Singularity Analysis and Avoidance of Variable-Speed Control Moment Gyros – Part I : No Power Constraint Case

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Single-gimbal control moment gyros (CMGs) have several advantages over other actuators for attitude control of spacecraft. For instance, they act as torque amplifiers and, thus, are suitable for slew maneuvers. However, their use as torque actuators is hindered by the presence of singularities, which, when encountered, do not allow a CMG cluster to generate torques about arbitrary directions. A method to overcome this drawback is the use variable-speed single-gimbal control moment gyros (VSCMGs). Whereas the wheel speed of a conventional CMG is constant, VSCMGs are allowed to have variable wheel speed. Therefore, VSCMGs have extra degrees of freedom that can be used to achieve additional objectives, such as singularity avoidance and/or power tracking, in addition to attitude control. In the present article, a gradient method using null motion to avoid singularities is presented. It is proved that there exists a null motion for all singularities, if a certain number of CMG wheels operate in VSCMG mode. In addition, the analysis is adapted to the case of conventional CMGs. The result verifies the already known conditions for the existence of null motions for CMG clusters.

I. Introduction

A control moment gyro (CMG) is a device used as a torque actuator for attitude control of spacecraft. It generates torques through angular momentum transfer to and from the main spacecraft body. This is achieved by changing the direction of the angular momentum vector of a gimballed flywheel. Since a CMG operates in a continuous manner – contrary to the gas jet's on/off operation – it can achieve precise attitude control. Moreover, as with other momentum exchange devices (e.g. reaction wheels), it does not consume any propellant, thus prolonging the operational life of the spacecraft. Single-gimbal CMGs essentially act as torque amplifiers.[1] This torque amplification property makes them particularly advantageous as attitude control actuators for large space spacecraft and space structures, e.g., a space station. In fact, single-gimbal and double-gimbal CMGs have been used for attitude control of the Skylab, the MIR and the International Space Station (ISS).

A disadvantage when using a CMG system in practice is the existence of singular gimbal angle states for which the CMGs cannot generate a torque along certain directions. At each singular state, all admissible torque directions lie on a two-dimensional surface in the three-dimensional angular momentum space. As a result, the CMG system cannot generate a torque normal to this surface. The CMG singularities can be classified into two categories: 1) external or saturated singularities, in which the total angular momentum sum of the CMGs lies on the maximum momentum envelope, and 2) internal singularities, in which the total momentum lies inside this envelope. The external singularities can be easily anticipated from the given CMG configuration and mission profile, therefore they can be taken into account at the design step. A properly designed momentum management scheme can also relieve the external singularity problem. The internal CMG singularities, on the other hand, are in general difficult to anticipate. Avoiding such internal singularities has thus been a long-standing problem in the CMG attitude control literature.[1–3]

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Two types of control moment gyros are typically used in practice, namely, the single gimbal CMG (SGCMG) and the double gimbal CMG (DGCMG). The SGCMGs have several advantages over DGCMGs. Firstly, they have a simpler mechanical structure. Secondly, their "torque amplification property" allows a small gimbal motor control (input) torque to generate a large (output) torque on the spacecraft. On the other hand, a SGCMG system has the disadvantage of nontrivial singular gimbal states. Although both SGCMGs and DGCMGs have singular gimbal angle states, the singularities of DGCMGs are trivial (i.e., external) or easily avoidable.[4,5] On the other hand, SGCMGs have internal singularities which cannot be avoided without generating an output torque error. Although one can reduce the possibility of encountering singularities by increasing the number of CMG wheels, it is generally impossible to completely eliminate the possibility of encountering singular states. Only for the special type of a "roof type," or "multiple type" SGCMG configuration,[4,6] which has no less than six wheels with each group of three wheels having an identical gimbal axis direction, it has been shown that all internal singularities are avoidable without generating torque error; see Refs. 4, 7. This wheel configuration is highly inefficient, however, in the sense that the radius of the momentum envelope is small despite the redundant number of wheels.[8] To make matters worse, the most popular steering law for SGCMGs (i.e., the minimum norm steering law) renders the singular states attractive thus exacerbating the steering problem.[1,2,9]

The internal unavoidable singularities make SGCMGs less popular for typical space missions despite their other advantages. DGCMGs have been used onboard Skylab[4,10,11] and in the International Space Station (ISS),[11] whereas the russian space station MIR used a cluster of six SGCMGs.[11,12] MIR used a skew-configured, redundant, six-wheel SGCMGs cluster, with only four CMGs being active at a time. In such a system there still exist unavoidable internal singularities, but all of them lie near the momentum envelope. This configuration is intentionally oversized in order to place all unavoidable singularities outside the required momentum reservoir.[5] In order to keep the terminology simple in the sequel the term CMG will be synonymous to a SGCMG system unless specifically stated otherwise.

A theory for the singularities of a CMG system was first rigorously established by Margulies and Aubrun.[1] They examined the geometric properties of the singular states, and introduced some important concepts, such as the momentum envelope. They also introduced the *null motion* technique to avoid certain singular states. The term null motion refers to any strategy that changes the gimbal angles without generating any torque. Based on the work of Ref. 1 several authors have refined and improved the analysis of the CMG singularity problem. Tokar and Platonov[3] showed that a CMG system in a pyramid configuration has singular states, at which a null motion strategy does not work. Kurokawa[4,7,8,13] published several papers studying the characteristics of the singularities both from a mathematical and a geometric point of view. Specifically, Kurokawa presented a method to distinguish between hyperbolic (i.e., avoidable) and elliptical (unavoidable) singularities. Bedrossian et al[14] also introduced a method for classifying singularities and demonstrated an analogy between the singularities of a CMG system and those of a robot manipulator. Most recently, Wie[15] provided a comprehensive mathematical treatment of the singularities of a CMG system, emphasizing the characterization and the visualization of the physical and geometrical nature of the singularities.

Most methods developed so far to tackle the CMG singularity problem can be divided into three categories. The first category consists of the so-called "singularity robust (SR)" methods,[11,16,17] which produce a torque error when encountering singular states. These methods are relatively simple but they cannot generate the commanded torque exactly in the vicinity of the singular states. The second category uses local gradient methods which rely on null motion. If the CMG cluster has a redundant number of wheels (i.e., more than three), there exist null motions which do not affect the output torque. The gradient methods search for a direction along which an objective function, containing information about the singularity, increases (or decreases) locally. One then applies a null motion along this direction. This method produces an output torque exactly equal to the required torque. However, as mentioned above, it is known that there are singular states which cannot be avoided using null motion methods.[3,4,14] The third category uses global avoidance methods, which include path planning,[5] preferred gimbal angle,[18] workspace restriction,[4,12] etc. These global methods anticipate the singular states and subsequecntly they steer the gimbal angles so that the CMG system does not encounter any singularities. However, some of these methods need off-line calculation,[5,18] while others do not fully utilize the angular momentum capacity of the CMG cluster.[4,12]

In the present article, a singularity avoidance method using single gimbal *variable speed* control moment gyros (VSCMGs) is presented. The concept of a VSCMG was first introduced by Ford and Hall[19] where it was called "gimballed momentum wheel." The term VSCMG was coined in Ref. 20, emphasizing the fact that these devices typically function as conventional CMGs. Whereas the wheel speed of a conventional

CMG is kept constant, the wheel speed of a VSCMG is allowed to vary continuously. A VSCMG can thus be considered as a hybrid device between a reaction wheel and a conventional CMG. The extra degree of freedom of a VSCMG, owing to the wheel speed changes, can be used to avoid the singularities.

The present article complements the results of Ref. 21 as well as those of Refs. 15 and 22 in several aspects. First, we provide a mathematical analysis for the singularities of a VSCMG cluster and present a singularity avoidance method using null motion for the VSCMG case. When applied to conventional CMGs, this analysis provides new insights into the issue of CMG singularities. In addition, it is more straightforward than existing ones.[14,15] Second, we offer a singularity avoidance and escape method (using null motion) and we characterize the conditions for its validity. It should be mentioned here that although Ref. 22 has also introduced a singularity avoidance method using VSCMGs, nonetheless, no conditions are provided in Ref. 22 under which such a strategy is possible.

This paper is organized as follows. In Section II, we provide the equations of a VSCMG actuator and define the singularity issue of a VSCMG cluster, as well as the related key terminology. In Section III, we provide a condition in which null motion can exist at a singular configuration. It is shown that a CMG cluster with no less than two wheels operating in VSCMG mode will avoid all singularities using null motion. In Section IV, a singularity avoidance scheme using null motion, and based on the gradient method, is also proposed. Finally, Section V presents numerical examples for the verification of the singularity analysis and the proposed singularity avoidance method for VSCMGs.

II. System Model and Singularity of VSCMGs

Consider a rigid spacecraft with a cluster of N single-gimbal VSCMGs, which are used to produce internal torques onboard a spacecraft. The mutually orthogonal unit vectors of the ith VSCMG are shown in Fig. 1 and are defined as follows:

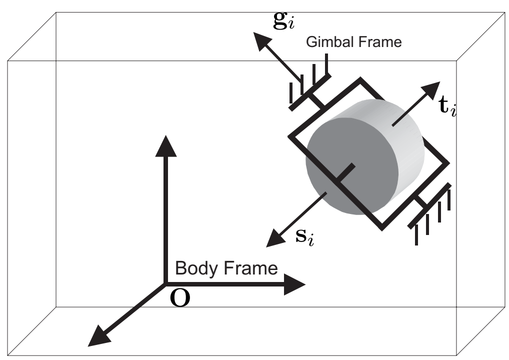

Figure 1. Spacecraft Body with a Single VSCMG

$\mathbf{g}_i$: gimbal axis vector

$\mathbf{s}_i$: spin axis vector

$\mathbf{t}_i$: transverse axis vector (torque vector), given as $\mathbf{t}_i = \mathbf{g}_i \times \mathbf{s}_i$.

The wheel of the VSCMG can rotate about the gimbal axis $\mathbf{g}_i$ with a gimbal angle γ_i. The wheel can also rotate about the spin axis $\mathbf{s}_i$ with an angular speed Ω_i. The unit vectors $\mathbf{s}_i$ and $\mathbf{t}_i$ depend on the gimbal angle γ_i, while the gimbal axis vector $\mathbf{g}_i$ is fixed in the body frame. The relationship between the derivatives of these unit vectors can be written as

$$\dot{\mathbf{s}}_i = \dot{\gamma}_i \mathbf{t}_i, \quad \dot{\mathbf{t}}_i = -\dot{\gamma}_i \mathbf{s}_i, \quad \dot{\mathbf{g}}_i = 0, \quad i = 1, \cdots, N. \tag{1}$$

In deriving the equations for a VSCMG actuator we will assume that the gimbal rates $\dot{\gamma}_i$ are much smaller than the wheel speeds Ω_i, so that $\dot{\gamma}_i$ do not contribute to the total angular momentum. We will also neglect the moments of inertia of the gimbal frame structures. These assumptions are common in the studies of

CMGs/VSCMGs systems and they are accurate for typical CMG/spacecraft configurations. For the exact equations of motion of a spacecraft with VSCMG actuators without these assumptions, see Refs. 21 and 20.

With a slight abuse of notation, in the sequel we use bold letters to denote both a vector and its elements in the standard basis. The angular momentum vector of each wheel can be expressed as $h_i\mathbf{s}_i$, for $i = 1, \ldots, N$, where $h_i = I_{\mathrm{ws}_i}\Omega_i$ and with I_{ws_i} denoting the moment of inertia of the ith VSCMG wheel about its spin axis. The total angular momentum $\mathbf{H}$ of the VSCMG system is the vector sum of the individual momenta of each wheel

$$\mathbf{H}(\gamma_1, \cdots, \gamma_N, \Omega_1, \cdots, \Omega_N) = \sum_{i=1}^{N} h_i\mathbf{s}_i. \tag{2}$$

The time derivative of $\mathbf{H}$ is equal to the torque $\mathbf{T}$ applied from the spacecraft main body to the VSCMG system, which is equal and opposite to the output torque from the VSCMG to the spacecraft body. This relation is written as

$$\mathbf{T} = \dot{\mathbf{H}} = \sum_{i=1}^{N} h_i\mathbf{t}_i\dot{\gamma}_i + \sum_{i=1}^{N} \mathbf{s}_i I_{\mathrm{ws}_i}\dot{\Omega}_i, \tag{3}$$

and in matrix form,

$$[C(\boldsymbol{\Omega}, \boldsymbol{\gamma}) \quad D(\boldsymbol{\gamma})] \begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\Omega}} \end{bmatrix} = \mathbf{T}, \tag{4}$$

where $C : \mathbb{R}^N \times [0, 2\pi)^N \to \mathbb{R}^{3\times N}$ and $D : [0, 2\pi)^N \to \mathbb{R}^{3\times N}$ are matrix-valued functions given by

$$C(\boldsymbol{\Omega}, \boldsymbol{\gamma}) \triangleq [I_{\mathrm{ws}_1}\Omega_1\mathbf{t}_1, \cdots, I_{\mathrm{ws}_N}\Omega_N\mathbf{t}_N] \tag{5}$$

$$D(\boldsymbol{\gamma}) \triangleq [I_{\mathrm{ws}_1}\mathbf{s}_1, \cdots, I_{\mathrm{ws}_N}\mathbf{s}_N] \tag{6}$$

and where $\boldsymbol{\gamma} \triangleq (\gamma_1, \ldots, \gamma_N)^T \in [0, 2\pi)^N$ and $\boldsymbol{\Omega} \triangleq (\Omega_1, \ldots, \Omega_N)^T \in \mathbb{R}^N$.

Notice that equations (4) are also valid for a conventional CMG system, if we set the wheel speeds Ω_i to be constant ($\dot{\boldsymbol{\Omega}} = 0$), and for a reaction wheel system, if we set the gimbal angles γ_i to be constant ($\dot{\boldsymbol{\gamma}} = 0$). For a conventional CMG system, without loss of generality, we may assume that $h_i = 1$ for $i = 1, \cdots, N$. Then the torque equation (4) becomes

$$C(\boldsymbol{\gamma})\,\dot{\boldsymbol{\gamma}} = \mathbf{T}, \tag{7}$$

where $C(\boldsymbol{\gamma}) = [\mathbf{t}_1, \cdots, \mathbf{t}_N]$. In order to generate a torque $\mathbf{T}$ along an arbitrary direction, we need rank $C(\boldsymbol{\gamma})$ = 3 for all $\boldsymbol{\gamma} \in [0, 2\pi)^N$. If rank $C(\boldsymbol{\gamma}_s) \neq 3$ for some $\boldsymbol{\gamma}_s$, however, $\dot{\boldsymbol{\gamma}}$ cannot be calculated for arbitrary torque commands.* Thus, henceforth we define the singular states of a CMG system as the gimbal states $\boldsymbol{\gamma}_s$ for which rank $C(\boldsymbol{\gamma}_s) = 2$.† In the singular states all unit vectors $\mathbf{t}_i$ lie on the same plane, and we can thus define a singular direction vector $\mathbf{u}$ which is normal to this plane. That is,

$$\mathbf{u}^T\mathbf{t}_i = 0, \quad \forall\, i = 1, \cdots, N. \tag{8}$$

The extra degree of freedom provide by changing the wheel speed of a VSCMG can be used to regulate the kinetic energy stored in a VSCMG. Therefore, VSCMG has been recently proposed as the actuators of choice for Integrated Power and Attitude Control Systems (IPACS) onboard spacecraft. Since the total kinetic energy stored in the wheels of the VSCMG cluster is

$$E \triangleq \tfrac{1}{2}\boldsymbol{\Omega}^T I_{\mathrm{ws}}\boldsymbol{\Omega}, \tag{9}$$

where $I_{\mathrm{ws}} \triangleq \mathrm{diag}[I_{\mathrm{ws}_1}, \cdots, I_{\mathrm{ws}_N}] \in \mathbb{R}^{N\times N}$, it follows that the power (rate of change of the energy) is given by

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \boldsymbol{\Omega}^T I_{\mathrm{ws}}\dot{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & \boldsymbol{\Omega}^T I_{\mathrm{ws}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\Omega}} \end{bmatrix}. \tag{10}$$

For an IPACS system the last equation has to be augmented to (7) and the control inputs $\dot{\boldsymbol{\gamma}}$ and $\dot{\boldsymbol{\Omega}}$ are to be determined for given $\mathbf{T}$ and P. In this paper we will not deal with the power tracking problem. The problem of combined attitude tracking and power tracking using a cluster of VSCMGs is treated instead in Ref. 23.

*Even in this case, there may exist a solution $\dot{\boldsymbol{\gamma}}$ to (7), if the required torque $\mathbf{T}$ lies in the two-dimensional range of $C(\boldsymbol{\gamma}_s)$, but this can be treated as an exceptional case.

†Rank $C = 1$ can happen only in very special configuration, for example, in roof-type configuration,[4] so we neglect this case.

A. Singular Configurations of a VSCMG System

If there are at least two wheels and their (fixed) gimbal axes are not parallel to each other (as in a pyramid configuration), and if none of the wheel speeds becomes zero, the column vectors of $[C\ D]$ in Eq. (4) always span the three-dimensional space, i.e., $\text{rank}([C\ D]) = 3$.[20] This means that we can always solve Eq. (4) for any given torque command $\mathbf{T}$. It follows that such a VSCMG system is always able to generate control torques along an arbitrary direction. In other words, such a VSCMG system never falls into a singularity, owing to the extra degrees of freedom provided by wheel speed control.[20–22] However, the norm of the column vectors of the matrix C is much larger than those of the matrix D. It is therefore preferable to generate the required torque using gimbal angle changes (i.e., as in CMGs) rather than using wheel speed changes (i.e., as in reaction wheels).‡ Also, for high wheel speeds it is power-inefficient to produce torques via wheel acceleration/deceleration. Therefore, in practice, it is desirable to keep $\text{rank}\, C = 3$. We thus define a VSCMG singularity as follows.

Definition 1 *The singularity of a VSCMG cluster is defined as the gimbal angle configuration, in which the rank of the matrix C is less than 3.*

Note that we have defined as a singularity the rank deficiency of the matrix C, even though in this case the VSCMGs will be able to generate an arbitrary torque. Notice also that if none of the wheel speeds is zero, the matrix C defined in Eq.(5) becomes singular if and only if the unit vectors $\mathbf{t}_i$ span a two-dimensional plane, similarly to the conventional CMG case. Hence, the rank of the matrix $C(\boldsymbol{\Omega}, \boldsymbol{\gamma})$ in (5) is independent of the (nonzero) wheel speeds. This observation leads us to the conclusion that the singularities of VSCMGs occur at a similar condition as the singularities of the conventional CMGs.

III. Singularity Analysis of VSCMGs

While the gimbal rates $\dot{\gamma}_i$ are the only control input variables in a CMG system, the wheel accelerations $\dot{\Omega}_i$ offer additional control variables in the case of a VSCMG system. The torque equation for the VSCMG case is

$$[C(\boldsymbol{\Omega}, \boldsymbol{\gamma}) \quad D(\boldsymbol{\gamma})] \begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\Omega}} \end{bmatrix} = \mathbf{T}.$$

When $\text{rank}\, C(\boldsymbol{\Omega}, \boldsymbol{\gamma}) = 2$ there exists a singular direction $\mathbf{u}$ perpendicular to this plane, that is, $\mathbf{u}^T \mathbf{t}_i = 0$ for all $i = 1, \cdots, N$ and the condition for singularity remains therefore the same as for the CMG case.

Schaub et al[20] introduced a technique to cope with this type of singularity of the matrix C using the "weighted" minimum norm solution of (4), namely,

$$\begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\Omega}} \end{bmatrix} = WQ^T (QWQ^T)^{-1} \mathbf{T}, \tag{11}$$

where $Q \triangleq [C\ D]$ and W is a weighting matrix which is function of the singularity index of the matrix C,[20,21] for instance,

$$W \triangleq \begin{bmatrix} w_1 e^{-w_2 \kappa(C)} \mathbf{I}_N & 0_N \\ 0_N & \mathbf{I}_N \end{bmatrix}, \tag{12}$$

where $\kappa(C)$ is the condition number of the matrix C and w_1 and w_2 are positive constants. According to this approach, the VSCMGs operate as CMGs to take full advantage of the torque amplification effect under normal conditions (i.e., $\kappa(C)$ is small) but as the singularity is approached, $\kappa(C)$ becomes large and the VSCMGs smoothly switch to a momentum wheel mode.[20,21] However, this technique is a passive method, which–by itself–does not ensure avoidance of singularities. Therefore, an active method to avoid the singularity is needed.

For this purpose, let us consider the possibility of null motion for a VSCMG system. Such a null motion must satisfy

$$[C \quad D] \begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\Omega}} \end{bmatrix}_{\text{null}} = \mathbf{0}_{3\times 1}. \tag{13}$$

‡This is the torque amplification effect of a CMG, which is the main advantage of the CMG system over other actuators.

Equivalently, a null motion strategy will not change the total angular momentum $\mathbf{H}$. Notice that $\dot{\boldsymbol{\gamma}} \in \mathcal{N}(C)$ and $\dot{\boldsymbol{\Omega}} \in \mathcal{N}(D)$ is a sufficient but not necessary condition for the existence of null motion. Even if $\dot{\boldsymbol{\gamma}} \notin \mathcal{N}(C)$ and $\dot{\boldsymbol{\Omega}} \notin \mathcal{N}(D)$, there still exists the possibility of satisfying Eq. (13). In fact, there always exists a null motion solution $[\dot{\boldsymbol{\gamma}}^T \ \dot{\boldsymbol{\Omega}}^T]^T_{\text{null}}$ satisfying Eq. (13), if $N > 2$.

Our objective is to investigate the possibility of escaping from a singularity using null motion. Mathematically, we are interested in conditions such that the following is true

$$\begin{aligned} C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}(t) + D(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\Omega}}(t) &= 0, \quad \text{at time } t \qquad (14) \\ C(\boldsymbol{\Omega}(t+\mathrm{d}t), \boldsymbol{\gamma}(t+\mathrm{d}t))\dot{\boldsymbol{\gamma}}(t+\mathrm{d}t) + D(\boldsymbol{\gamma}(t+\mathrm{d}t))\dot{\boldsymbol{\Omega}}(t+\mathrm{d}t) &= 0, \quad \text{at time } t+\mathrm{d}t \qquad (15) \end{aligned}$$

where $C(\boldsymbol{\Omega}, \boldsymbol{\gamma})$ and $D(\boldsymbol{\gamma})$ as in (5)-(6). Using Taylor's theorem, one obtains

$$\begin{aligned} \boldsymbol{\gamma}(t+\mathrm{d}t) &= \boldsymbol{\gamma}(t) + \dot{\boldsymbol{\gamma}}(t)\mathrm{d}t + r_1(\mathrm{d}t), \qquad \dot{\boldsymbol{\gamma}}(t+\mathrm{d}t) = \dot{\boldsymbol{\gamma}}(t) + \ddot{\boldsymbol{\gamma}}(t)\mathrm{d}t + r_2(\mathrm{d}t), \\ \boldsymbol{\Omega}(t+\mathrm{d}t) &= \boldsymbol{\Omega}(t) + \dot{\boldsymbol{\Omega}}(t)\mathrm{d}t + r_3(\mathrm{d}t), \qquad \dot{\boldsymbol{\Omega}}(t+\mathrm{d}t) = \dot{\boldsymbol{\Omega}}(t) + \ddot{\boldsymbol{\Omega}}(t)\mathrm{d}t + r_4(\mathrm{d}t). \end{aligned}$$

where $\lim_{\mathrm{d}t \to 0} \|r_i(\mathrm{d}t)\|/\mathrm{d}t = 0, i = 1, \ldots, 4$. The question of existence of null motion therefore reduces to one of finding $[\ddot{\boldsymbol{\gamma}}^T(t) \, \ddot{\boldsymbol{\Omega}}^T(t)]^T \in \mathbb{R}^{2N}$ such that (15) holds, given that (14) holds. Noticing that

$$\begin{aligned} C(\boldsymbol{\Omega}(t+\mathrm{d}t), \boldsymbol{\gamma}(t+\mathrm{d}t)) &= C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t)) + \sum_{i=1}^{N} \frac{\partial C}{\partial \gamma_i}\dot{\gamma}_i(t)\mathrm{d}t + \sum_{i=1}^{N} \frac{\partial C}{\partial \Omega_i}\dot{\Omega}_i(t)\mathrm{d}t + r_5(\mathrm{d}t) \\ D(\boldsymbol{\gamma}(t+\mathrm{d}t)) &= D(\boldsymbol{\gamma}(t)) + \sum_{i=1}^{N} \frac{\partial D}{\partial \gamma_i}\dot{\gamma}_i(t)\mathrm{d}t + r_6(\mathrm{d}t), \end{aligned}$$

where $\lim_{\mathrm{d}t \to 0} \|r_i(\mathrm{d}t)\|/\mathrm{d}t = 0, i = 5, 6$, we have from (15) that

$$\begin{aligned} 0 = {} & C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}(t) + D(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\Omega}}(t) \\ & + \left(\sum_{i=1}^{N} \frac{\partial C}{\partial \gamma_i}\dot{\gamma}_i(t)\mathrm{d}t\right)\dot{\boldsymbol{\gamma}}(t) + \left(\sum_{i=1}^{N} \frac{\partial C}{\partial \Omega_i}\dot{\Omega}_i(t)\mathrm{d}t\right)\dot{\boldsymbol{\gamma}}(t) + C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t))\ddot{\boldsymbol{\gamma}}(t)\mathrm{d}t \\ & + \left(\sum_{i=1}^{N} \frac{\partial D}{\partial \gamma_i}\dot{\gamma}_i(t)\mathrm{d}t\right)\dot{\boldsymbol{\Omega}}(t) + D(\boldsymbol{\gamma}(t))\ddot{\boldsymbol{\Omega}}(t)\mathrm{d}t + r_7(\mathrm{d}t) \end{aligned} \qquad (16)$$

with $\lim_{\mathrm{d}t \to 0} \|r_7(\mathrm{d}t)\|/\mathrm{d}t = 0$. Using Eq.(14), dividing with dt and taking the limit as d$t \to 0$, we have that a null motion exists if and only if there exist $\ddot{\boldsymbol{\gamma}}(t) \in \mathbb{R}^N$ and $\ddot{\boldsymbol{\Omega}}(t) \in \mathbb{R}^N$ such that the following is true

$$\begin{aligned} 0 = {} & C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t))\ddot{\boldsymbol{\gamma}}(t) + D(\boldsymbol{\gamma}(t))\ddot{\boldsymbol{\Omega}}(t) \\ & + \left(\sum_{i=1}^{N} \frac{\partial C}{\partial \gamma_i}\dot{\gamma}_i(t)\right)\dot{\boldsymbol{\gamma}}(t) + \left(\sum_{i=1}^{N} \frac{\partial C}{\partial \Omega_i}\dot{\Omega}_i(t)\right)\dot{\boldsymbol{\gamma}}(t) + \left(\sum_{i=1}^{N} \frac{\partial D}{\partial \gamma_i}\dot{\gamma}_i(t)\right)\dot{\boldsymbol{\Omega}}(t) \end{aligned} \qquad (17)$$

Condition (17) can be written as

$$[C(\boldsymbol{\Omega}(t), \boldsymbol{\gamma}(t)) \quad D(\boldsymbol{\gamma}(t))] \begin{bmatrix} \ddot{\boldsymbol{\gamma}}(t) \\ \ddot{\boldsymbol{\Omega}}(t) \end{bmatrix} = -2\sum_{i=1}^{N} I_{\mathrm{ws}i}\mathbf{t}_i\dot{\gamma}_i\dot{\Omega}_i + \sum_{i=1}^{N} I_{\mathrm{ws}i}\Omega_i\mathbf{s}_i\dot{\gamma}_i^2, \qquad (18)$$

where $[\dot{\boldsymbol{\gamma}}^T(t), \dot{\boldsymbol{\Omega}}^T(t)]^T \in \mathcal{N}([C\, D])$. Since the column vectors of $[C\, D]$ always span the 3-dimensional space, there always exist vectors $\ddot{\boldsymbol{\gamma}}(t) \in \mathbb{R}^N$ and $\ddot{\boldsymbol{\Omega}}(t) \in \mathbb{R}^N$ which satisfy (18). Thus, a null motion always exists for the VSCMG case. Most interestingly, we do not need all N wheels to operate as VSCMGs in order to avoid singularities. Only two out of all N wheels need to operate as VSCMGs, while the remaining $N-2$ may operate as conventional CMGs. This is due to the fact that any two inner products $\mathbf{u} \cdot \mathbf{s}_i$ cannot be zero at a singularity simultaneously, provided that no two gimbal directions are identical. We conclude that every singularity (in terms of the rank deficiency of C) is escapable with null motion $[\dot{\boldsymbol{\gamma}}^T, \dot{\boldsymbol{\Omega}}^T]^T_{\text{null}} \in \mathcal{N}([C\ D])$, if we have no less than two VSCMGs out of a total of N wheels.

Remark: The above analysis can be easily adapted to investigate the existence of null motions for a conventional CMGs cluster. By setting $h_i = 1$ for all $i = 1, \ldots, N$ and $\dot{\mathbf{\Omega}}(t) \equiv \ddot{\mathbf{\Omega}}(t) \equiv \mathbf{0}_{N\times 1}$ in Eq. (17) one yields the following condition for the existence of null motion for the CMG case as

$$C(\boldsymbol{\gamma}(t))\,\ddot{\boldsymbol{\gamma}}(t) = D(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}^2(t), \tag{19}$$

where $\dot{\boldsymbol{\gamma}}^2 \triangleq [\dot{\gamma}_1^2, \;\cdots\;, \dot{\gamma}_N^2]^T$ and $C(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}(t) = 0$. There exists $\dot{\boldsymbol{\gamma}}(t) \in \mathcal{N}[C(\boldsymbol{\gamma}(t))]$ such that equation (19) has a solution for some $\ddot{\boldsymbol{\gamma}}(t) \in \mathbb{R}^N$ if and only if there exists $\dot{\boldsymbol{\gamma}}(t) \in \mathcal{N}[C(\boldsymbol{\gamma}(t))]$ such that $D(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}^2(t) \in \mathcal{R}[C(\boldsymbol{\gamma}(t))]$, equivalently, $\mathbf{v}^T D(\boldsymbol{\gamma}(t))\dot{\boldsymbol{\gamma}}^2(t) = 0$, for all nonzero $\mathbf{v} \in \mathcal{R}^{\perp}[C(\boldsymbol{\gamma}(t))]$. Now recall that $\mathcal{R}^{\perp}[C(\boldsymbol{\gamma}(t))] = \text{span}\{\mathbf{u}\}$ hence the condition for existence of a solution to (19) is that

$$\exists \dot{\boldsymbol{\gamma}} \in \mathcal{N}[C(\boldsymbol{\gamma}_s)] \quad \text{such that} \quad \mathbf{u}^T D(\boldsymbol{\gamma}_s)\dot{\boldsymbol{\gamma}}^2 = 0. \tag{20}$$

Notice now that $\mathbf{u}^T D(\boldsymbol{\gamma}_s)\dot{\boldsymbol{\gamma}}^2 = \dot{\boldsymbol{\gamma}}^T \mathcal{P} \dot{\boldsymbol{\gamma}}$ where $\mathcal{P} \triangleq \text{diag}[\mathbf{u}^T\mathbf{s}_1, \cdots, \mathbf{u}^T\mathbf{s}_N]$. Therefore, (20) takes the form

$$\exists \dot{\boldsymbol{\gamma}} \in \mathcal{N}[C(\boldsymbol{\gamma}(t))] \quad \text{such that} \quad \dot{\boldsymbol{\gamma}}^T \mathcal{P} \dot{\boldsymbol{\gamma}} = 0. \tag{21}$$

Condition (21) for the existence of null motion is identical to the condition obtained by Ref. 14. However, the previous method is more straightforward and avoids using the concept of "virtual" gimbal angle displacements, which is to some degree, a mathematical artifact; see also Ref. 15.

IV. Singularity Avoidance Using Null Motion of VSCMGs

Let $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$ denote a measure of the singularity of the matrix $C(\mathbf{\Omega}, \boldsymbol{\gamma})$ which is a function of the gimbal angles and wheel speeds. Without loss of generality, let $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$ denote the condition number of the matrix C[§]. The condition number of C has to be kept small to avoid any singularities. The proposed method, commonly known as the "gradient method,"[4] adds a null motion which does not have any effect on the generated output torque but it decreases the singularity measure $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$. For example, if we let $Q = [C\ D]$, then any null motion can be written as

$$\begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\mathbf{\Omega}} \end{bmatrix}_{\text{null}} = [\mathbf{I}_{2N} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q]\tilde{W}\,\mathbf{d}, \qquad \mathbf{d} \in \mathbb{R}^{2N\times 1} \tag{22}$$

where $\tilde{W} > 0$ is some weighting matrix that can be used to distribute the control input between gimbal rate and wheel speed acceleration. It can be easily shown that $Q[\dot{\boldsymbol{\gamma}}^T, \dot{\mathbf{\Omega}}^T]^T_{\text{null}} = 0$, and the matrix $[\mathbf{I}_{2N} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q]\tilde{W}$ is positive semi-definite (See Appendix A for the proof.). If the vector $\mathbf{d}$ is selected as

$$\mathbf{d} = -k \begin{bmatrix} \frac{\partial \kappa}{\partial \boldsymbol{\gamma}}^T \\ \frac{\partial \kappa}{\partial \mathbf{\Omega}}^T \end{bmatrix}, \quad k > 0 \tag{23}$$

then, the rate of change of $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$ due to (22) is

$$\begin{aligned} \dot{\kappa}_{\text{null}} &= \begin{bmatrix} \frac{\partial \kappa}{\partial \boldsymbol{\gamma}} & \frac{\partial \kappa}{\partial \mathbf{\Omega}} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\mathbf{\Omega}} \end{bmatrix}_{\text{null}} \\ &= -k \begin{bmatrix} \frac{\partial \kappa}{\partial \boldsymbol{\gamma}} & \frac{\partial \kappa}{\partial \mathbf{\Omega}} \end{bmatrix} [\mathbf{I}_{2N} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q]\tilde{W} \begin{bmatrix} \frac{\partial \kappa}{\partial \boldsymbol{\gamma}}^T \\ \frac{\partial \kappa}{\partial \mathbf{\Omega}}^T \end{bmatrix} \leq 0. \end{aligned} \tag{24}$$

Therefore, it is expected that the singularity will be avoided. However, this method does not necessarily guarantee singularity avoidance, since the change of $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$ due to the torque-generating solution of Eq. (11) may dominate the change due to the null motion of Eq. (22). Nevertheless, singularity avoidance methods based on null motions have been successfully used in practice.[4,22] If κ represents the condition number as before, Ref. 22 provides an algorithm to quickly compute $\frac{\partial \kappa}{\partial \boldsymbol{\gamma}}$. The other sensitivity $\frac{\partial \kappa}{\partial \mathbf{\Omega}}$ can also be computed in a similar fashion; see also Ref. 25. Finally, we point out that although Eq. (22) looks similar to Eq. (16) of Ref. 22, the latter is missing the post-multiplication by the matrix $\tilde{W}$. Without it, one cannot ensure that $\dot{\kappa}_{\text{null}} \leq 0$ (See Appendix A.).

[§]Among the several choices of $\kappa(\boldsymbol{\gamma}, \mathbf{\Omega})$, the condition number has been selected as a measure of the singularity in this article, since it is known that it is a more reliable measure of rank-deficiency of a matrix than, say, the determinant of the matrix.[24]

V. Numerical Examples

A numerical example is provided to test the proposed singularity avoidance method in Eqs. (22) and (23). A spacecraft with four VSCMGs in a regular pyramid configuration is used for all numerical simulations. Table 1 contains the parameters used for the simulations. They closely parallel those used in Refs. 20, 26 and 21. In Table 1 ${}^{B}I$ denotes the spacecraft moment of inertia matrix without the VSCMG cluster and $I_{g\star}$ ($\star = \mathrm{g}, \mathrm{t}, \mathrm{s}$) denote the inertias of the gimbal frame. For more information on the exact equations on motion for a spacecraft with a VSCMG cluster, see Ref. 21.

Table 1. Simulation Parameters

Symbol	Value	Units
N	4	–
θ	54.75	deg
$\boldsymbol{\gamma}(0)$	$[\pi/2, -\pi/2, -\pi/2, \pi/2]^T$	rad
$\dot{\boldsymbol{\gamma}}(0)$	$[0, 0, 0]^T$	rad/sec^2
${}^{B}I$	$\begin{bmatrix} 15053 & 3000 & -1000 \\ 3000 & 6510 & 2000 \\ -1000 & 2000 & 11122 \end{bmatrix}$	kg m^2
I_{ws}	diag{0.7, 0.7, 0.7, 0.7}	kg m^2
$I_{\mathrm{wt}}, I_{\mathrm{wg}}$	diag{0.4, 0.4, 0.4, 0.4}	kg m^2
$I_{\mathrm{gs}}, I_{\mathrm{gt}}, I_{\mathrm{gg}}$	diag{0.1, 0.1, 0.1, 0.1}	kg m^2

The exact equations of motion from Ref. 21 are used in all simulations in order to validate our approach. For all simulations the initial reference attitude is assumed to be aligned with the inertial frame and the angular velocity of the reference attitude is chosen as

$$\boldsymbol{\omega}_{\mathrm{d}}(t) = \begin{bmatrix} 2 \times 10^{-3} \sin(2\pi t/9000) \\ -3 \times 10^{-3} \sin(2\pi t/12000) \\ 1 \times 10^{-3} \sin(2\pi t/10000) \end{bmatrix} \text{(rad/sec)}$$

The initial attitude of the spacecraft body frame is chosen as

$$\mathbf{q}_0 = [-1, 1, -1, 1]^T/\sqrt{4} \tag{25}$$

where $\mathbf{q}$ is the Euler parameter vector with respect to the inertial frame. The initial attitude in (25) corresponds to the 3-2-1 Euler angle set $\phi_0 = -90°$, $\theta_0 = 0°$, $\psi_0 = -90°$. The initial angular velocity of the body frame is set to zero.

The results from two simulations are presented below. In the first case only the attitude tracking control of Eq. (11) is applied. In the second case the singularity avoidance control of Eqs. (22) and (23) is used simultaneously with the torque generating solution of (11). In particular, only two CMGs are allowed to operate in VSCMG mode in this scenario. The gain in the singularity avoidance control is chosen as $k = 0.005$, and the weighting matrix $\tilde{W}$ as

$$\tilde{W} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 1 \times 10^4 \mathbf{I} \end{bmatrix}.$$

Figure 2 shows the reference and actual attitude histories. In these plots the subscript d designates the desired quaternion history. The spacecraft attitude tracks the desired attitude exactly after a short period of time. Figure 3 shows that the matrix C becomes close to being singular after approximately $t = 8000$ sec without any singularity avoidance algorithm. The control input $\dot{\boldsymbol{\Omega}}$ becomes very large during this period, since the weighting matrix W in Eq. (11) makes the VSCMGs operate in reaction wheel mode, and so $\dot{\boldsymbol{\Omega}}$ has to generate the required output torque. Note that without the weighting matrix, the gimbal rate input $\dot{\boldsymbol{\gamma}}$ would become very large, instead of $\dot{\boldsymbol{\Omega}}$. Both cases are undesirable.

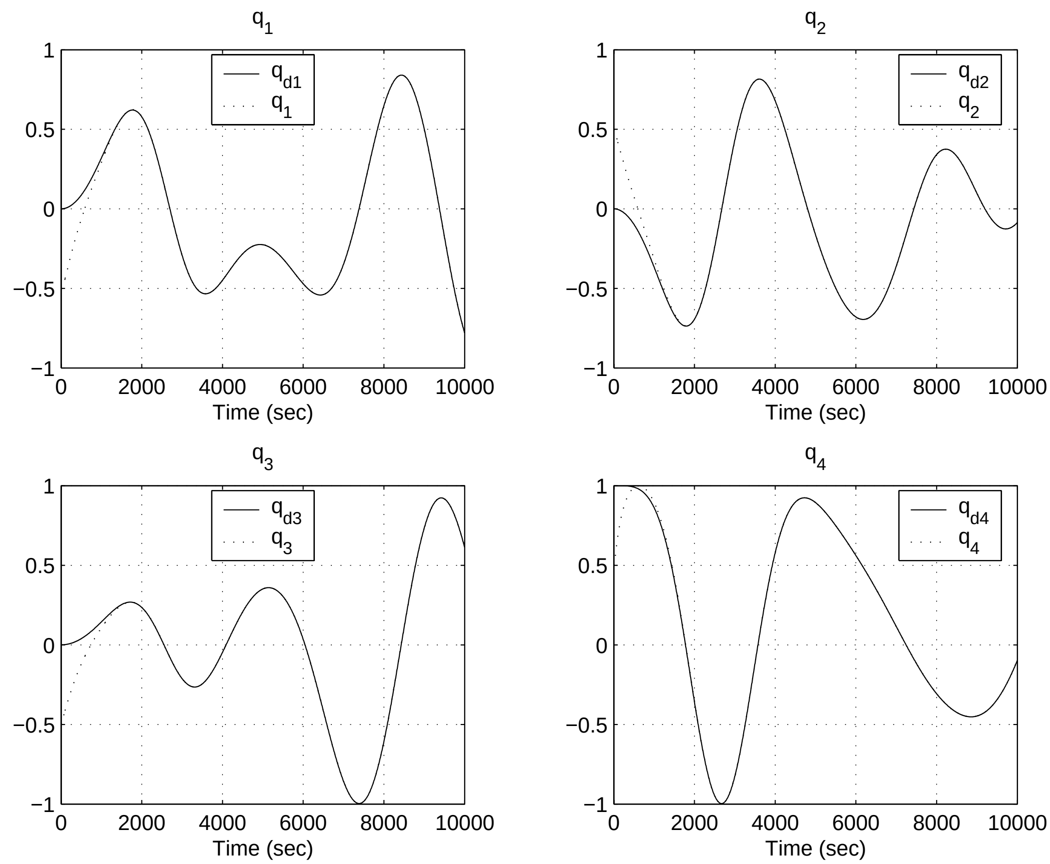

Figure 2. Reference and Actual Attitude Trajectory

On the other hand, Fig. 4 shows that singularities are successfully avoided using the null motion algorithm of Eq. (22). Although slightly larger control inputs $\dot{\boldsymbol{\gamma}}$ are needed to reconfigure the gimbal angles as the matrix C approach the singular states, the overall magnitudes of both $\dot{\boldsymbol{\gamma}}$ and $\dot{\boldsymbol{\Omega}}$ are kept within a reasonable range, contrary to the case without a singularity avoidance strategy. Note that the singularities are avoided even though two of the wheel speeds, Ω_3 and Ω_4, are kept constant.

The attitude profiles are exactly the same as in the previous case, and are shown in Figs. 2. It should be pointed out that the attitude time histories with null motion are identical to those without null motion, that is, the null motion has not affected the output torque delivered to the spacecraft bus, as expected.

VI. Conclusions

In this article the singularity problem associated with a VSCMGs system is introduced and studied in detail. A VSCMG system has more degrees of freedom than a conventional CMG system, so in theory it can generate arbitrary torques. However, in practice it is still desirable to keep the gimbal angles away from singular configurations in order to make the best use of the torque amplification effect of the CMGs.

A gradient-based method using null motion has been introduced to avoid the singularities of a VSCMG cluster. It has been shown that this method will work if no less than two wheels operate in variable wheel speed mode and there is no power requirement. Our proof for identifying singular states can be adapted to the conventional CMG case, yielding the same condition for null motion found in the literature. Nonetheless, our new proof is more straightforward and gives new insights into the CMG/VSCMG singularity problem.

Appendix A

In this appendix, it is proved that the matrix P_{null} defined as

$$P_{\text{null}} \triangleq [\mathbf{I}_{2N} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q]\tilde{W} \tag{A.1}$$

in Eq. (22) is positive semi-definite. The matrix P_{null} can be written as

$$\begin{aligned} P_{\text{null}} &= \tilde{W} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W} \\ &= \tilde{W}^{\frac{1}{2}}[\mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}}]\tilde{W}^{\frac{1}{2}} \end{aligned} \tag{A.2}$$

where $\tilde{W}^{\frac{1}{2}}\tilde{W}^{\frac{1}{2}} = \tilde{W}$ and $\tilde{W}^{\frac{T}{2}} = \tilde{W}^{\frac{1}{2}}$. Let $P_1 = \mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}}$, then $P_1^T = P_1$, and

$$\begin{aligned} P_1P_1 &= [\mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}}][\mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}}] \\ &= \mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}} \\ &\quad + \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}} \\ &= \mathbf{I}_{2N} - \tilde{W}^{\frac{1}{2}}Q^T(Q\tilde{W}Q^T)^{-1}Q\tilde{W}^{\frac{1}{2}} \\ &= P_1, \end{aligned} \tag{A.3}$$

so P_1 is symmetric and idempotent and thus positive semi-definite. It follows that that P_{null} is also positive semi-definite.

It should be pointed out that the matrix $P'_{\text{null}} \triangleq \mathbf{I}_{2N} - \tilde{W}Q^T(Q\tilde{W}Q^T)^{-1}Q$ used in Ref. 22 is missing the postmultiplication by the matrix $\tilde{W}$, and thus it is not positive semi-definite in general, because it is idempotent but not symmetric.

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References

[1]Margulies, G. and Aubrun, J. N., "Geometric Theory of Single-Gimbal Control Moment Gyro Systems," *Journal of the Astronautical Science*, Vol. 26, No. 2, 1978, pp. 159–191.

[2]Cornick, D. E., "Singularity Avoidance Control Laws for Single Gimbal Control Moment Gyros," *Proceedings of the AIAA Guidance and Control Conference*, AIAA, New York, 1979, pp. 20–33.

[3]Tokar, E. N. and Platonov, V. N., "Singular Surfaces in Unsupported Gyrodyne Systems," *Cosmic Research*, Vol. 16, No. 5, 1979, pp. 547–555.

[4]Kurokawa, H., "A Geometry Study of Single Gimbal Control Moment Gyros - Singularity Problem and Steering Law," Tech. Rep. Report No.175, Mechanical Engineering Laboratory, Japan, Jan. 1998.

[5]Paradiso, J. A., "Global Steering of Single Gimballed Control Moment Gyroscopes Using a Directed Search," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, Sep.-Oct. 1992, pp. 1236–1244.

[6]Yoshikawa, T., "Steering Law for Roof Type Configuration Control Moment Gyro System," *Automatica*, Vol. 13, No. 4, July 1979, pp. 359–368.

[7]Kurokawa, H., Yajima, N., and Usui, S., "A Study of Single Gimbal CMG System," *Proceedings of 15th International Symposium on Space Technology and Science*, Japan, 1986, pp. 1219–1224.

[8]Kurokawa, H. and Yajima, N., "A Study of CMG Systems - for Selection and Evaluation," *Proceedings of 16th International Symposium on Space Technology and Science*, Japan, 1988, pp. 1243–1249.

[9]Heiberg, C. J., Bailey, D., and Wie, B., "Precision Spacecraft Pointing Using Single-Gimbal Control Moment Gyroscopes with Disturbance," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 1, 2000, pp. 77–85.

[10]Wertz, J. R., editor, *Spacecraft Attitude Determination and Control*, D. Reidel Publishing Company, Dordrecht, Netherlands, 1986 (reprinted), pp. 196–203.

[11]Wie, B., Bailey, D., and Heiberg, C. J., "Singularity Robust Steering Logic for Redundant Single-Gimbal Control Moment Gyros," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 5, 2001, pp. 865–872.

[12]Kurokawa, H., "Constrained Steering Law of Pyramid-Type Control Moment Gyros and Ground Tests," *Journal of Guidance, Control, and Dynamics*, Vol. 20, No. 3, 1997, pp. 445–449.

[13]Kurokawa, H., "Exact Singularity Avoidance Control of the Pyramid Type CMG System," *Proceedings of the AIAA Guidance, Navigation and Control Conference*, Scottsdale, AZ, 1994, pp. 170–180, AIAA Paper 94-3559.

[14]Bedrossian, N. S., Paradiso, J., Bergmann, E. V., and Rowell, D., "Redundant Single Gimbal Control Moment Gyroscope Singularity Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 6, 1990, pp. 1096–1101.

[15]Wie, B., "Singularity Analysis and Visualization for Single-Gimbal Control Moment Gyro Systems," *AIAA Guidance, Navigation, and Control Conference*, Austin, TX, August 2003, Paper No. 2003-5658.

[16]Oh, H. and Vadali, S., "Feedback Control and Steering Laws for Spacecraft Using Single Gimbal Control Moment Gyro," *The Journal of the Astronautical Sciences*, Vol. 39, No. 2, 1991, pp. 183–203.

[17]Ford, K. A. and Hall, C. D., "Singular Direction Avoidance Steering for Control-Moment Gyros," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 4, 2000, pp. 648–656.

[18]Vadali, S. R., Oh, H. S., and Walker, S. R., "Preferred Gimbal Angles for Single Gimbal Control Moment Gyros," *Journal of Guidance Control, and Dynamics*, Vol. 13, No. 6, 1990, pp. 1090–1095.

[19]Ford, K. A. and Hall, C. D., "Flexible Spacecraft Reorientations Using Gimbaled Momentum Wheels," *Advances in the Astronautical Sciences, Astrodynamics*, edited by F. Hoots, B. Kaufman, P. J. Cefola, and D. B. Spencer, Vol. 97, Univelt, San Diego, 1997, pp. 1895–1914.

[20]Schaub, H., Vadali, S. R., and Junkins, J. L., "Feedback Control Law for Variable Speed Control Moment Gyroscopes," *Journal of the Astronautical Sciences*, Vol. 46, No. 3, 1998, pp. 307–328.

[21]Yoon, H. and Tsiotras, P., "Spacecraft Adaptive Attitude And Power Tracking With Variable Speed Control Moment Gyroscopes," *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 6, Nov.-Dec. 2002, pp. 1081–1090.

[22]Schaub, H. and Junkins, J. L., "Singularity Avoidance Using Null Motion and Variable-Speed Control Moment Gyros," *Journal of Guidance Control, and Dynamics*, Vol. 23, No. 1, 2000, pp. 11–16.

[23]Yoon, H. and Tsiotras, P., "Singularity Analysis and Avoidance of Variable-Speed Control Moment Gyros – Part II : Power Constraint Case," *Proceeding of AIAA/AAS Astrodynamics Specialist Conference and Exhibit*, Providence, Rhode Island, Aug. 2004.

[24]Horn, R. and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, Cambridge, United Kingdom, 1985, pp. 427, 449.

[25]Junkins, J. L. and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures*, AIAA, New York, 1993, pp. 48–49.

[26]Richie, D. J., Tsiotras, P., and Fausz, J. L., "Simultaneous Attitude Control and Energy Storage using VSCMGs : Theory and Simulation," *Proceedings of American Control Conference*, 2001, pp. 3973–3979.

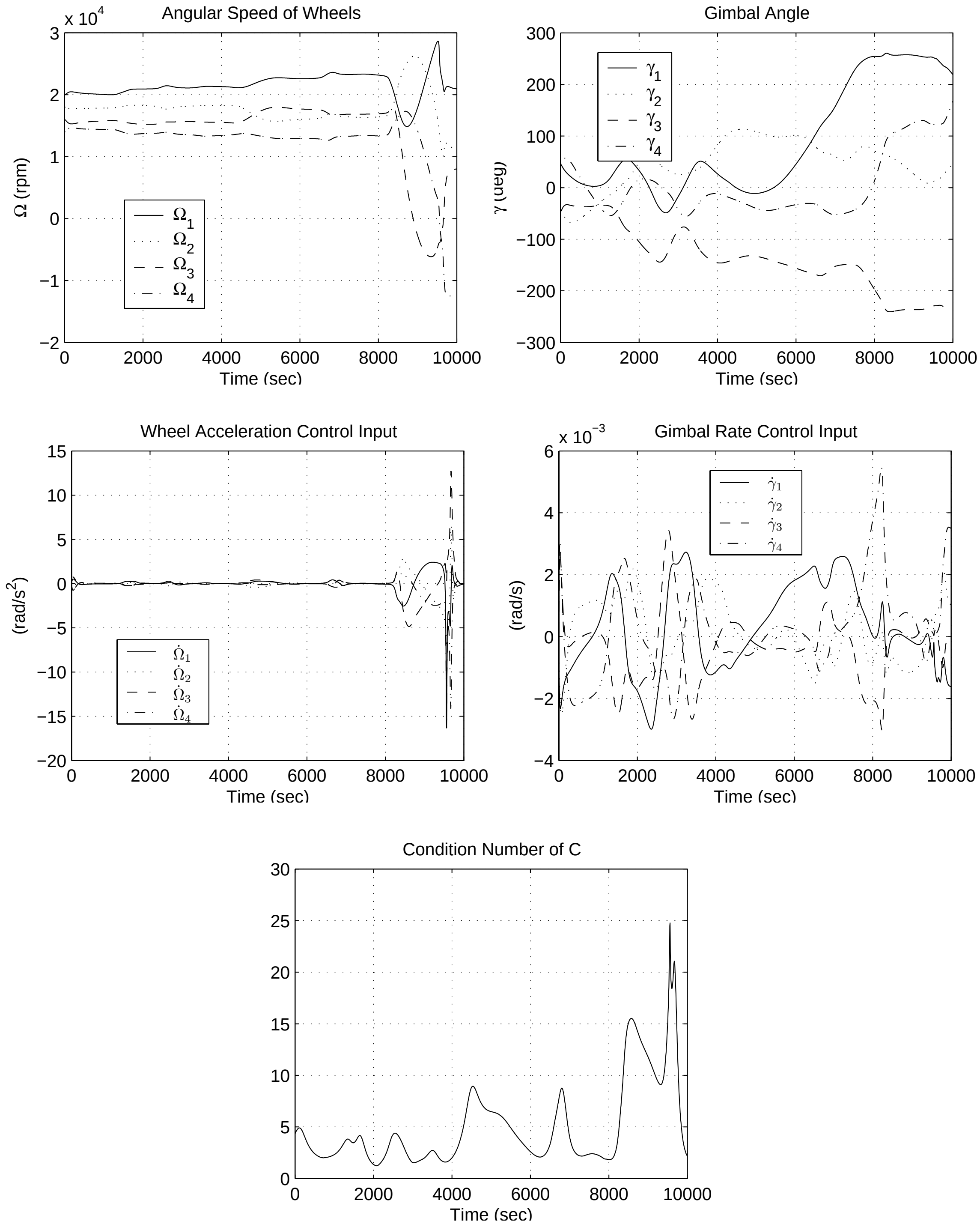

Figure 3. Simulation Without Singularity Avoidance

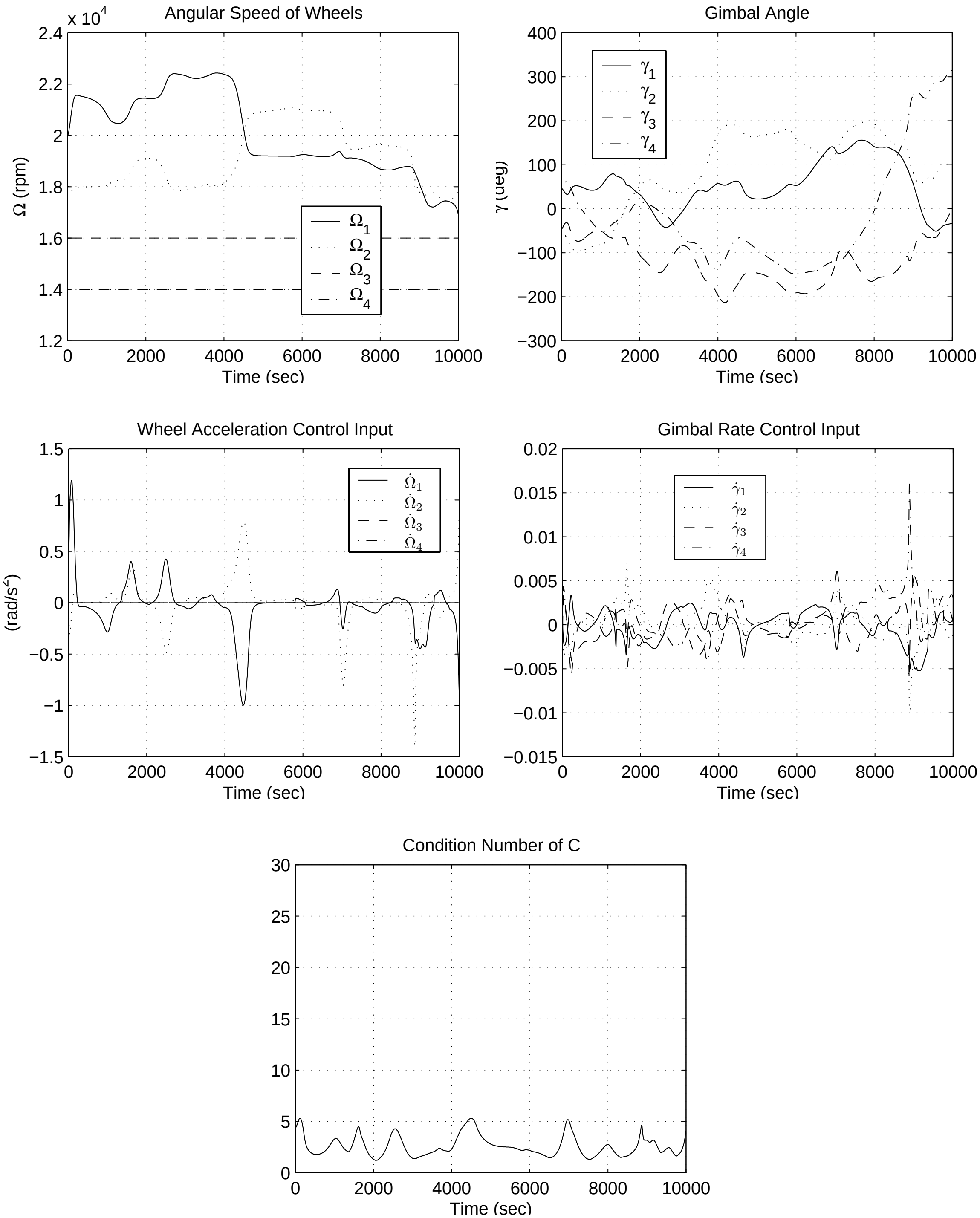

Figure 4. Simulation With Singularity Avoidance.