{BL}}_{\mathrm{D/BL}} = [10, 0, 0]^{\mathrm{T}}$m, resembling a situation in which we might wish to observe a specific face of a tumbling satellite, or simply prepare the spacecraft for the rendezvous phase of proximity operations. In order to show the benefits of the backstepping controller proposed herein, its performance is compared with the feedback controller of Ref. 28, with gains set to $k_p = 10$ and $k_d = 10$. These values helped ensure convergence of the controller and a settling time similar to that of the proposed controller. The results are shown below.

Figure 2 shows the relative pose between the body of the follower spacecraft and the body of the leader spacecraft as the follower maneuvers around it. Even though both controllers possess similar attitude performance, the performance of the translational part of the controller is significantly better for the proposed backstepping controller. Here, it is worth pointing out that different units are used to plot $\bar{r}^{\mathrm{BF}}_{\mathrm{BF/BL}}$ (and others below): m for the proposed controller and km for the velocity-feedback controller from Ref. 28.

Figure 3 shows the relative pose between the follower body frame and the desired reference frame. Since the control objective is to superimpose the two, the attitude tends to $q_{\mathrm{BF/D}} = \mathbb{1}$ and the position vector $\bar{r}^{\mathrm{BF}}_{\mathrm{BF/D}}$ tends to zero. Again, it can be observed how the follower spacecraft orbits at a distance of up to 500 m in the body x- and y-axes, relative to the desired frame, when using the velocity-feedback controller. In the case of the proposed backstepping controller, the spacecraft stays within 30 m of the target.

Figure 4 shows the relative angular and linear velocities of the follower with respect to the leader. In both cases these stabilize to the origin, as is expected, but the backstepping controller keeps the linear velocities closer to the desired relative velocity throughout the maneuver, and especially during the convergence phase.

Finally, we have the forces and torques applied by the follower spacecraft in Figure 5. While there is no substantial difference in the torques being applied, the forces for the proposed controller are one order of magnitude smaller than those of the controller from Ref. 28. For completeness purposes, the backstepping signals $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$ are shown in Figure 6, and they converge to the origin as is expected from the theoretical derivation.

The proposed controller possesses highly desirable characteristics since in general it reduces the relative distances to the target, by also reducing the relative velocities between the satellites. This decreased the risk of maneuvering around the leader and diminishes the consequences of a potential collision, which at high relative speeds could be catastrophic. The small relative distances during the transient of the approach also enhance relative navigation and aide sensing equipment that relies on proximity to the leader, such as vision cameras or infrared sensors.

A reason for the improved performance is that the proposed controller directly accounts for the effects of gravity on the leader, while the feedback controller of Ref. 28 only makes use of the dual acceleration of the leader to construct the desired reference.

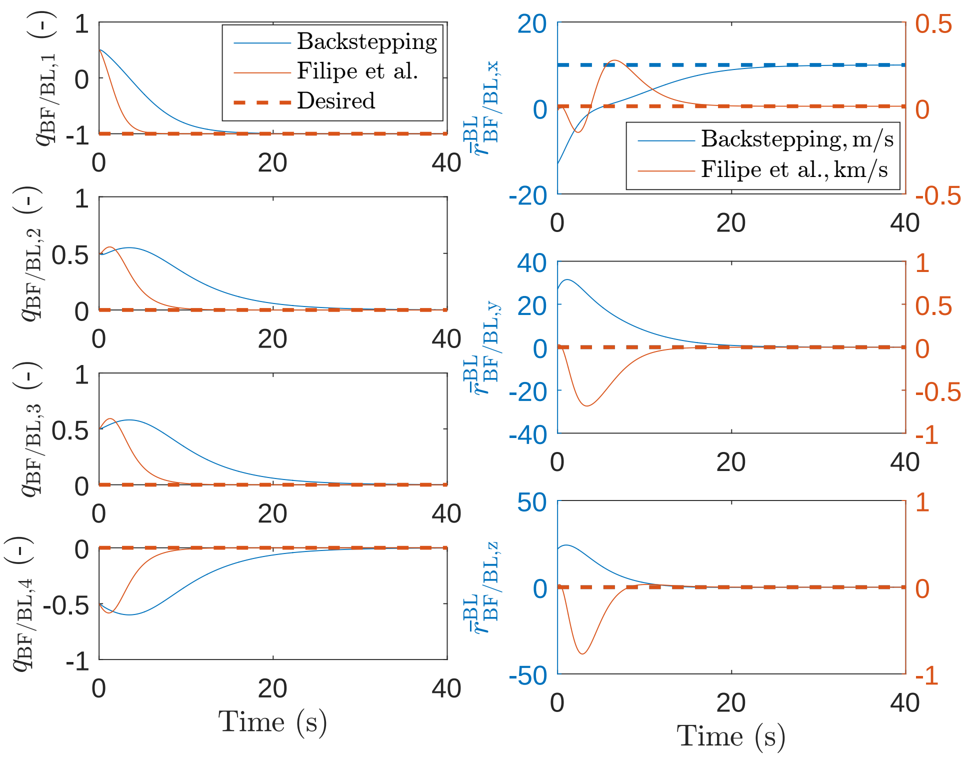

Figure 2. Relative pose between follower and leader spacecraft body frames. (Dashed lines show desired frame pose: $q_{\mathrm{D/BL}}$)

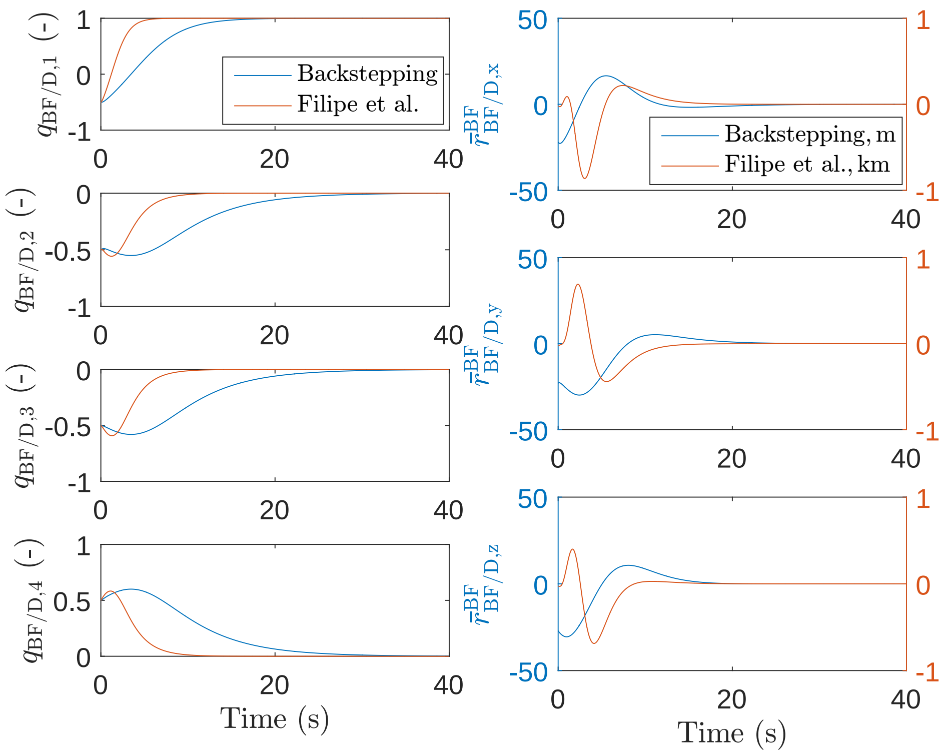

Figure 3. Relative pose between follower and desired reference frames.

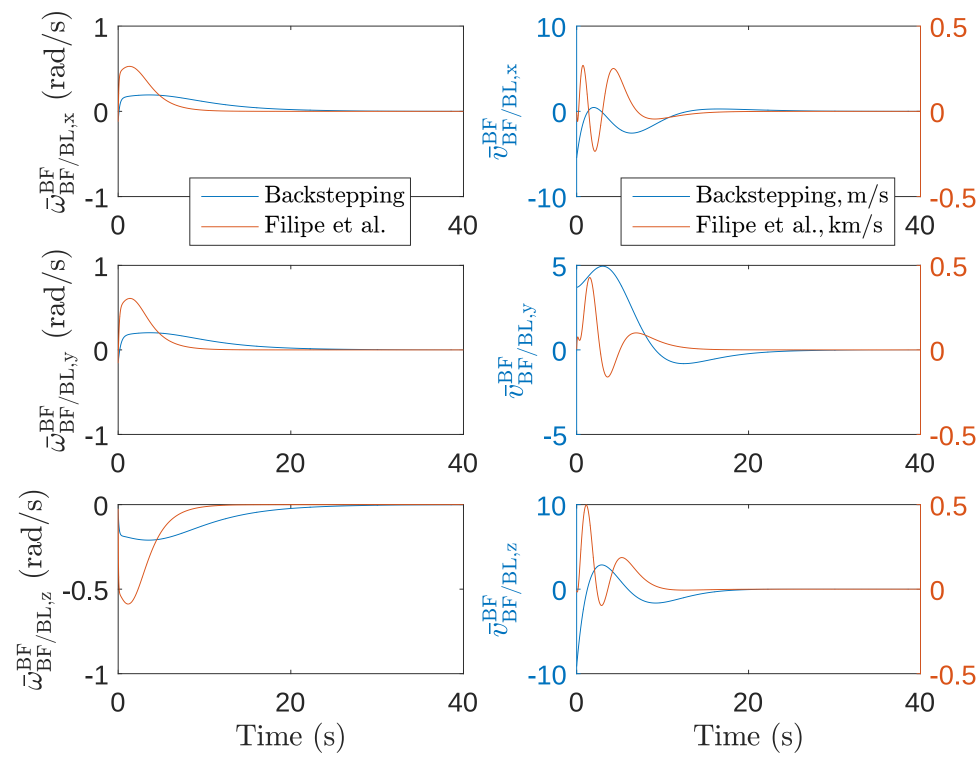

Figure 4. Relative linear and angular velocities between follower and leader spacecraft body frames.

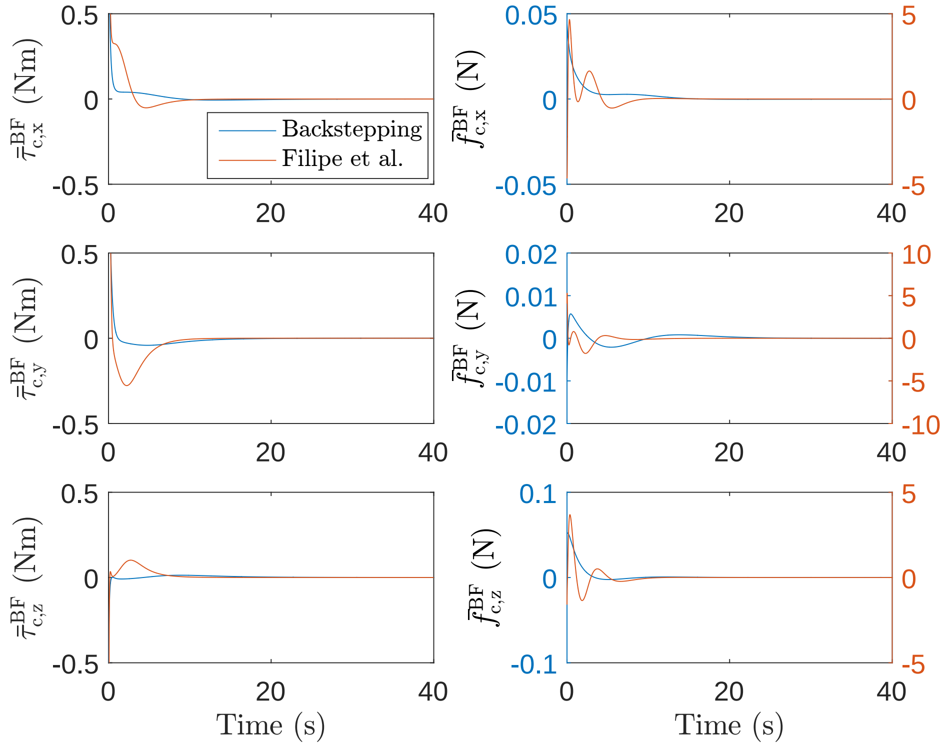

Figure 5. Control forces and torques applied by the follower spacecraft.

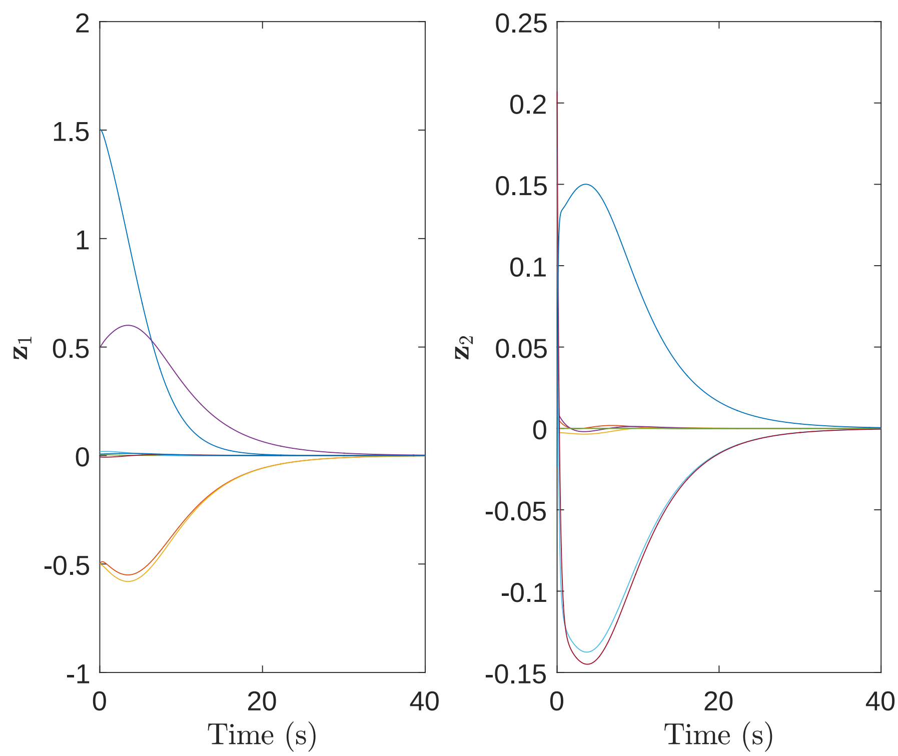

Figure 6. Backstepping variables.

V. Conclusion

In this paper backstepping has been used to design a pose-stabilizing controller for a satellite in a leader-follower spacecraft scenario. Through the extension of an attitude-stabilizing controller, dual quaternions allowed for an analogous result for pose-stabilization to be achieved. The controller allows for a fixed pose to be commanded relative to the leader satellite as the desired reference. The follower pose will asymptotically stabilize to this commanded pose, and the relative dual velocity will asymptotically stabilize to zero thus maintaining the same relative attitude and constant distance. The controller uses knowledge of the forces and torques affecting the dynamics of the leader as feedfoward, which previous results in pose tracking using dual quaternions have not taken into account. This results in smaller relative distances during the transient phase of proximity operations, and lower relative velocities.

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